INFINITE HAT PROBLEM AS A CHOICE PRINCIPLE

HANUL JEON

Abstract. We show that a variant of Feferman's model satisfies $\mathsf{ZF} + \mathsf{AC}(2)$ with the non-existence of a non-principal ultrafilter over ω. We also see that the same model thinks there is a solution to the infinite hat problem but has no E_0-transversal.

1. Introduction

The solution of the infinite hat problem is closely related to constructing a parity function $p\colon 2^\omega \to 2$ that satisfies the following property: if $s, t \in 2^\omega$ are differing at exactly one place, then $p(s) = 1 - p(t)$. It is known by Lubarsky [1] that we can provide a solution to the infinite hat problem by using $\mathsf{AC}(2)$:

Definition 1.1. $\mathsf{AC}(n)$ is the assertion that if $\mathcal{A}$ is a family of n-element sets then we can find a choice function $f\colon \mathcal{A} \to \bigcup \mathcal{A}$.

Proposition 1.2 (Lubarsky [1], $\mathsf{AC}(2)$). *There is a parity function.*

Proof. Let E_0 be the equivalence relation over 2^ω given by identifying two binary sequences that are the same up to finitely many digits. Define E_{parity} by

$$(s,t) \in E_{parity} \text{ iff } (s,t) \in E_0 \text{ and } \#\{n < \omega \mid s(n) \neq t(n)\} \text{ is even.}$$

Then E_{parity} is also an equivalence relation. Furthermore, for given $x \in 2^\omega$, $[x]_{E_0}$ is decomposed into two E_{parity}-equivalence classes.

Now let us consider the family $\mathcal{A} = \{A_a \mid a \in 2^\omega/E_0\}$ given by $A_a = \{[s]_{E_{parity}} \mid s \in a\}$. Clearly $|A_a| = 2$ for each $a \in 2^\omega/E_0$. By $\mathsf{AC}(2)$, we have a choice function f for $\mathcal{A}$. Then we can define p by using f by

$$p(s) = \begin{cases} 1 & \text{if } s \in f([s]_{E_0}), \\ 0 & \text{otherwise.} \end{cases}$$

p is clearly a parity function. □

It was open whether there can be a parity function when there is no ultrafilter over ω or there is no E_0-transversal. The answer to this question is positive and was shown first by Serafin [7] using geometric set theory. In this short draft, we sketch how to prove this independence by means of symmetric models.

2. Infinitely many prisoners without ultrafilters over ω

The main idea of this section is to modify Feferman's model for the independence of the existence of ultrafilters over ω from ZF. We do not delve into the definition of symmetric models. It is available in other materials (like [4].) I sometimes follow notations in modern materials like [5].

The model and proof are available in [2], §17. For the reader's convenience, let us provide the definition of the model:

Definition 2.1. Let $\mathbb{P} = \mathsf{Fn}(\omega\times\omega, 2)$ be the forcing adding ω many Cohen reals. Fix a $\mathbb{P}$-generic filter G over V. For each $X \subseteq \omega\times\omega$, define α_X acting on $\mathbb{P}$ by

$$\alpha_X(p)(i,j) = \begin{cases} 1 - p(i,j) & \text{if } (i,j) \in X \\ p(i,j) & \text{if } (i,j) \notin X. \end{cases}$$

I would like to thank Luke Serafin for introducing an interesting problem to me, and Asaf Karagila for how to make DC true over the model.

Consider the group $\mathcal{G}$ generated by all α_Xs, and consider the normal filter $\mathcal{F}$ generated by

$$\operatorname{fix}_{\mathcal{G}}(E \times \omega) := \{\alpha_X \mid X \cap (E \times \omega) = \varnothing\}$$

for some finite subset E of ω. Consider the model $N = \mathsf{HS}^{V[G]}_{\mathcal{F}}$.

Intuitively, $\mathbb{P}$ adds a generic infinite tabular whose cells are filled by either 0 or 1. The group $\mathcal{G}$ and the filter $\mathcal{F}$ impose the symmetry over the tabular, which allows permuting all but finitely many columns.

Unfortunately, this model has a drawback: it does not satisfy $\mathsf{AC}(2)$! However, a slight modification provides a way to allow $\mathsf{AC}(2)$. Before providing the definition, let us observe that the group given by α_Xs is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\omega\times\omega}$ with pointwise addition, under the map

$$\alpha_X \mapsto \lambda(i,j) \in \omega\times\omega.\ [\texttt{if}\ \ (i,j) \in X\ \texttt{ then 1 else 0}].$$

Definition 2.2. Now consider the forcing $\mathbb{P} = \mathsf{Fn}(\omega\times\omega, 3)$. The only difference is that the codomain of the generic function is 3 instead of 2. Fix a $\mathbb{P}$-generic filter G over V. For each $v \in (\mathbb{Z}/3\mathbb{Z})^{\omega\times\omega}$, define α_v by

$$\alpha_v(p)(i,j) = [\alpha_v(i,j) + p(i,j)]_3,$$

where $[i+j]_3$ is the addition as elements of $\mathbb{Z}/3\mathbb{Z}$ whose value is in $3 = \{0,1,2\}$.

Then consider $\mathcal{G}$ generated by α_vs. Then consider the filter $\mathcal{F}$ generated by

$$\operatorname{fix}_{\mathcal{G}}(E \times \omega) := \{\alpha_v \mid (E\times\omega) \subseteq \alpha_v^{-1}[\{0\}]\}.$$

The definition of N is the same as before.

We may view that the new forcing adds a generic infinite tabular whose cells are filled by elements of $\mathbb{Z}/3\mathbb{Z}$, and $\mathcal{G}$ and $\mathcal{F}$ imposes a symmetry for permuting all columns but finitely many ones. It is clear that $\mathcal{G}$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\omega\times\omega}$. By the same argument given in [2, §17], N does not have an ultrafilter over ω:

Proposition 2.3. *N thinks there is no ultrafilter over ω.*

Proof. For each n, define

$$\dot{a}_n = \{(\check{k}, p) \mid k < \omega,\ p \in \mathbb{P},\ p(n,k) = 1\}.$$

We can see that $\operatorname{sym}_{\mathcal{G}}(\dot{a}_n) := \{\alpha_v \mid \alpha_v(\dot{a}_n) = \dot{a}_n\} \supseteq \operatorname{fix}_{\mathcal{G}}(\{n\}\times\omega)$, so $a_n := \dot{a}_n^G \in N$.

Now assume that $U = \dot{U}^G \in N$. Suppose that $p \in G$ forces $\dot{U}$ is an ultrafilter over ω. Then we can find a finite $E \subseteq \omega$ such that $\operatorname{sym}_{\mathcal{G}}(\dot{U}) \supseteq \operatorname{fix}_{\mathcal{G}}(E\times\omega)$. Choose $l > \max E$ and $q \le p$ such that q decides $\dot{a}_l \in \dot{U}$. Without loss of generality, we only consider the case when $q \Vdash \dot{a}_l \in \dot{U}$.

Choose a large m_0 such that $(l,m) \notin \operatorname{dom} q$ for $m > m_0$, and take $v \in (\mathbb{Z}/3\mathbb{Z})^\omega$ by

$$v(i,j) = \begin{cases} 1 & \text{if } i = l \land j > m_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\alpha_v \in \operatorname{fix}_{\mathcal{G}}(E\times\omega)$ and $\alpha_v q = q$.

Let us consider $a_l^{(i)}$ for $i = 0,1,2$ defined by $a_l^{(i)} := (\alpha_v^i \dot{a}_l)^G$. Then we can see that the union of $\{a_l^{(0)}, a_l^{(1)}, a_l^{(2)}\}$ is ω and the intersection of each two elements is a subset of m_0. Furthermore, $a_l^{(0)} = a_l$.

By acting α_v to $q \Vdash \dot{a}_l \in \dot{U}$, we have $q \Vdash \alpha_v \dot{a}_l \in \dot{U}$. Thus $\alpha_l^{(0)}, \alpha_1^{(1)} \in U$, and so $m \in U$. This shows U is principal. ☐

It remains to show that N satisfies $\mathsf{AC}(2)$. To do this, we borrow an idea for separating $\mathsf{AC}(2)$ and $\mathsf{AC}(3)$ presented in [4, §7.4].

Proposition 2.4. *N satisfies $\mathsf{AC}(2)$.*

Proof. It suffices to show the following claim holds: for any two-element set $S \in N$, we can find $x \in S$ such that $\operatorname{sym}(x) \supseteq \operatorname{sym}(S)$. The above claim implies $\mathsf{AC}(2)$ over N (if $V[G]$ is a model of choice, of course), and its proof is presented on [4, Example 7.13], so we omit its details.

Let $S = \{x,y\}$ and $H = \operatorname{sym}(S)$. If we take $K = \operatorname{sym}(x) \cap \operatorname{sym}(S)$, then either $K = H$ or $[H:K] = 2$ since for $\pi, \rho \in H$, $\pi x = \rho x$ iff $\pi K = \rho K$. Now we exclude the case $[H:K] = 2$.

Observe that our $\mathcal{G}$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^\omega$, and H and K are subgroups of it. We can view them as $\mathbb{Z}/3\mathbb{Z}$-vector spaces. Then if $[H:K]$ is finite then it must be equal to $3^{\dim H/K}$, which cannot be 2. ☐

Remark 2.5. By replacing 3 with any finite values, we can prove that $\mathsf{AC}(n)$ is independent of the existence of a non-principal ultrafilter over ω. This shows the hat problem for finitely many hat colors can have a solution even when there is no non-principal ultrafilter over ω.

3. Miscellaneous results

3.1. **Guessing colors without E_0-transversal.** Recall that the E_0-transversal is a set of representatives out of each element of $2^\omega/E_0$, where E_0 is the equivalence relation identifying two binary sequences up to finite differences. Let us briefly sketch how to obtain a model without an E_0-transversal by using symmetric models. Then the desired result follows from modifying its proof by using previous techniques.

Proposition 3.1. *Feferman's model thinks there is no E_0-transversal.*

Proof. Let us use the same notations we provided. Suppose that we have an E_0-transversal $f \in N$. Then we can find a finite set $E \subseteq \omega$ such that $\operatorname{sym}(\dot f) \supseteq \operatorname{fix}(E \times \omega)$, and fix some $l > \max E$. Suppose that $p \in G$ forces $\dot f$ is an E_0-transversal over $\mathsf{HS}_{\dot{\mathcal{F}}}$, and let $q \le p$ be the condition that decides the value of $\dot f(\dot a_l)$.

Now take $m < \omega$ such that $\operatorname{dom} q \subseteq \omega \times m$, and $X = \{(l,m)\}$. Then we have $\alpha_X q = q$ and $\alpha_X \dot f = \dot f$. If $q \Vdash \dot f(\dot a_l) = \dot a_l \bigtriangleup \check b$ for some finite $b \subseteq \omega$, then by acting α_X, we have $q \Vdash \dot f(\alpha_X \dot a_l) = \alpha_X \dot a_l \bigtriangleup \check b$. However, a_l and $\alpha_X a_l$ only differ at a single point, so they must have the same E_0-transversal, contradicting with that $q \Vdash \dot a_l \bigtriangleup \check b \neq \alpha_X \dot a_l \bigtriangleup \check b$. □

3.2. **DC over Feferman's model.** It is not easy to show directly that the original model satisfies DC. Fortunately, we can get around this problem by adding uncountably many reals instead of adding countably many Cohen reals.

Definition 3.2. Let $\mathbb{P} \subseteq \mathsf{Fn}(\omega \times \omega_1, 3)$ Fix a $\mathbb{P}$-generic filter G over V. For each $v \in (\mathbb{Z}/3\mathbb{Z})^{\omega\times\omega_1}$, define α_v as we defined it in Definition 2.2. We can similarly define $\mathcal{G}$ and $\operatorname{fix}_{\mathcal{G}}(E \times \omega)$ for a *countable* subset $E \subseteq \omega_1$. Consider the model $N = \mathsf{HS}_{\mathcal{F}}^{V[G]}$.

By using a similar proof, we can see that N has no free ultrafilter over ω and also satisfies $\mathsf{AC}(2)$. Furthermore, we have that DC holds over the new model. Instead of providing a proof for DC over the specific model, we will characterize a general property for symmetric systems implying DC and are valid over N. The following notion is due to Karagila [6]:

Definition 3.3. Let $(\mathbb{P}, \mathcal{G}, \mathcal{F})$ be a symmetric system. We call $(\mathbb{P}, \mathcal{G}, \mathcal{F})$ is *mixable* if whenever $\{p_\alpha \mid \alpha \in I\}$ is an antichain over $\mathbb{P}$ and we have $\dot x_\alpha \in \mathsf{HS}$ for $\alpha \in I$, then we can find $\dot x \in \mathsf{HS}$ such that for each $\alpha \in I$, $p_\alpha \Vdash \dot x_\alpha = \dot x$.

Theorem 3.4. *If a symmetric system $(\mathbb{P}, \mathcal{G}, \mathcal{F})$ is mixable, $\mathbb{P}$ is of countable chain condition and $\mathcal{F}$ is ω_1-complete, then $1 \Vdash^{\mathsf{HS}} \mathsf{DC}$.*

Proof. We prove the following claim:

Lemma 3.5. *If $p \Vdash^{\mathsf{HS}} \forall x \in \dot A \exists y \in \dot A \phi(x,y)$ and $p \Vdash \dot a \in \dot A$ for $\dot a, \dot A \in \mathsf{HS}$, then we can find $\dot b \in \mathsf{HS}$ such that $p \Vdash \dot b \in \dot A \wedge \phi(\dot a, \dot b)$.*

From the assumption, we have $p \Vdash^{\mathsf{HS}} \exists y \in \dot A \phi(\dot a, y)$. Hence we can find an antichain $\{p_i \mid i < \omega\}$ below p and $\dot b_i \in \mathsf{HS}$ such that $p_i \Vdash^{\mathsf{HS}} \dot b_i \in \dot A \wedge \phi(\dot a, \dot b_i)$. Now combine $\dot b_i$ into a single name $\dot b$ so that each p_i forces $\dot b = \dot b_i$. Then we have $p \Vdash^{\mathsf{HS}} \dot b \in \dot A \wedge \phi(\dot a, \dot b)$, showing the claim.

Now we claim that $1 \Vdash^{\mathsf{HS}} \mathsf{DC}$. Suppose that we have $p \in \mathbb{P}$, $\dot A, \dot a \in \mathsf{HS}$ such that $p \Vdash^{\mathsf{HS}} \dot a \in \dot A$ and $p \Vdash^{\mathsf{HS}} \forall x \in \dot A \exists y \in \dot A \phi(x,y)$. By the previous claim, we can find $\dot a_i$ ($i < \omega$) such that $\dot a_0 = \dot a$, $p \Vdash \dot a_n \in \dot A \wedge \phi(\dot a_n, \dot a_{n+1})$ for all $n < \omega$. Now set

$$\dot f = \{\langle \mathsf{op}(\check n, \dot a_n), 1\rangle : n < \omega\},$$

where op is a canonical function for ordered pairs for names, that is, a function such that $1 \Vdash \langle \dot x, \dot y\rangle = \mathsf{op}(\dot x, \dot y)$ for all $\dot x, \dot y \in V^{\mathbb{P}}$. We can see that $\operatorname{sym}(\dot f) \supseteq \bigcap_{n<\omega} \operatorname{sym}(\dot a_n)$, so $\dot f \in \mathsf{HS}$. Furthermore, it is easy to see that $p \Vdash^{\mathsf{HS}} \dot f$ is a function of domain ω and $p \Vdash^{\mathsf{HS}} \dot f(\check n) = \dot a_n$ for each $n < \omega$. From this, we have $p \Vdash^{\mathsf{HS}} \forall n < \omega \phi(\dot f(n), \dot f(n+1))$. This shows $1 \Vdash^{\mathsf{HS}} \mathsf{DC}$. □

Proposition 3.6. *The symmetric model we defined in Definition 3.2 is mixable and the filter is ω_1-complete.*

Proof. First, let us show that the given symmetric system is mixable. Note that $\mathbb{P}$ is of countable chain condition, and let $\{p_n \mid n < \omega\}$ be an antichain. Let $\langle \dot{x}_n \mid n < \omega\rangle$ be a sequence over HS, and suppose that we have a countable ordinal α_n such that $\operatorname{sym}(\dot{x}_n) \supseteq \operatorname{fix}(\alpha_n \times \omega)$. We may further assume that $\operatorname{dom} p_n \subseteq \alpha_n \times \omega$ for each n. Let $\alpha = \sup_{n<\omega} \alpha_n$ and let $\dot{x}$ be a canonical fusion of names, that is,

$$\dot{x} = \{\langle \dot{y}, q\rangle \mid \exists r \in \mathbb{P} \exists n < \omega[\langle \dot{y}, r\rangle \in \dot{x}_n \land q \le r, p_n]\}.$$

We can see that if $\pi \in \operatorname{fix}(\alpha \times \omega)$, then $\pi\dot{x} = \dot{x}$, so $\dot{x} \in \mathsf{HS}$. Furthermore, by definition, $p_n \Vdash \dot{x}_n = \dot{x}$ for each n.

Finally, remind that $\mathcal{F}$ is generated by $\operatorname{fix}(E \times \omega)$ for countable $E \subseteq \omega_1$. Hence the ω_1-completeness of $\mathcal{F}$ follows from the fact that the countable union of countable sets is also countable. ☐

3.3. Definable strategy.

We may ask whether we can have a definable strategy for the infinite hat problem. Let us formally define the meaning of the strategy before answering this question. The following definition is due to [3], but we only consider its special case since we are particularly interested in the hat problem with countably many prisoners lined up as the ordertype of ω.

Definition 3.7. Let C be the set of colors. A function $S\colon \omega \times C^\omega \to C$ is a *strategy* if for each $n < \omega$ and colorings $g, h \in C^\omega$, if $g \upharpoonright]n, \omega[= h \upharpoonright]n, \omega[$, then $S(n, g) = S(n, h)$.

Thus, the strategy function takes the position of the prisoners and the current coloring of the line of prisoners, and it returns some color that only depends on the colors nth prisoner can see. If C is at most countable, then we can view C as a subset of ω, and so C^ω as a subset of reals.

Now suppose that $\phi(n, x, c)$ is a formula defining a strategy, where $n < \omega$, x is a real, and c is the resulting color. $\phi(n, x, c)$ actually defines a strategy if it satisfies the following conditions:

- $\phi(n, x, c)$ defines a function, that is, $\forall^0 n \forall^1 x \exists^0 ! c \phi(n, x, c)$.
- For each $n < \omega$ and reals $x, y \in \omega^\omega$, if $x \upharpoonright]n, \omega[= y \upharpoonright]n, \omega[$, and for all $c, d \in \omega$, $\phi(n, x, c)$ and $\phi(n, y, d)$ implies $c = d$.

Now observe that if ϕ is $\mathbf{\Sigma}^1_1$, then all formulas have the complexity $\mathbf{\Pi}^1_1$. Also, if ϕ is $\mathbf{\Pi}^1_1$, then all formulas have the complexity $\mathbf{\Pi}^1_2$. From this, we can see that there is no $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$-definable strategy function.

Proposition 3.8. *There is no $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$-definable strategy for the infinite hat problem with two hat colors.*

Proof. Let us only consider the case when the defining formula $\phi(n, x, c)$ for the strategy is $\mathbf{\Pi}^1_1$. Then the claim that the formula defines the strategy is $\mathbf{\Pi}^1_2$. It is known by Shelah ([8], Theorem 7.16) that we can force every first-order definable set of reals with a real and ordinal parameter to have the Baire property. Let us say the generic extension given by Shelah $V[G]$. Then $L(\mathbb{R})^{V[G]}$ thinks all sets of reals have the Baire property. However, [3] proved that if all sets of reals have the Baire property, then the infinite hat problem with two hat colors does not have a strategy function.

Let $\phi(n, x, c)$ be a $\mathbf{\Pi}^1_1$ formula with parameters from $(\omega^\omega)^V$. Observe that the claim $\phi(n, x, c)$ does not define a strategy function is $\mathbf{\Sigma}^1_2$, so it is absolute between V and $V[G]$, and also absolute between $V[G]$ and $L(\mathbb{R})^{V[G]}$. Since the claim that $\phi(n, x, c)$ does not define a strategy function is valid over $L(\mathbb{R})^{V[G]}$, it is also valid over V. ☐

Question 3.9. Can we generalize the above result for $n \le \omega$ many hat colors?

However, the above bound is ZFC-optimal since $V = L$ shows the global well-order $<_L$ restricted to ω^ω is a $\mathbf{\Delta}^1_2$-definable well-order of ω^ω:

Example 3.10. Consider the function $f\colon \omega^\omega \to \omega^\omega$ choosing the $<_L$-least E_0-equivalent real. That is, we define f by

$$f(x) = y \iff \exists^0 n(x \upharpoonright]n, \omega[= y \upharpoonright]n, \omega[) \land y \le_L x.$$

The above definition is $\mathbf{\Delta}^1_2$, and we can use f to define a $\mathbf{\Delta}^1_2$-definable strategy for the infinite hat problem.

Remark 3.11. We can extend this result further under the suitable large cardinal hypothesis. For example, if every set has a sharp, then we get $\mathbf{\Sigma}^1_3$-generic absoluteness. Repeating the previous argument, no strategy function for infinite hat problems with two hat colors is $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$ definable.

4. QUESTIONS

Question 4.1. What is the difference between the solution of the hat problem obtained by forcing (presented in [7]) and by $\mathsf{AC}(2)$? Can we say that the latter is 'wilder' than the former? Also, which uniformity principles can be compatible with the existence of the solution to the hat problem? For example, can it be compatible with the perfect set property?

Question 4.2. What happens if hats are allowed to have infinitely many colors? Can we find a definable strategy for hat problems, such as arbitrary Borel hat colorings or analytic hat colorings? (Borel hat coloring is a sequence of hat colors that is Borel.)

Question 4.3. Can we separate the existence of a solution to the hat problem from $\forall n < \omega \mathsf{AC}(n)$?

Question 4.4. Can we separate $\mathsf{AC}(\omega)$, Choice for collection of countably infinite sets from the existence of an ultrafilter over ω? This seems related to Social Welfare Orders.

REFERENCES

[1] Stefan Geschke, Robert Lubarsky, and Mona Rahn. "Choice and the Hat Game". In: *RIMS Kôkyûroku.* Vol. 1949. Infinitary combinatorics in set theory and its applications. Research Institute for Mathematical Sciences, Kyoto University, 2015, pp. 34–44.
[2] Lorenz J. Halbeisen. *Combinatorial set theory: With a gentle introduction to forcing.* Springer Monographs in Mathematics. Springer, 2017.
[3] Christopher S. Hardin and Alan D. Taylor. "An introduction to infinite hat problems". In: *Math. Intelligencer* 30.4 (2008), pp. 20–25.
[4] Thomas J. Jech. *The axiom of choice.* Studies in Logic and the Foundations of Mathematics, Vol. 75. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
[5] Asaf Karagila. "Iterating symmetric extensions". In: *J. Symb. Log.* 84.1 (2019), pp. 123–159.
[6] Asaf Karagila. "Realizing realizability results with classical constructions". In: *Bull. Symb. Log.* 25.4 (2019), pp. 429–445.
[7] Luke Serafin. "Hat Game Independence". Manuscript in preparation.
[8] Saharon Shelah. "Can you take Solovay's inaccessible away?" In: *Israel J. Math.* 48.1 (1984), pp. 1–47.

Email address: `hj344@cornell.edu`
URL: `https://hanuljeon95.github.io`

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853