Constructive Ackermann's interpretation

Hanul Jeon

Cornell University

2022/01/14

The 2nd Korean Logic day

Table of Contents

Set theory and arithmetic

- Peano arithmetic PA (Peano 1889)

Set theory and arithmetic

- Peano arithmetic PA (Peano 1889)
- Zermelo-Fraenkel set theory ZF (Zermelo 1908, Fraenkel and Skolem 1922)

Set theory and arithmetic

- Peano arithmetic PA (Peano 1889)
- Zermelo-Fraenkel set theory ZF (Zermelo 1908, Fraenkel and Skolem 1922)
- Both theories provide a foundation for mathematics

Set theory and arithmetic

- Peano arithmetic PA (Peano 1889)
- Zermelo-Fraenkel set theory ZF (Zermelo 1908, Fraenkel and Skolem 1922)
- Both theories provide a foundation for mathematics, but PA is incapable of representing an actual infinity.

Set theory and arithmetic, constructively

- Constructivism (Brouwer 1907, Markov 1954, Bishop 1967)

Set theory and arithmetic, constructively

- Constructivism (Brouwer 1907, Markov 1954, Bishop 1967)
- Heyting arithmetic HA (Heyting 1930)

Set theory and arithmetic, constructively

- Constructivism (Brouwer 1907, Markov 1954, Bishop 1967)
- Heyting arithmetic HA (Heyting 1930)
- Intuitionistic set theory IZF (Friedman 1973)

Set theory and arithmetic, constructively

- Constructivism (Brouwer 1907, Markov 1954, Bishop 1967)
- Heyting arithmetic HA (Heyting 1930)
- Intuitionistic set theory IZF (Friedman 1973)
- Constructive set theory CZF (Aczel 1978)

Set theory and arithmetic, constructively

- Constructivism (Brouwer 1907, Markov 1954, Bishop 1967)
- Heyting arithmetic HA (Heyting 1930)
- Intuitionistic set theory IZF (Friedman 1973)
- Constructive set theory CZF (Aczel 1978)
  HA + Excluded Middle = PA
  IZF + Excluded Middle = CZF + Excluded Middle = ZF

Differences between IZF and CZF

IZF

CZF

Differences between IZF and CZF

IZF

1. Full separation

CZF

1. Bounded separation

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset

CZF

1. Bounded separation
2. Subset collection

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative
4. Equiconsistent with ZF

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation
4. Far more weaker than ZF

Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative
4. Equiconsistent with ZF

and more...

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation
4. Far more weaker than ZF

Ackermann's intrepretation

Question

Is there any relationship between PA and ZF?

Ackermann's intrepretation

Question

Is there any relationship between PA and ZF?

Theorem (Ackermann 1937)

PA *can interpret* ZF *without Infinity.*

Ackermann's intrepretation

Question

Is there any relationship between PA and ZF?

Theorem (Ackermann 1937)

PA *can interpret* ZF *without Infinity.*

Theorem (Kaye and Wong 2007)

PA *is bi-interpretable with* $\mathsf{ZF}^{\mathrm{fin}}$.

Here $\mathsf{ZF}^{\mathrm{fin}} = (\mathsf{ZF} - \text{Infinity}) + \neg\text{Infinity} + \forall x \exists TC(x)$.
(Alternatively, $\in$-induction instead of $\forall x \exists TC(x)$.)

Ackermann's intrepretation, constructively?

Question

Is there any relationship between HA and some set theory?

- Unlike classical case, we have at least two candidates: $\mathsf{IZF}^{\mathsf{fin}}$ and $\mathsf{CZF}^{\mathsf{fin}}$.

Ackermann's intrepretation, constructively?

Question

Is there any relationship between HA and some set theory?

- Unlike classical case, we have at least two candidates: $\mathsf{IZF}^{\mathsf{fin}}$ and $\mathsf{CZF}^{\mathsf{fin}}$.

Theorem (McCarty and Shapiro, J.)

HA *is bi-interpretable with* $\mathsf{CZF}^{\mathsf{fin}}$*.*

Heyting arithmetic

Definition (Heyting arithmetic)

Language $= \{0, S, <, +, \cdot\}$
Axioms:

1. S is injective,
2. Every natural number is 0 or a successor,
3. Defining formulas for $+$, $\cdot$, $<$, and
4. The Induction scheme: if $\phi(0)$ and $\phi(n) \to \phi(Sn)$ for all n, then $\forall n \phi(n)$

Theorem (Recursion theorem)

Let $f(\cdot)$ and $g(\cdot,\cdot)$ be definable functions. Then we can also define h satisfying the following conditions:

1. $h(0,y) = f(y)$, *and*
2. $h(Sx,y) = g(h(x,y),y)$.

Theorem (Recursion theorem)

Let $f(\cdot)$ and $g(\cdot,\cdot)$ be definable functions. Then we can also define h satisfying the following conditions:

1. $h(0, y) = f(y)$, *and*
2. $h(Sx, y) = g(h(x, y), y)$.

Theorem

If $\phi(x)$ is a bounded formula, i.e., every quantifier of $\phi(x)$ is of the form

- $(\forall x < a) \equiv (\forall x : x < a \to \cdots)$, *or*
- $(\exists x < a) \equiv (\exists x : x < a \wedge \cdots)$,

then $\phi(x) \vee \neg\phi(x)$.

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function,
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function,
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists,
7. Regularity: every set has a $\in$-minimal element,

Axioms of ZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Replacement: $\{F(x) \mid x \in a\}$ exists if F is a class function,
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists,
7. Regularity: every set has a $\in$-minimal element,
8. Infinity: $\mathbb{N}$ exists.

Axioms of IZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Separation: $\{x \in a \mid \phi(x)\}$ exists,
5. Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$
6. Power set: $\mathcal{P}(a) = \{x \mid x \subseteq a\}$ exists,
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Axioms of CZF

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Bounded Separation: $\{x \in a \mid \phi(x)\}$ exists if ϕ is bounded,
5. Strong Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$,
6. Subset Collection: There is a full subset of $\mathrm{mv}(a, b)$,
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. Infinity: $\mathbb{N}$ exists.

Axioms of $\mathrm{CZF}^{\mathrm{fin}}$

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Bounded Separation: $\{x \in a \mid \phi(x)\}$ exists if ϕ is bounded,
5. Strong Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$,
6. Subset Collection: There is a full subset of $\mathrm{mv}(a, b)$,
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. V=Fin: every set is finite

Simplified axioms of $\mathsf{CZF}^{\mathrm{fin}}$

Definition

1. Extensionality: $a = b \iff \forall x(x \in a \leftrightarrow x \in b)$,
2. Pairing: $\{a, b\}$ exists,
3. Union: $\bigcup a$ exists,
4. Binary intersection: $a \cap b$ exists,
5. ~~Strong Collection: if $\forall x \in a \exists y \phi(x, y)$, then there is b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$,~~
6. ~~Subset Collection: There is a full subset of $\mathrm{mv}(a, b)$,~~
7. Set Induction: $\forall a[[\forall x \in a \phi(x)] \to \phi(a)] \to \forall a \phi(a)$
8. V=Fin: every set is finite

Consequences of simplified $\mathsf{CZF}^{\mathsf{fin}}$, $\mathbb{T}$

Definition

Let $\mathbb{T}$ be a theory comprises Extensionality, Pairing, Union, Binary intersection, Set Induction and V=Fin.

Consequences of simplified CZF$^{\text{fin}}$, $\mathbb{T}$

Definition

Let $\mathbb{T}$ be a theory comprises Extensionality, Pairing, Union, Binary intersection, Set Induction and V=Fin.

Then $\mathbb{T}$ proves the following theorems:

Theorem (Primitive recursion over natural numbers)

Let A and B be classes and $F : B \to A$, $G : B \times \mathbb{N} \times A \to A$ be class functions. Then there is a definable class function $H : B \times \mathbb{N} \to A$ such that

1. $H(b, 0) = F(b)$, *and*
2. $H(b, Sn) = G(b, n, H(b, n))$.

Theorem (Set recursion)

Let $G : V^{n+2} \to V$ be an $(n+2)$-ary class function. Then there is a $(n+1)$-ary class function F such that

$$F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) \mid z \in y \rangle).$$

Theorem (Set recursion)

Let $G : V^{n+2} \to V$ be an $(n+2)$-ary class function. Then there is a $(n+1)$-ary class function F such that

$$F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) \mid z \in y \rangle).$$

Theorem (Bounded Excluded middle)

Let $\phi(x)$ be a bounded formula, i.e., every quantifier is of the form $\forall x \in y$ or $\exists x \in y$, we have $\phi(x) \vee \neg\phi(x)$.

Theorem (Set recursion)

Let $G : V^{n+2} \to V$ be an $(n+2)$-ary class function. Then there is a $(n+1)$-ary class function F such that

$$F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) \mid z \in y \rangle).$$

Theorem (Bounded Excluded middle)

Let $\phi(x)$ be a bounded formula[1], i.e., every quantifier is of the form $\forall x \in y$ or $\exists x \in y$, we have $\phi(x) \vee \neg\phi(x)$.

[1] also called Δ_0-formula

Theorem (Set recursion)

Let $G : V^{n+2} \to V$ be an $(n+2)$-ary class function. Then there is a $(n+1)$-ary class function F such that

$$F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) \mid z \in y \rangle).$$

Theorem (Bounded Excluded middle)

Let $\phi(x)$ be a bounded formula[1], i.e., every quantifier is of the form $\forall x \in y$ or $\exists x \in y$, we have $\phi(x) \vee \neg\phi(x)$.

Theorem

$\mathbb{T}$ proves Strong Collection, Subset Collection and Powerset. Moreover, $\mathsf{CZF}^{\mathrm{fin}}$ and $\mathbb{T}$ prove the same sentences.

[1] also called Δ_0-formula

Interpretation

There are various possible formulations of interpretations. However, we will only consider the following form of interpretations:

Definition

Let T_i ($i = 0, 1$) be theories over languages $\mathcal{L}_i$. Then the map $\mathfrak{t} : \varphi \mapsto \varphi^{\mathfrak{t}}$, which sends $\mathcal{L}_0$-formulas to $\mathcal{L}_1$-formulas, is an interpretation from T_0 to T_1 (notation: $\mathfrak{t} : T_0 \to T_1$) if the following holds:

- There is a $\mathcal{L}_1$-formula $\pi_\forall(x)$ (*domain of* T_0) such that $T_1 \vdash \exists x \pi_\forall(x)$,
- For each n-ary function symbol f of $\mathcal{L}_0$, there is a $(n+1)$-ary formula $\pi_f(\vec{x}, y)$ such that T_1 proves π_f is functional, and
- $\mathfrak{t}$ sends predicate symbols P to a corresponding formula π_P

Definition (cont'd)

- Let $s_0, \cdots, s_{n-1}, t_1, \cdots, t_m$ be terms, f a function symbol, and P a predicate symbol or $=$, then $\mathfrak{t}$ sends $(P(f(s_0, \cdots, s_{n-1}), t_1, \cdots, t_m))$ to

$$\exists x_0 \cdots \exists x_{n-1} \exists y \left[\bigwedge_{0 \le i < n} (x_i = s_i)^{\mathfrak{t}} \wedge \pi_f(x_0, \cdots, x_{n-1}, y) \wedge (P(y, t_1, \cdots, t_m))^{\mathfrak{t}} \right]$$

- $\mathfrak{t}$ respects logical connectives, and sends $\forall x \phi(x)$ and $\exists \phi(x)$ to $\forall x(\pi_\forall(x) \to \phi^{\mathfrak{t}}(x))$ and $\exists x(\pi_\forall(x) \wedge \phi^{\mathfrak{t}}(x))$ respectively.
- If $T_0 \vdash \phi(\vec{x})$ then $T_1 \vdash \phi^{\mathfrak{t}}(\vec{x})$.

Interpretation: an example

Example

- T_0 = The Theory of monoids : Language $\{e, *\}$ with axioms
  1. $\forall x[(x * e) = (e * x) = x]$, and
  2. $\forall xyz[x * (y * z) = (x * y) * z]$.
- T_1 = PA.

Interpretation: an example

Example

- T_0 = The Theory of monoids : Language $\{e, *\}$ with axioms
  1. $\forall x[(x * e) = (e * x) = x]$, and
  2. $\forall xyz[x * (y * z) = (x * y) * z]$.
- T_1 = PA.

Define $\mathfrak{s} : T_0 \to T_1$ by

1. $\pi_\forall(x) \equiv (x \neq 0)$
2. $\pi_e(x) \equiv (x = S0)$
3. $\pi_*(x, y, z) \equiv (x \cdot y = z)$

$\pi_\forall$ states our 'monoid' does not have 0 as an element,

Interpretation: an example

Example

- T_0 = The Theory of monoids : Language $\{e, *\}$ with axioms
  1. $\forall x[(x * e) = (e * x) = x]$, and
  2. $\forall xyz[x * (y * z) = (x * y) * z]$.
- T_1 = PA.

Define $\mathfrak{s} : T_0 \to T_1$ by

1. $\pi_\forall(x) \equiv (x \neq 0)$
2. $\pi_e(x) \equiv (x = S0)$
3. $\pi_*(x, y, z) \equiv (x \cdot y = z)$

$\pi_\forall$ states our 'monoid' does not have 0 as an element, and $\pi_e(x)$ and $\pi_*(x, y, z)$ interprets $x = e$ and $x * y = z$ respectively.

Bi-interpretation

Convention

For $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_2$, $\mathfrak{t}\mathfrak{s}$ is a composition of two interpretations, given by

$$\phi^{\mathfrak{t}\mathfrak{s}} \equiv (\phi^{\mathfrak{s}})^{\mathfrak{t}}.$$

Definition

Let $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_0$ be interpretations. Then $\mathfrak{s}$ is an inverse of $\mathfrak{t}$ if $T_0 \vdash \phi^{\mathfrak{t}\mathfrak{s}} \leftrightarrow \phi$ and $T_1 \vdash \phi^{\mathfrak{s}\mathfrak{t}} \leftrightarrow \phi$.

Bi-interpretation

Convention

For $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_2$, $\mathfrak{ts}$ is a composition of two interpretations, given by

$$\phi^{\mathfrak{ts}} \equiv (\phi^{\mathfrak{s}})^{\mathfrak{t}}.$$

Definition

Let $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_0$ be interpretations. Then $\mathfrak{s}$ is an <u>inverse</u> of $\mathfrak{t}$ if $T_0 \vdash \phi^{\mathfrak{ts}} \leftrightarrow \phi$ and $T_1 \vdash \phi^{\mathfrak{st}} \leftrightarrow \phi$.
If $\mathfrak{s}$ has an inverse, then $\mathfrak{s}$ is a <u>bi-interpretation</u>.

Bi-interpretation

Convention

For $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_2$, $\mathfrak{ts}$ is a composition of two interpretations, given by

$$\phi^{\mathfrak{ts}} \equiv (\phi^{\mathfrak{s}})^{\mathfrak{t}}.$$

Definition

Let $\mathfrak{s} : T_0 \to T_1$ and $\mathfrak{t} : T_1 \to T_0$ be interpretations. Then $\mathfrak{s}$ is an <u>inverse</u> of $\mathfrak{t}$ if $T_0 \vdash \phi^{\mathfrak{ts}} \leftrightarrow \phi$ and $T_1 \vdash \phi^{\mathfrak{st}} \leftrightarrow \phi$.
If $\mathfrak{s}$ has an inverse, then $\mathfrak{s}$ is a <u>bi-interpretation</u>.

In other words, $\mathfrak{s}$ is an inverse of $\mathfrak{t}$ if $\mathfrak{ts} = 1_{T_0}$ and $\mathfrak{st} = 1_{T_1}$, where $1_T : T \to T$ is the identity interpretation

Interpretating Set theory into Arithmetic

First, we will interpret the membership relation $\in$.

Interpretating Set theory into Arithmetic

First, we will interpret the membership relation $\in$.

Definition (Ackermann)

Work over HA. Let a and b be natural numbers. Define $a \mathrel{\mathtt{E}} b$ as follows:

$$a \mathrel{\mathtt{E}} b \iff \exists r < 2^a \exists m < b[b = (2m+1)\cdot 2^a + r]. \qquad (1)$$

Intuitively, $a \mathrel{\mathtt{E}} b$ means the ath digit of the binary expansion of b is 1.

Theorem

$\mathfrak{a} : \mathbb{T} \to \mathsf{HA}$ *is an interpretation, which is defined by* $(x \in y)^{\mathfrak{a}} \equiv (x \mathrel{\mathrm{E}} y)$ *and* $\pi_\forall(x) \equiv (x = x)$.

Theorem

$\mathfrak{a} : \mathbb{T} \to \mathsf{HA}$ *is an interpretation, which is defined by* $(x \in y)^{\mathfrak{a}} \equiv (x \mathrel{\mathtt{E}} y)$ *and* $\pi_\forall(x) \equiv (x = x)$.

Proof.

- Extensionality: $a = b$ if and only if a and b have the same binary expansion.
- Set Induction: Follows from the usual induction.
- Pairing, Union, Binary Intersection: We can directly construct an instance witnessing each axiom.

Proof. (cont'd).

For example, if we define

$$\mathrm{pair}(a, b) = \begin{cases} 2^a & \text{if } a = b, \\ 2^a + 2^b & \text{if } a \neq b. \end{cases}$$

then $c = \mathrm{pair}(a, b)$ satisfies $(c = \{a, b\})^{\mathfrak{a}}$.

Proof. (cont'd).

For example, if we define

$$\text{pair}(a,b) = \begin{cases} 2^a & \text{if } a = b, \\ 2^a + 2^b & \text{if } a \neq b. \end{cases}$$

then $c = \text{pair}(a,b)$ satisfies $(c = \{a,b\})^{\mathfrak{a}}$.

- V=Fin:
  1. Define $v(n)$ inductively by $v(0) = 0$, $v(n+1) = 2^{v(n)} + v(n)$. Then show that $(a \in \mathbb{N})^{\mathfrak{a}}$ if and only if $a = v(n)$ for some n.

Proof. (cont'd).

For example, if we define

$$\mathrm{pair}(a,b) = \begin{cases} 2^a & \text{if } a = b, \\ 2^a + 2^b & \text{if } a \neq b. \end{cases}$$

then $c = \mathrm{pair}(a,b)$ satisfies $(c = \{a,b\})^{\mathfrak{a}}$.

- V=Fin:
  1. Define $v(n)$ inductively by $v(0) = 0$, $v(n+1) = 2^{v(n)} + v(n)$. Then show that $(a \in \mathbb{N})^{\mathfrak{a}}$ if and only if $a = v(n)$ for some n.
  2. Prove that $\exists n(c$ and $v(n)$ have the same size$)^{\mathfrak{a}}$ by induction on c. □

Interpretating Arithmetic into Set theory

We will take Kaye and Wong's ordinal interpretation

Definition (Ordinal interpretation)

The interpretation $\mathfrak{o} : \mathsf{HA} \to \mathbb{T}$ sends relations and fucntions to a corresponding operations of $\mathbb{N}$ defined by $\mathbb{T}$, and $\pi_\forall(x) \equiv (x \in \mathbb{N})$.

Interpretating Arithmetic into Set theory

We will take Kaye and Wong's ordinal interpretation

Definition (Ordinal interpretation)

The interpretation $\mathfrak{o} : \mathsf{HA} \to \mathbb{T}$ sends relations and fucntions to a corresponding operations of $\mathbb{N}$ defined by $\mathbb{T}$, and $\pi_\forall(x) \equiv (x \in \mathbb{N})$.

- Kaye and Wong use ordinals instead of $\mathbb{N}$, but $\mathbb{T}$ proves the class of all ordinals is $\mathbb{N}$.

Interpretating Arithmetic into Set theory

We will take Kaye and Wong's ordinal interpretation

Definition (Ordinal interpretation)

The interpretation $\mathfrak{o} : \mathsf{HA} \to \mathbb{T}$ sends relations and fucntions to a corresponding operations of $\mathbb{N}$ defined by $\mathbb{T}$, and $\pi_\forall(x) \equiv (x \in \mathbb{N})$.

- Kaye and Wong use ordinals instead of $\mathbb{N}$, but $\mathbb{T}$ proves the class of all ordinals is $\mathbb{N}$.
- However, $\mathfrak{o}$ is not a bi-interpretation.

Salvaging the ordinal interpretation

Definition

$\hat{\Sigma} : \mathbb{N} \times \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ is a function defined recursively as follows:

$$\hat{\Sigma}(c+1, x) = \begin{cases} \hat{\Sigma}(c, x), & \text{if } c+1 \notin x, \\ \hat{\Sigma}(c, x) + (c+1), & \text{if } c+1 \in x, \end{cases}$$

Take $\Sigma(x) = \hat{\Sigma}(x, \bigcup x)$ and

$$\mathfrak{p}(x) = \Sigma(\{2^{\mathfrak{p}(y)} \mid y \in x\}).$$

Salvaging the ordinal interpretation

Definition

$\hat{\Sigma} : \mathbb{N} \times \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ is a function defined recursively as follows:

$$\hat{\Sigma}(c+1, x) = \begin{cases} \hat{\Sigma}(c, x), & \text{if } c+1 \notin x, \\ \hat{\Sigma}(c, x) + (c+1), & \text{if } c+1 \in x, \end{cases}$$

Take $\Sigma(x) = \hat{\Sigma}(x, \bigcup x)$ and

$$\mathfrak{p}(x) = \Sigma(\{2^{\mathfrak{p}(y)} \mid y \in x\}).$$

- Intuitively, $\Sigma(x)$ is the sum of all elements of x, and
- $\mathfrak{p}(x)$ codes a given set to its corresponding binary expansion.

Example

$\mathfrak{p}(\varnothing) = 0$, $\mathfrak{p}(\{\varnothing\}) = 2^0 = 1$, and $\mathfrak{p}(\{\varnothing, \{\varnothing\}\}) = 2^0 + 2^1$.

Example

$\mathfrak{p}(\varnothing) = 0$, $\mathfrak{p}(\{\varnothing\}) = 2^0 = 1$, and $\mathfrak{p}(\{\varnothing, \{\varnothing\}\}) = 2^0 + 2^1$.

Theorem

$\mathbb{T}$ proves $\mathfrak{p}$ is a bijection between V and $\mathbb{N}$.

where V is the class of all sets.

Example

$\mathfrak{p}(\varnothing) = 0$, $\mathfrak{p}(\{\varnothing\}) = 2^0 = 1$, and $\mathfrak{p}(\{\varnothing, \{\varnothing\}\}) = 2^0 + 2^1$.

Theorem

$\mathbb{T}$ proves $\mathfrak{p}$ is a bijection between V and $\mathbb{N}$.

where V is the class of all sets.

Theorem

If $\mathfrak{b}$ is defined by $(\phi(\vec{x}))^{\mathfrak{b}} \equiv \phi^{\mathfrak{o}}(\mathfrak{p}(\vec{x}))$, then $\mathfrak{b} : \mathsf{HA} \to \mathbb{T}$. Moreover, $\mathfrak{a}$ and $\mathfrak{b}$ are inverses of each other.

Review: Σ_n and Π_n

Definition

Let $\Delta_0 = \Sigma_0 = \Pi_0$ be the set of all bounded formulas. Define Σ_n and Π_n as follows:

1. ϕ is Σ_n if it is equivalent to $\exists x_1 \cdots \exists x_n \psi$ for some $\psi \in \Pi_{n-1}$, and
2. ϕ is Π_n if it is equivalent to $\forall x_1 \cdots \forall x_n \psi$ for some $\psi \in \Sigma_{n-1}$.

Review: Σ_n and Π_n

Definition

Let $\Delta_0 = \Sigma_0 = \Pi_0$ be the set of all bounded formulas. Define Σ_n and Π_n as follows:

1. ϕ is Σ_n if it is equivalent to $\exists x_1 \cdots \exists x_n \psi$ for some $\psi \in \Pi_{n-1}$, and
2. ϕ is Π_n if it is equivalent to $\forall x_1 \cdots \forall x_n \psi$ for some $\psi \in \Sigma_{n-1}$.

Σ_n and Π_n measure the complexity of a given formula based on its quantifiers.

Review: Σ_n and Π_n

Definition

Let $\Delta_0 = \Sigma_0 = \Pi_0$ be the set of all bounded formulas. Define Σ_n and Π_n as follows:

1. ϕ is Σ_n if it is equivalent to $\exists x_1 \cdots \exists x_n \psi$ for some $\psi \in \Pi_{n-1}$, and
2. ϕ is Π_n if it is equivalent to $\forall x_1 \cdots \forall x_n \psi$ for some $\psi \in \Sigma_{n-1}$.

Σ_n and Π_n measure the complexity of a given formula based on its quantifiers.

Proposition

1. $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$ for each n.
2. $\bigcup_n \Sigma_n = \bigcup_n \Pi_n$ is the set of all formulas.

Lévy-Fleischmann hierarchy

We will define classes of formulas that are constructive analogue of Σ_n and Π_n classes.

Lévy-Fleischmann hierarchy

We will define classes of formulas that are constructive analogue of Σ_n and Π_n classes.

Definition

Let Φ and Ψ be a set of formulas over the language of set theory or arithmetic. Then $\mathcal{E}(\Phi)$ is the smallest set containing Φ which is closed under $\land$, $\lor$, $\exists$ and bounded quantifications.

Lévy-Fleischmann hierarchy

We will define classes of formulas that are constructive analogue of Σ_n and Π_n classes.

Definition

Let Φ and Ψ be a set of formulas over the language of set theory or arithmetic. Then $\mathcal{E}(\Phi)$ is the smallest set containing Φ which is closed under $\land$, $\lor$, $\exists$ and bounded quantifications.

$\mathcal{U}(\Phi, \Psi)$ is the smallest set containing Φ such that

1. $\Phi \subseteq \mathcal{U}(\Phi, \Psi)$,
2. $\mathcal{U}(\Phi, \Psi)$ is closed under $\land$, $\lor$, $\forall$, and bounded quantifications, and
3. if $\psi \in \Psi$ and $\phi \in \mathcal{U}(\Phi, \Psi)$ then $\psi \to \phi$ is in $\mathcal{U}(\Phi, \Psi)$.

Definition

$\mathcal{E}_0 = \mathcal{U}_0$ is the class of all bounded formulas. Define $\mathcal{E}_n$ and $\mathcal{U}_n$ recursively as follows:

- $\mathcal{E}_{n+1} = \mathcal{E}(\mathcal{U}_n)$, and
- $\mathcal{U}_{n+1} = \mathcal{U}(\mathcal{E}_n, \mathcal{E}_n)$.

Definition

$\mathcal{E}_0 = \mathcal{U}_0$ is the class of all bounded formulas. Define $\mathcal{E}_n$ and $\mathcal{U}_n$ recursively as follows:

- $\mathcal{E}_{n+1} = \mathcal{E}(\mathcal{U}_n)$, and
- $\mathcal{U}_{n+1} = \mathcal{U}(\mathcal{E}_n, \mathcal{E}_n)$.

Theorem

1. *$\mathcal{E}_n$ and $\mathcal{U}_n$ are monotone, i.e., $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$ and $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$,*
2. *$\mathcal{E}_n \subseteq \mathcal{U}_{n+1}$ and $\mathcal{U}_n \subseteq \mathcal{E}_{n+1}$,*
3. *$\bigcup_{n=0}^{\infty} \mathcal{E}_n = \bigcup_{n=0}^{\infty} \mathcal{U}_n$ is the set of all formulas.*
4. *Assuming the full excluded middle, we have $\mathcal{E}_n = \Sigma_n$ and $\mathcal{U}_n = \Pi_n$.*

Subtheories $\mathsf{I}\mathcal{E}_n$ and $\mathsf{SI}\mathcal{E}_n$

Definition

- $\mathsf{I}\mathcal{E}_n$ is a subtheory of HA obtained by restricting Induction scheme to $\mathcal{E}_n$-formulas.

Subtheories $\mathrm{I}\mathcal{E}_n$ and $\mathrm{SI}\mathcal{E}_n$

Definition

- $\mathrm{I}\mathcal{E}_n$ is a subtheory of HA obtained by restricting Induction scheme to $\mathcal{E}_n$-formulas.
- $\mathrm{SI}\mathcal{E}_n$ is a subtheory of $\mathbb{T}$ obtained by restricting Set Induction scheme to $\mathcal{E}_n$-formulas.

Subtheories $\mathrm{I}\mathcal{E}_n$ and $\mathrm{SI}\mathcal{E}_n$

Definition

- $\mathrm{I}\mathcal{E}_n$ is a subtheory of HA obtained by restricting Induction scheme to $\mathcal{E}_n$-formulas.
- $\mathrm{SI}\mathcal{E}_n$ is a subtheory of $\mathbb{T}$ obtained by restricting Set Induction scheme to $\mathcal{E}_n$-formulas.

Observation

Almost all notions (e.g., $\mathbb{N}$, $\mathfrak{a}$, $\mathfrak{b}$) that are necessary for our proof are definable by $\mathcal{E}_1$-formulas.

Defining notions over $\mathsf{I}\mathcal{E}_1$ and $\mathsf{SI}\mathcal{E}_1$

$\mathsf{I}\mathcal{E}_1$ and $\mathsf{SI}\mathcal{E}_1$ are strong enough to allow recursive definition for $\mathcal{E}_1$-formulas.

Defining notions over $\mathsf{I}\mathcal{E}_1$ and $\mathsf{SI}\mathcal{E}_1$

$\mathsf{I}\mathcal{E}_1$ and $\mathsf{SI}\mathcal{E}_1$ are strong enough to allow recursive definition for $\mathcal{E}_1$-formulas. For example, $\mathsf{SI}\mathcal{E}_1$ can prove

Theorem ($\mathcal{E}_1$-primitive recursion over natural numbers)

Let A and B be $\mathcal{E}_1$-definable classes and $F : B \to A$, $G : B \times \mathbb{N} \times A \to A$ be $\mathcal{E}_1$-definable class functions. Then there is a $\mathcal{E}_1$-definable definable class function $H : B \times \mathbb{N} \to A$ such that

1. $H(b, 0) = F(b)$, *and*
2. $H(b, Sn) = G(b, n, H(b, n))$.

The same holds for set recursion over $\mathsf{SI}\mathcal{E}_1$ and recursion over $\mathsf{I}\mathcal{E}_1$.

Bi-interpretation between subtheories

Theorem

Let $n \geq 1$. Then $\mathfrak{a} : \mathsf{SI}\mathcal{E}_n \to \mathsf{I}\mathcal{E}_n$, $\mathfrak{b} : \mathsf{I}\mathcal{E}_n \to \mathsf{SI}\mathcal{E}_n$ and $\mathfrak{a}$ are $\mathfrak{b}$ are inverses of each others.

Bi-interpretation between subtheories

Theorem

Let $n \geq 1$. Then $\mathfrak{a} : \mathsf{SI}\mathcal{E}_n \to \mathsf{I}\mathcal{E}_n$, $\mathfrak{b} : \mathsf{I}\mathcal{E}_n \to \mathsf{SI}\mathcal{E}_n$ and $\mathfrak{a}$ are $\mathfrak{b}$ are inverses of each others.

Proof.

Since $\mathsf{I}\mathcal{E}_1 \subseteq \mathsf{I}\mathcal{E}_n$ and $\mathsf{SI}\mathcal{E}_1$ and $\mathsf{SI}\mathcal{E}_n$, both $\mathsf{I}\mathcal{E}_n$ and $\mathsf{SI}\mathcal{E}_n$ can define necessary notions we need for the proof. Hence we can carry on the same proof for HA and $\mathbb{T}$. □

Relation between Kaye and Wong's result

Theorem (Kaye and Wong 2007)

- $\mathsf{I}\Sigma_n$: *Subtheory of* PA *by restricting the induction scheme for* Σ_n*-formulas.*
- Σ_n-Sep: *Extensionality, Pairing, Empty Set, Union,* $\neg$*Infinity,* Δ_0*-Collection,* $(\Sigma_1 \cup \Pi_1)$*-Set Induction,* Σ_n*-Separation.*

Then $\mathfrak{a} : \Sigma_\mathrm{n}\text{-Sep} \to \mathsf{I}\Sigma_n$ *and* $\mathfrak{b} : \mathsf{I}\Sigma_n \to \Sigma_\mathrm{n}\text{-Sep}$ *are inverses of each other.*

Relation between Kaye and Wong's result

Theorem (Kaye and Wong 2007)

- $\mathrm{I}\Sigma_n$: *Subtheory of* PA *by restricting the induction scheme for* Σ_n*-formulas.*
- Σ_n-Sep: *Extensionality, Pairing, Empty Set, Union,* $\neg$*Infinity,* Δ_0*-Collection,* $(\Sigma_1 \cup \Pi_1)$*-Set Induction,* Σ_n*-Separation.*

Then $\mathfrak{a} : \Sigma_\mathrm{n}\text{-Sep} \to \mathrm{I}\Sigma_n$ *and* $\mathfrak{b} : \mathrm{I}\Sigma_n \to \Sigma_\mathrm{n}\text{-Sep}$ *are inverses of each other.*

Theorem

- $\mathrm{I}\mathcal{E}_n$ + *Full Excluded middle* $= \mathrm{I}\Sigma_n$, *and*
- $\mathrm{SI}\mathcal{E}_n$ + *Full Excluded middle* $= \Sigma_\mathrm{n}$-Sep.

IZF vs CZF

Theorem

$\mathsf{IZF}^{\mathrm{fin}}$ *proves the law of excluded middle. Hence* $\mathsf{IZF}^{\mathrm{fin}} = \mathsf{ZF}^{\mathrm{fin}}$.

IZF vs CZF

Theorem

$\mathsf{IZF}^{\mathsf{fin}}$ *proves the law of excluded middle. Hence* $\mathsf{IZF}^{\mathsf{fin}} = \mathsf{ZF}^{\mathsf{fin}}$*.*

- Unlike $\mathsf{CZF}^{\mathsf{fin}}$, $\mathsf{IZF}^{\mathsf{fin}}$ is a classical theory.

IZF vs CZF

Theorem

$\mathsf{IZF}^{\mathsf{fin}}$ *proves the law of excluded middle. Hence* $\mathsf{IZF}^{\mathsf{fin}} = \mathsf{ZF}^{\mathsf{fin}}$.

- Unlike $\mathsf{CZF}^{\mathsf{fin}}$, $\mathsf{IZF}^{\mathsf{fin}}$ is a classical theory.
- Hence $\mathsf{IZF}^{\mathsf{fin}}$ is not bi-interpretable with HA, unlike $\mathsf{CZF}^{\mathsf{fin}}$.

IZF vs CZF

Theorem

$\mathsf{IZF}^{\mathsf{fin}}$ *proves the law of excluded middle. Hence* $\mathsf{IZF}^{\mathsf{fin}} = \mathsf{ZF}^{\mathsf{fin}}$.

- Unlike $\mathsf{CZF}^{\mathsf{fin}}$, $\mathsf{IZF}^{\mathsf{fin}}$ is a classical theory.
- Hence $\mathsf{IZF}^{\mathsf{fin}}$ is not bi-interpretable with HA, unlike $\mathsf{CZF}^{\mathsf{fin}}$.
- A possible reason for the philosophical preference of CZF over IZF as a constructive counterpart of ZF?

Comparison with McCarty and Shapiro's SST

McCarty and Shapiro also provided a set theory which is bi-interpretable with HA.

Comparison with McCarty and Shapiro's SST

McCarty and Shapiro also provided a set theory which is bi-interpretable with HA.

Definition (Small Set Theory SST)

SST comprises the following axioms:

1. Extensionality,
2. Empty set,
3. y-successor of x: $x \cup \{y\}$ exists for all x and y, and
4. Induction: If $\phi(\varnothing)$ and if

$$\forall x, y[y \notin x \land \phi(x) \land \phi(y) \to \phi(x \cup \{y\})],$$

then $\forall x \phi(x)$ holds.

Theorem

SST *proves every axiom of* $\mathsf{CZF}^{\mathsf{fin}}$.

Theorem

SST *proves every axiom of* $\mathsf{CZF}^{\mathrm{fin}}$.

Hence my result and that of McCarty and Shapiro is almost the same.

Theorem

SST *proves every axiom of* $\mathrm{CZF}^{\mathrm{fin}}$.

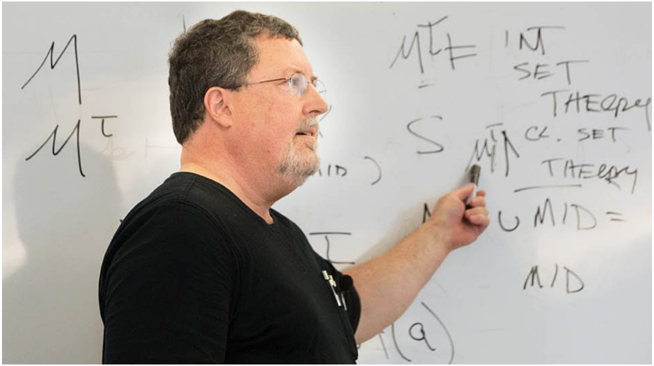
Hence my result and that of McCarty and Shapiro is almost the same.
Why almost? Heyting arithmetic I used and they used are different!

Remark

McCarty and Shapiro uses the following definition of HA: the language of HA contains symbols for each primitive recursive functions, and its definitions as axioms.
I only consider $+$, $\cdot$ and $\leq$ as a part of the language of HA.

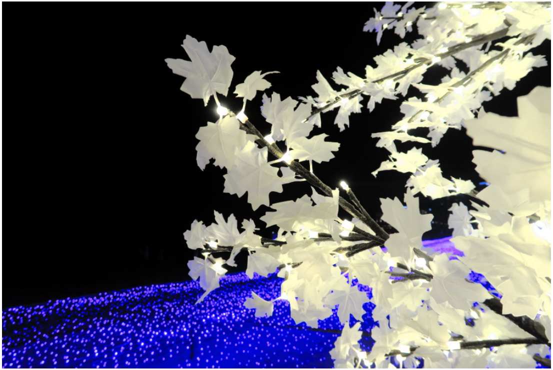
Acknowledgements

- I would like to thank my advisor, Otto van Koert for ceaseless support and care on my thesis and graduation,
- Andrés E. Caicedo for helpful comments on preparing my paper,
- Charles McCarty and David Shapiro for their attention to my work.
- Lastly, I am so glad for my thesis committees to make their time.



David C. McCarty (1953 - 2020)

Questions



The end

Thank you for listening to my presentation!