Consequences of the axiom of choice

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Olivetti seminar

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What is... the Axiom of Choice?

Definition

The Axiom of Choice (AC) is the following claim: for every non-empty collection X of non-empty sets, we can find a <u>choice function</u> $f\colon X \to \bigcup X$ such that $x \in f(x)$ for all $x \in X$.

$\bigcup X$ is the union of X, meaning the set of all elements of sets in X.

A brief history

- Introduced by Zermelo (1904) in his paper about a proof of the Well-ordering principle.
- Caused a controversy because of its non-constructive nature: so how to construct/define an object given by the Axiom of Choice?
- (Gödel 1938) Axiom of choice is compatible with ZF.
- (Cohen 1963) Axiom of choice is not provable from ZF.
- Now widely accepted as an essential axiom of mathematics.

Equivalent forms

- For every $\{X_i \mid i \in I\}$, $\prod_{i\in I} X_i$ is non-empty.
- Zorn's lemma
- Well-ordering principle: every set admits a well-order.
- Every surjective function has a right inverse.

Forms of Choice

Definition (Countable Choice)

The Axiom of Countable Choice is the following claim: for every non-empty countable collection X of non-empty sets, we can find a choice function $f : X \to \bigcup X$ such that $x \in f(x)$ for all $x \in X$.

In other words: $\{A_n \mid n \in \mathbb{N}\}$ has a choice function $f : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} A_n$ such that $f(n) \in A_n$ for all n.

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Countable Choice is weaker than the full Choice.

Definition (Dependent Choice)

The Axiom of Dependent Choice is the following claim: if $\prec$ is a binary relation over a set X such that

$$\forall x \in X \exists y (x \prec y).$$

For $x_0 \in X$, we can find a $\prec$-chain $\langle x_n \mid x \in \mathbb{N} \rangle$, that is, $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$.

The choice of x_{n+1} 'depends on' x_n, hence we call it Dependent Choice.
Dependent Choice implies Countable Choice. Dependent Choice is weaker than the Full Choice.

Boolean Prime Ideal Theorem

Definition

A ring R is Boolean if $a^2 = a$ for all $a \in R$.

Remark

Every Boolean ring is commutative. A typical example of a Boolean ring is $(\mathcal{P}(X), \triangle, \cap, \varnothing, X)$. ($A \triangle B := (A \setminus B) \cup (B \setminus A)$.)

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Definition (Boolean Prime Ideal Theorem, BPIT)

BPIT is the following statement: every Boolean ring has a nontrivial prime ideal.

BPIT is weaker than Choice.

Choice in Set theory

Theorem

Let $\{X_n \mid n \in \mathbb{N}\}$ be a countable collection of countable sets. Then the union $\bigcup_{n\in\mathbb{N}} X_n$ is also countable.

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Proof.

Since X_n is countable, there is a surjection $f : \mathbb{N} \to X_n$ for each $n \in \mathbb{N}$. For each n, choose a surjection $f_n : \mathbb{N} \to X_n$, then define $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n$ by

$$f(n, m) = f_n(m).$$

Then f is a surjection. □

Theorem

Countable Choice proves the following claims are all equivalent: for a given set X,

1. *X is infinite, that is, for every natural number n, $|\{0, 1, \cdots, n-1\}| \leq |X|$.*
2. *X is Dedekind-infinite, that is, X has a proper subset $S \subsetneq X$ which has the same 'size' with X.*

$1 \implies 2$ must require Countable Choice. Proving $2 \implies 1$ needs no Choice.

Proof.

Assume that X is infinite. For each $n \in \mathbb{N}$, consider

$$\mathcal{X}_n = \{A \subseteq X \mid |A| = n\}.$$

Since X is infinite, $\mathcal{X}_n \neq \varnothing$ for each n.

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Choose $A_n \in \mathcal{X}_n$ for each n, and consider $A = \bigcup_{n\in\mathbb{N}} A_n$. A is countable.
Let $A = \{a_n \mid n \in \mathbb{N}\}$ be an enumeration of A. Define

$$f(x) = \begin{cases} a_{n+1} & \text{if } x = a_n, \\ x & \text{if } x \notin A. \end{cases}$$

Then f is a bijection between X and $X \setminus \{a_0\}$. □

Independence

Theorem

The following statements are not provable without Countable Choice:

1. *Every infinite set is Dedekind-infinite.*
2. *A countable union of countable sets is countable.*

Cardinal Arithmetic

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Its proof is not too hard, but requires the familiarity with ordinals. So let me skip it!

Choice in Analysis

We know there are two equivalent definition of the continuity...

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for each $a \in \mathbb{R}$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all x,

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

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A function $f : \mathbb{R} \to \mathbb{R}$ is <u>continuous</u> if for each $a \in \mathbb{R}$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all x,

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Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is <u>sequentially continuous</u> if for every $a \in \mathbb{R}$ and a sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ converging to a, we have $f(a_n) \to f(a)$ as $n \to \infty$.

We can see that sequential continuity implies continuity under Countable Choice:

Proof.

Suppose that f is not continuous. Then there must be $a \in \mathbb{R}$ and $\varepsilon > 0$ such that for every $\delta > 0$ we can find x such that

$$|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon.$$

For each $n \in \mathbb{N} \setminus \{0\}$, choose a_n such that $|a_n - a| < 1/n$ and $|f(a_n) - f(a)| \geq \varepsilon > 0$. Then $a_n \to a$, but $f(a_n)$ does not converge to $f(a)$. □

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Theorem

We cannot prove continuity from sequential continuity without Countable Choice.

Choice in Algebra

One of main implications of Zorn's lemma in algebra class is:

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Theorem

1. *Choice is equivalent to the claim that every commutative ring has a maximal ideal.*
2. *BPIT is equivalent to the claim that every commutative ring has a prime ideal.*

Basis

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Theorem (Blass 1984)

If every vector space has a basis, then Choice holds.

Open Problem

Assume that if every F-vector space has a basis. Does Choice hold?

Guest axiom: Regularity

Blass' proof relies a less-well known axiom named regularity

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Being well-founded means resembling well-orderness.
Blass' proof is divided into two parts:

1. He proved that if every vector space has a basis, then the Axiom of Multiple Choice (AMC) holds.
2. Assuming Regularity, Multiple Choice implies Choice.

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Definition

The axiom of regularity is the claim that the membership relation $\in$ is well-founded.

Being well-founded means resembling well-orderness.
Blass' proof is divided into two parts:

1. He proved that if every vector space has a basis, then the Axiom of Multiple Choice (AMC) holds.
2. Assuming Regularity, Multiple Choice implies Choice.

Open Problem

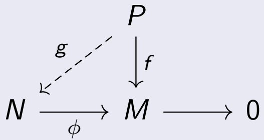
Can we prove it without Regularity if every vector space has a basis, then Choice holds?

Projective and Injective modules

Let R be a ring.

Definition

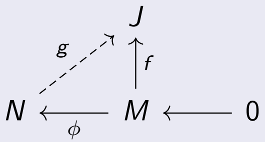
An R-module P is projective if for every epimorphism $\phi\colon N \to M$ and $f\colon P \to M$, we can find $g\colon P \to N$ such that $\phi \circ g = f$.



Free modules are a main example of projective modules.

Definition

An R-module J is injective if for every monomorphism $\phi\colon N \to M$ and $f\colon P \to M$, we can find $g\colon P \to N$ such that $g \circ \phi = f$.



Divisible abelian groups are a main example of injective $\mathbb{Z}$-modules.

Definition

Let R be a ring.

1. The claim 'there are enough projective R-modules' is the following: every R-module is an image of a projective module.
2. The claim 'there are enough injective R-modules' is the following: every R-module is a submodule of an injective module.

Enough projectivity and injectivity is necessary to develop homological algebra.

There might be no enough projective or injective modules

Theorem (Blass 1979)

The following are equivalent:

1. *Every free abelian group is projective.*
2. *Every divisible abelian group is injective.*
3. *Choice.*

Theorem (Blass 1979)

1. *We cannot prove there is an injective abelian group without Choice.*
2. *If there is enough projective abelian groups, then Dependent Choice holds.*

Algebraic closure

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Remark (Banaschewski 1992)

BPIT suffices to prove it.

Theorem (Läuchli 1962)

If we do not assume Choice or BPIT, then we cannot prove:

1. *Every field has an algebraic closure, and*
2. *The algebraic closure of $\mathbb{Q}$ is unique.*

Choice in Topology

Theorem (Tychonoff)

Assuming Choice, a product of compact spaces is also compact.

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Theorem (Tychonoff)

Assuming Choice, a product of compact spaces is also compact.

Theorem

1. *Choice is equivalent to Tychonoff theorem.*
2. *BPIT is equivalent to Tychonoff theorem for Hausdorff spaces.*

Theorem (Baire's category theorem)

For every complete metric space X, the intersection of a countable family of open dense sets is also dense.

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Theorem (Blair 1977)

Baire's category theorem is equivalent to Dependent Choice.

Choice in Measure theory

We know that Choice implies the existence of a non-Lebesgue measurable subset of $\mathbb{R}$.

Theorem (Vitali)

There is a non-Lebesgue measurable subset of $\mathbb{R}$.

Remark

BPIT also implies there is a non-Lebesgue measurable subset of $\mathbb{R}$.

A mathematical world for measure theorists

Theorem (Solovay 1970)

If ZFC + 'there is an inaccessible cardinal' is consistent, then the combination of the following statements are also consistent:

1. *ZF + Dependent Choice.*
2. *Every subset of $\mathbb{R}$ is Lebesgue measurable.*

A mathematical world for measure theorists

Theorem (Solovay 1970)

If ZFC + ‘there is an inaccessible cardinal’ is consistent, then the combination of the following statements are also consistent:

1. *ZF + Dependent Choice.*
2. *Every subset of $\mathbb{R}$ is Lebesgue measurable.*

Remark (Shelah 1984)

The existence of an inaccessible cardinal is necessary to establish this result.

Borel sets

Definition

Let X be a complete separable metric space. The collection of all Borel sets is the smallest collection that satisfies:

1. It contains all open subsets of X, and
2. It is closed under countable unions and complements.

Borel sets have a focal role in measure theory and descriptive set theory.

Theorem

Assuming Dependent Choice, there is a subset of $\mathbb{R}$ that is not Borel.

Theorem

It is consistent with ZF that every subset of $\mathbb{R}$ is a countable union of countable sets.
Especially, every subset of $\mathbb{R}$ is Borel.

Ways of proving the independence

1. Reducing to a known axiom, like Countable Choice, Dependent Choice, or BPIT.
   e.g., Tychonoff's theorem for Hausdorff spaces implies BPIT.
2. Permutation models and symmetric models.
3. Some other inner models, e.g., $L(\mathbb{R})$.

Permutation model

Start from ZFA: ZF with atoms (or, urelements.)
Basic idea: sets in a permutation model must have some symmetry.

Definition

An atom (or urelement) is an object that are not the empty set but contains no element.

Let A be a set of atoms, and G be a subgroup of a symmetric group of A. (That is, G comprises bijection from A to itself.)

Definition

A collection $\mathcal{F}$ of subsets of G is a normal filter if it satisfies the following condition:

1. $G \in \mathcal{F}$,
2. $X \in \mathcal{F}, X \subseteq Y \implies Y \in \mathcal{F}$,
3. $\mathcal{F}$ is closed under finite intersections, and
4. If $X \in \mathcal{F}$ and $g \in G$, then $gXg^{-1} \in \mathcal{F}$.
5. For $a \in A$, $\{\pi \in G \mid \pi(a) = a\} \in \mathcal{F}$.

We can extend a permutation π for atoms to an $\in$-automorphism:

$$\pi(x) := \{\pi(y) \mid y \in x\}.$$

Definition

- $\mathrm{sym}(x) = \{\pi \mid \pi(x) = x\}$.
- A set x is symmetric if $\mathrm{sym}(x) \in \mathcal{F}$.
- The permutation model is the class of all hereditarily symmetric sets.
  That is, x is in the permutation model if x is symmetric and every element of x is hereditarily symmetric.

Some weird sets

Example (Basic Fraenkel model)

Start with a countable set of atoms $A = \{a_n \mid n \in \mathbb{N}\}$. Let G be the all permutations over A, and let $\mathcal{F}$ be a filter generated by the stabilizers of finite subsets of A, that is, sets of the form

$$\mathrm{fix}_G(S) = \{\pi \mid \forall a \in S(\pi(a) = a)\}$$

for a finite $S \subseteq A$.
Then the resulting permutation model thinks A satisfies the following: every subset of A is finite or cofinite.
We call a set with this property <u>amorphous</u>.

Example (Second Fraenkel model)

Work with the same A, but now let G be the set of all permutations of A that switches a_{2n} and a_{2n+1}. Use the same filter we described before.
Then the resulting permutation model contains a set $R = \{P_n \mid n \in \mathbb{N}\}$, where $P_n = \{a_{2n}, a_{2n+1}\}$. However, the permutation model thinks $\prod_{n\in\mathbb{N}} P_n = \varnothing$.

Coda

- Permutation models are not models of ZF because it does not satisfy Regularity.
- However, there is a way to 'transform' a permutation model to a model of ZF while pertaining the desired failure of Choice: Jech-Sochor theorem.
- Jech-Sochor theorem states for any given permutation model M and a given ordinal α, we can construct a submodel N of a forcing extension $V[G]$ that is isomorphic to the permutation model M up to α.

Questions

Thank you!