FIXED POINT RECURSIVE ORDINAL FUNCTIONS

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The initial motivation for this note is to formulate a natural class of ordinal functions including Veblen functions. I am planning to expand this note by adding the definition of 'fixed point recursive set function' that I am still working on. I welcome any comments or ideas.

1. Fixed point recursive ordinal functions

Definition 1.1. A class of *fixed point recursive ordinal function* (abbr. FPRO functions) is defined inductively as follows:

(1) Projections, Identity, zero, the successor operator is FPRO.
(2) The composition of FPRO functions is also FPRO.
(3) (Primitive recursion) If f is FPRO, then so is g satisfying the following:

$$g(\alpha, \vec{\pi}) = f(\sup_{\xi<\alpha} g(\xi, \vec{\pi}), \vec{\pi}).$$

(4) (Fixed point recursion) If f is FPRO, then $\mathsf{Fix}_f = \varphi$ is also FPRO, where φ satisfies the following: For each α and β with a parameter $\vec{\pi}$, $\varphi(\alpha, \beta, \vec{\pi})$ is the β-th fixed point for functions $\vec{\zeta} \mapsto f(\vec{\zeta}, \vec{\pi})$ and $\eta \mapsto \varphi(\xi, \eta, \vec{\pi})$ for all $\xi < \alpha$.
   Equivalently, $\varphi(\alpha, \beta, \vec{\pi})$ is the least ordinal γ satisfying the following:
   - For each $\eta < \beta$, $\varphi(\alpha, \eta, \vec{\pi}) < \gamma$.
   - If $\vec{\zeta} < \gamma$, then $f(\vec{\zeta}, \vec{\pi}) < \gamma$.
   - If $\xi < \alpha$ and $\eta < \gamma$, then $\varphi(\xi, \eta, \vec{\pi}) < \gamma$.

Lemma 1.2. *The Veblen function is FPRO.*

Proof. Let φ be the function obtained from the succcessor function by the fixed point recursion. Then we can see that $\varphi(0, \xi)$ is the ξth ordinal closed under the successor operator, so $\varphi(0, \xi) = \omega \cdot \xi$. Similarly, $\varphi(1, \xi)$ is the ξth ordinal closed under $\eta \mapsto \omega \cdot \eta$, so $\varphi(1, \xi) = \omega^{\omega \cdot \xi}$ for $\xi \geq 1$. ($\varphi(1, 0) = 0$.) By the same logic, we can see that $\varphi(2, \xi) = \varepsilon_\xi$, so we can see by induction that $\varphi(1 + \alpha, \beta) = \varphi_\alpha(\beta)$ for $\alpha \geq 1$. □

FPRO functions can represent not only Veblen functions but also finitary Veblen functions:

Lemma 1.3. *For each n, $(\xi_0, \cdots, \xi_{n-1}) \mapsto \varphi(\xi_0, \cdots, \xi_{n-1})$ is FPRO.*

Proof. An essentially the same proof as the previous lemma works. □

However, the expressive power of the FPRO function is bounded by finitary Veblen functions:

Proposition 1.4. *Let f be an FPRO function. Then there is a natural number n such that for every $\alpha_0, \cdots \alpha_{k-1}$, we have*

$$f(\alpha_0, \cdots \alpha_{k-1}) \leq \varphi(\max(\alpha_0, \cdots, \alpha_{k-1}), \underbrace{0, \ldots, 0}_{n \text{ times}}). \qquad (1)$$

Proof. Let us follow the proof presented in [1]. We prove it by induction on FPRO functions, and ignore trivial cases. For a notational convenience, let us use the following convention:

$$\varphi\binom{\alpha}{n} = \varphi(\alpha, \underbrace{0, \ldots, 0}_{n \text{ times}})$$

For composition, suppose that f, g_0, $\cdots$, g_{m-1} be FPRO functions of arity m and suppose that they satisfy (1) by n. Then

$$f(g_0(\vec{\alpha}), \cdots, g_{m-1}(\vec{\alpha})) \le \varphi\binom{\max(g_0(\vec{\alpha}), \cdots, g_{m-1}(\vec{\alpha}))}{n}$$
$$\le \varphi\binom{\varphi\binom{\max\vec{\alpha}}{n}}{n} \le \varphi\binom{\max\vec{\alpha}}{n+1}.$$

For primitive recursion, suppose that f satisfies (1) by n, and let g be a function defined by primitive recursion by f with a parameter $\vec{\pi}$. Let us prove (1) by induction on α: Then we have

$$g(\alpha, \vec{\pi}) = f(\sup_{\xi<\alpha} g(\xi, \vec{\pi}), \vec{\pi}) \le \varphi\binom{\max(\sup_{\xi<\alpha} g(\xi, \vec{\pi}), \vec{\pi})}{n}$$
$$\le \varphi\binom{\max\left(\sup_{\xi<\alpha}\varphi\binom{\max(\xi,\vec{\pi})}{n+1}, \vec{\pi}\right)}{n} \le \varphi\binom{\max(\alpha, \vec{\pi}))}{n+1}$$

The last inequality holds since $\varphi\binom{\max(\alpha,\vec{\pi}))}{n+1}$ is greater than $\sup_{\xi<\max(\alpha,\vec{\pi})}\varphi\binom{\xi}{n+1}$ and is closed under the map $\varphi\binom{\cdot}{n}$.

Lastly, let f be FPRO and g be a function given by the fixed point recursion on f with parameter $\vec{\pi}$. Suppose that f satisfies (1) by n. Now let us prove the following inequality by induction on $(\alpha, \beta) \in \mathrm{Ord}^2$ under the lexicographical order:

$$g(\alpha, \beta, \vec{\pi}) \le \varphi\binom{\max(\alpha, \beta, \vec{\pi})}{n+1}. \tag{2}$$

Suppose that (2) holds for all $(\xi, \eta) < (\alpha, \beta)$. Recall that $\gamma = g(\alpha, \beta, \vec{\pi})$ is the least ordinal satisfying the following:

- For each $\eta < \beta$, $g(\alpha, \eta, \vec{\pi}) < \gamma$.
- If $\vec{\zeta} < \gamma$, then $f(\vec{\zeta}, \vec{\pi}) < \gamma$.
- If $\xi < \alpha$ and $\eta < \gamma$, then $g(\xi, \eta, \vec{\pi}) < \gamma$.

Now we can see that $\delta = \varphi\binom{\max(\alpha,\beta,\vec{\pi})}{n+1}$ satisfies the following:

- For each $\eta < \beta$, $\varphi\binom{\max(\alpha,\eta,\vec{\pi})}{n+1} < \delta$.
- If $\vec{\zeta} < \delta$, then $\varphi\binom{\max(\vec{\zeta},\vec{\pi})}{n} < \delta$.
- If $\xi < \alpha$ and $\eta < \delta$, then $\varphi\binom{\max(\xi,\eta,\vec{\pi})}{n+1} < \delta$.

By the inductive assumption, we have (2) for (α, η) for all $\eta < \beta$, or (ξ, η) for every $\xi < \alpha$ and $\eta \in \mathrm{Ord}$. Hence we get $\gamma \le \delta$. □

However, we have not proved that FPRO functions are total. It turns out that KP_0 with Π_2-Set Induction proves every FPRO function terminates, where KP_0 is KP with Set Induction restricted to bounded formulas. Before jumping into the problem, let us provide the following useful characterization of Π_n-Set Induction, which says it allows us to induct over a certain type of class well-order:

Proposition 1.5 ([4, Lemma 4.4]). *Let $E_0 = \varepsilon_{\mathrm{Ord}+1}$ be a class order for the least epsilon number greater than* Ord. *(See Definition 4.1 of [4] for the details.) Then we have the following:*

(1) KP_0 with Σ_1-Set Induction proves E_0 is linear.
(2) Working over KP_0 with Π_n-Set Induction, let $A(x)$ be a Π_n-formula. If we have

$$\forall s \in E_0[((\forall t <_{E_0} s\ A(t)) \to A(s)) \to A(s)],$$

then for every (standard) natural number k, we have

$$\forall s \in E_0[(\forall t <_{E_0} s\ A(t)) \to (\forall t <_{E_0} s + \mathrm{Ord}^k\ A(t))].$$

Hence we have the following:

Corollary 1.6. KP_0 *with Π_2-Set Induction proves the Π_2-induction over Ord^k for each (standard) natural number k.*

Proposition 1.7. KP_0 *with* Π_2*-Set Induction proves the following: The class of* Σ_1*-definable functions is closed under fixed point recursion on ordinals.*

To show Proposition 1.7, we need to find a Σ_1-formula defining a fixed point recursion of a Σ_1-definable function. It involves the following variant of Gödel's diagonal lemma:

Lemma 1.8. *Let* $\phi(n,x)$ *be a* Σ_1*-formula. Then we can find a* Σ_1*-formula* ψ *such that*

$$\mathsf{PRS} \vdash \forall x[\psi(x) \leftrightarrow \phi(\ulcorner\psi\urcorner, x)].$$

Proof. Consider the following computable function for formulas θ with two free variables:

$$d(\ulcorner\theta\urcorner) = \ulcorner\theta(\underline{\ulcorner\theta\urcorner}, x)\urcorner$$

Here $\underline{n}$ means the numeral for n, i.e., the symbol for $1 + \cdots + 1$ with n many 1. d is computable, so it is Δ_0-definable with a parameter ω over PRS. Now for a given Σ_1-formula $\phi(n,x)$, let $\phi'(n,x)$ be a Σ_1-formula that is PRS-provably equivalent to

$$\exists m < \omega(d(n) = m \land \phi(m,x)).^{1}$$

Then for every Σ_1-formula θ with two variables we have

$$\mathsf{PRS} \vdash \phi'(\underline{\ulcorner\theta\urcorner}, x) \leftrightarrow \phi(\ulcorner\theta(\underline{\ulcorner\theta\urcorner}, x)\urcorner, x).$$

Now take $\theta \equiv \phi'$, and let $\psi(x) \equiv \phi'(\underline{\ulcorner\phi'\urcorner}, x)$. Then we get

$$\mathsf{PRS} \vdash \psi(x) \leftrightarrow \phi(\ulcorner\psi\urcorner, x).$$

Since x occurs free, we have the desired result by universal generalization. □

Proof of Proposition 1.7. We will prove the following: Let f be a Σ_1-definable function on ordinals and $\vec{\pi}$ be parameter ordinals. Then we can find a Σ_1-definable class function φ on ordinals satisfying the definition of fixed point recursion on f and $\vec{\pi}$.

We want to find a Σ_1-formula $\psi(\alpha,\beta,\vec{\pi},\gamma)$ defining φ. Then ψ must satisfy the following: $\psi(\alpha,\beta,\vec{\pi},\gamma)$ holds if and only if γ is the least ordinal satisfying the conjunction of the following:

(1) For every $\eta < \beta$, there is $\nu < \gamma$ such that $\psi(\alpha,\eta,\vec{\pi},\nu)$.
(2) For every $\vec{\zeta} < \eta$, we have $f(\vec{\zeta},\vec{\pi}) < \gamma$.
(3) For every $\xi < \alpha$ and $\eta < \gamma$, there is $\nu < \gamma$ such that $\psi(\xi,\eta,\vec{\pi},\nu)$.

The above statement looks complex, but it is a rephrase of the statement 'ψ defines a fixed point recursion of f.' Also, ν always means the value of φ for some input in the above formulas.

We can see that the first and the third expressions only involve bounded quantifiers. The second expression is more subtle because f is Σ_1-definable and not Δ_0-definable. However, observe that the expression $f(\vec{\xi},\vec{\pi}) < \gamma$ is provably Δ_1 over KP_1 + Σ_1-Set Induction because it is equivalent to one of the following:

- For every ν, if $f(\vec{\xi},\vec{\pi}) = \nu$ then $\nu < \gamma$.
- There is ν such that $f(\vec{\xi},\vec{\pi}) = \nu$ and $\nu < \gamma$.

Here $f(\vec{\xi},\vec{\pi}) = \nu$ should be understood as a Σ_1-definition of f, so each formulas are Π_1 and Σ_1 respectively.

To find the desired ψ, consider the following template formula $\phi(\ulcorner\theta\urcorner,\alpha,\beta,\vec{\pi},\gamma)$ that is the conjunction of the following, which is obtained from the condition for ψ by replacing all occurrences of ψ with $\vDash_{\Sigma_1} \ulcorner\theta\urcorner$.

(1) For every $\eta < \beta$, there is $\nu < \gamma$ such that $\vDash_{\Sigma_1} \ulcorner\theta\urcorner(\alpha,\eta,\vec{\pi},\nu)$.
(2) For every $\vec{\zeta} < \eta$, we have $f(\vec{\zeta},\vec{\pi}) < \gamma$.
(3) For every $\xi < \alpha$ and $\eta < \gamma$, there is $\nu < \gamma$ such that $\vDash_{\Sigma_1} \ulcorner\theta\urcorner(\xi,\eta,\vec{\pi},\nu)$.
(4) For every $\delta < \gamma$, we have one of the following:
  (a) There is $\eta < \beta$ such that for every $\nu < \delta$, we can find μ such that $\vDash_{\Sigma_1} \ulcorner\theta\urcorner(\alpha,\eta,\vec{\pi},\mu)$ and $\mu \neq \nu$, or
  (b) There is $\vec{\zeta} < \eta$ such that $f(\vec{\zeta},\vec{\pi}) \geq \delta$, or
  (c) There is $\xi < \alpha$ and $\eta < \delta$ such that for every $\nu < \delta$, we have μ such that $\vDash_{\Sigma_1} \ulcorner\theta\urcorner(\xi,\eta,\vec{\pi},\mu)$ and $\mu \neq \nu$.

[1] I suspect it works even over a weaker fragment of set theory, like provident set theory.

The reader may wonder why the last three formulas do not use a more direct expression $\vDash_{\Sigma_1} \ulcorner\theta\urcorner$. However, $\vDash_{\Sigma_1} \ulcorner\theta\urcorner$ is a Π_1-expression and we want to have its Σ_1-form. There is no way to express $\vDash_{\Sigma_1} \ulcorner\theta\urcorner$ in a Σ_1-way, but observe that our formula ψ will define a function. This means we will have the following equivalence:

$$\psi(\alpha,\beta,\vec{\pi},\gamma) \iff \forall\mu[\psi(\alpha,\beta,\vec{\pi},\mu)\to\gamma=\mu].$$

If ψ is Σ_1, then the latter is Π_1. Thus we can use the negation of

$$\forall\mu[\psi(\alpha,\beta,\vec{\pi},\mu)\to\gamma=\mu]$$

as a substitute of $\neg\psi$. Also, the template formula ϕ is a Σ-formula. However, we can find its Σ_1-equivalence due to Σ_1-Collection, so we can conflate ϕ with its Σ_1-equivalence.

Now apply Lemma 1.8 to ϕ to get a Σ_1-formula ψ trying to define the fixed point iteration of f. Now we claim by induction on $\mathrm{Ord}\cdot\alpha+\beta$ that $\exists\gamma\psi(\alpha,\beta,\vec{\pi},\gamma)$ holds.

Suppose that $\exists\gamma\psi(\xi,\eta,\vec{\pi},\gamma)$ holds for all $(\mathrm{Ord}\cdot\xi+\eta)<(\mathrm{Ord}\cdot\alpha+\beta)$. By the inductive assumption, we have defined

- $\varphi(\xi,\eta,\vec{\pi})$ for every $\xi<\alpha$ and $\eta\in\mathrm{Ord}$.
- $\varphi(\alpha,\eta,\vec{\pi})$ for every $\eta<\beta$.

via a Σ_1-formula ψ. By Σ_1-Replacement, $\gamma_0=\sup\{\varphi(\alpha,\eta,\vec{\pi})\mid\eta<\beta\}$ is a set. By the usual Σ_1-recursion, define

$$\gamma_{n+1}=\sup(\{\varphi(\xi,\eta,\vec{\pi})\mid\xi<\alpha\land\eta<\gamma_n\}\cup\{f(\vec{\eta},\vec{\pi})\mid\eta<\gamma_n\})$$

and take $\gamma=\sup_{n<\omega}\gamma_n$. Then γ satisfies $\psi(\alpha,\beta,\vec{\pi},\gamma)$. □

2. Fixed point recursion with dilator parameters

Is there any way to get ordinal functions to grow faster than ordinal functions obtained by fixed point recursion? It turns out that dilator-based iteration gives such functions. I will assume the familiarity with dilators. The reader may refer to [2] or [3] to learn about dilators. I will define an iteration scheme, but not prove its convergence.

Definition 2.1. For an ordinal function f and a dilator D, let us define Iter_f^D recursively as follows:

(1) $\mathsf{Iter}_f^{\underline{0}}(\alpha)=\alpha$.
(2) $\mathsf{Iter}_f^{D+\underline{1}}(\alpha)=f(\mathsf{Iter}_f^D(\alpha))$.
(3) $\mathsf{Iter}_f^{\sum_{\xi<\beta}D_\beta}(\alpha)=\sup_{\xi<\beta}\mathsf{Iter}_f^{\sum_{\eta<\xi}D_\eta}(\alpha)$ if $D_\xi\neq\underline{0}$ for all $\xi<\beta$.
(4) If D' is a perfect dilator, then $f^{D+D'}(\alpha)$ is the least ordinal γ greater than all of $f^{D+D'}(\beta)$ for all $\beta<\alpha$ and closed under $f^{D+\mathsf{Sep}(D')(\xi,\cdot)}$ for all $\xi<\gamma$.

Example 2.2. Let S be the successor function, i.e., $S(\alpha)=\alpha+1$. Then we have

$$\mathsf{Iter}_S^{\underline{\beta}}(\alpha)=\alpha+\beta.$$

Also, $\mathsf{Iter}_S^{\mathsf{Id}}(\alpha)$ is the least ordinal γ greater than all of $\mathsf{Iter}_S^{\mathsf{Id}}(\beta)$ for all $\beta<\alpha$ closed under the addition. Hence $\mathsf{Iter}_S^{\mathsf{Id}}(\alpha)=\omega^\alpha$ for $\alpha\geq1$. ($\mathsf{Iter}_S^{\mathsf{Id}}(0)=0$.) Similarly, we have that $\mathsf{Iter}_S^{\mathsf{Id}+\mathsf{Id}}(1+\alpha)=\varepsilon_\alpha$, $\mathsf{Iter}_S^{\mathsf{Id}+\mathsf{Id}+\mathsf{Id}}(1+\alpha)=\varphi_2(\alpha)$, and so on. (Remind that $\mathsf{Sep}(\mathsf{Id})(\alpha,\beta)=\alpha$.) In general, we have

$$\mathsf{Iter}_S^{\mathsf{Id}\cdot\underline{(1+\alpha)}}(1+\beta)=\varphi(\alpha,\beta).$$

Then what is $\mathsf{Iter}_S^{\mathsf{Id}^2}(\alpha)$? Well, $\gamma=\mathsf{Iter}_S^{\mathsf{Id}^2}(\alpha)$ must be closed under $\varphi(\beta,\cdot)$ for all $\beta<\gamma$, so we can guess $\mathsf{Iter}_S^{\mathsf{Id}^2}(1+\alpha)=\varphi(1,0,\alpha)$. In general, we get

$$\mathsf{Iter}_S^{\mathsf{Id}^{e_0}\cdot\underline{\alpha_0}+\cdots\mathsf{Id}^{e_n}\cdot\underline{\alpha_n}}(1+\beta)=\varphi\begin{pmatrix}\alpha_0&\cdots&\alpha_n&\beta\\ e_0&\cdots&e_n&0\end{pmatrix}.$$

We may push it further: $\mathsf{Iter}_S^{(1+\mathsf{Id})^{\mathsf{Id}}}(1+\beta)$ will be a fixed point for transfinite Veblen functions. I guess Iter_S^D is closely related to the ordinal collapsing function ψ_Ω, and if ε_X is the dilator returning the epsilon number, then $\mathsf{Iter}_S^{\varepsilon_{1+\mathsf{Id}}}(1)$ is the proof-theoretic ordinal of KP.

I guess reaching an ordinal at the level of the proof-theoretic ordinal of ID_2 (or around $\psi_\Omega(\varepsilon_{\Omega_2+1})$) requires recursion on ptykes of type $\mathsf{Dil}\to\mathsf{Dil}$, i.e., For a ptyx $P\colon\mathsf{Dil}\to\mathsf{Dil}$, and a dilator D, we need to define a new ptyx $\mathsf{Iter}_P^D\colon\mathsf{Dil}\to\mathsf{Dil}$. It seems to me that Girard's ptyx Λ describes something relevant.

References

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