Proof theory for higher pointclasses

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Table of Contents

A Journey to Consistency

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David Hilbert

A Journey to Consistency

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- Among them, Hibert's program provides a way to turn the quest for consistency into a mathematical problem.
- Formalizing mathematics within mathematics → Turning the consistency problem into a mathematical problem.

David Hilbert

Gödel's incompleteness, and its consequences

Theorem

No consistent, recursive T interpreting a fragment of arithmetic[a] *proves* $\mathrm{Con}(T)$.

- No 'reasonable' theory proves its own consistency. The failure of Hilbert's program.

Kurt Gödel

[a] Bounded arithmetic S^1_2 suffices

Gödel's incompleteness, and its consequences

Theorem

No consistent, recursive T interpreting a fragment of arithmetic[a] *proves* $\mathrm{Con}(T)$.

- No 'reasonable' theory proves its own consistency. The failure of Hilbert's program.
- We should have an 'unbounded hierarchy of theories' instead, like $T < T + \mathrm{Con}(T) < \cdots$.

[a] Bounded arithmetic S^1_2 suffices

Kurt Gödel

The consistency hierarchy

Definition

For two theories S and T extending a weak arithmetic A, define

1. $S \leq_{\mathsf{Con}} T \iff \mathsf{A} \vdash (\mathsf{Con}(T) \to \mathsf{Con}(S))$, and
2. $S <_{\mathsf{Con}} T \iff S \leq_{\mathsf{Con}} T$ and $T \not\leq_{\mathsf{Con}} S$.

Various logicians observed that $\leq_{\mathsf{Con}}$ for natural* theories is prewellordered.

*Its meaning is disputable, but let me follow Rathjen: "... theories which have something like an 'idea' to them."

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2. $S <_{\mathsf{Con}} T \iff S \leq_{\mathsf{Con}} T$ and $T \not\leq_{\mathsf{Con}} S$.

Various logicians observed that $\leq_{\mathsf{Con}}$ for natural* theories is prewellordered.
Moreover, the only way to show $S <_{\mathsf{Con}} T$ for natural S, T is to prove $T \vdash \mathsf{Con}(S)$.

*Its meaning is disputable, but let me follow Rathjen: "... theories which have something like an 'idea' to them."

Ill-behaviors of the consistency comparison

Theorem (Folklore)

1. *There are S, T such that neither $S \leq_{\mathsf{Con}} T$ nor $T \leq_{\mathsf{Con}} S$ hold.*
2. *For two S and T, there is U such that $S <_{\mathsf{Con}} U <_{\mathsf{Con}} T$.*
3. *There is a recursive sequence of theories $\langle T_n \mid n < \omega \rangle$ such that $T_0 >_{\mathsf{Con}} T_1 >_{\mathsf{Con}} \cdots$.*
4. *There are recursive theories S and T such that $S <_{\mathsf{Con}} T$ yet $T \nvdash \mathsf{Con}(S)$.*

Constructing such examples uses 'unnatural' constructions (e.g., diagonalization, Rosser's trick)

Consequence comparison vs. Consistency comparison

Steel and Woodin argued the following phenomenon:

Phenomenon

For two natural S, T extending ZFC,

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Pi^0_1} T \iff S \subseteq_{\Pi^0_\infty} T.$$

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For two natural S, T extending ZFC,

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This fails in general.

Example

Let ρ be a Rosser sentence for PA. Then $\mathrm{PA} + \rho \leq_{\mathrm{Con}} \mathrm{PA}$ yet $\mathrm{PA} + \rho \not\leq_{\Pi^0_1} \mathrm{PA}$.

What is the Ordinal analysis?

- Ordinal analysis calculates the proof-theoretic ordinal $|T|_{\Pi^1_1}$ of a given theory T.

Definition

$|T|_{\Pi^1_1} = \sup\{\mathrm{otp}(\alpha) \mid \alpha \text{ recursive linear order } \wedge T \vdash \mathrm{WO}(\alpha)\}$

- Gentzen 1934: $|\mathsf{PA}|_{\Pi^1_1} = \varepsilon_0$.
- Takeuti 1967: Ordinal analysis of Π^1_1-CA_0.

What does the proof-theoretic ordinal gauge?

We would hope $|T|_{\Pi^1_1}$ gauges the consistency strength of T precisely, but it does not. For example,

$$|T|_{\Pi^1_1} = |T + \mathrm{Con}(T)|_{\Pi^1_1}.$$

What does the proof-theoretic ordinal gauge?

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$$|T|_{\Pi^1_1} = |T + \mathrm{Con}(T)|_{\Pi^1_1}.$$

In general, we have the following:

Theorem (Kriesel)

Let T be a recursive, Π^1_1-sound theory extending ACA_0. If σ is a true Σ^1_1-sentence, then $|T|_{\Pi^1_1} = |T + \sigma|_{\Pi^1_1}$.

Definition

Let Γ be a pointclass (like Π^1_1 or Σ^1_2.)

1. $\check{\Gamma} = \{\neg\phi \mid \phi \in \Gamma\}$ is the dual pointclass.
2. $T \vdash^{\Gamma} \phi$ iff $T + \sigma \vdash \phi$ for some true Γ-sentence σ.
3. $S \subseteq_{\Gamma} T$ iff $S \vdash \phi \implies T \vdash \phi$ for every Γ-sentence ϕ.
4. $S \subseteq^{\check{\Gamma}}_{\Gamma} T$ iff $S \vdash^{\check{\Gamma}} \phi \implies T \vdash^{\check{\Gamma}} \phi$ for every Γ-sentence ϕ.

- $\vdash^{\check{\Gamma}}$: provability relation modulo true $\check{\Gamma}$ sentences.
- $\subseteq_{\Gamma}$ is the Γ-consequence comparison.
- $\subseteq^{\check{\Gamma}}_{\Gamma}$ is the Γ-consequence comparison modulo true $\check{\Gamma}$-sentences.

Definition

Γ-Rfn(T) is the assertion

$$\forall\phi \in \Gamma\left[T \vdash \phi \implies \vDash_\Gamma \phi\right].$$

(Cf. $\mathrm{Con}(T) \iff \Pi^0_1\text{-Rfn}(T)$.)

We also define the <u>Π^1_1-reflection comparison relation</u>

$$S \leq^{\Sigma^1_1}_{\Pi^1_1\text{-Rfn}} T \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^1_1\text{-Rfn}(T) \to \Pi^1_1\text{-Rfn}(S).$$

Walsh's characterization

Theorem (Walsh 2023)

For recursive Π^1_1-sound theories S, T extending ACA_0,

1. $|S|_{\Pi^1_1} \le |T|_{\Pi^1_1} \iff S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T$.
2. $|S|_{\Pi^1_1} \le |T|_{\Pi^1_1} \iff S \le^{\Sigma^1_1}_{\Pi^1_1\text{-Rfn}} T$.

Theorem

For recursive Π^1_1-sound theories S, T extending ACA_0,

$$S <^{\Sigma^1_1}_{\Pi^1_1\text{-Rfn}} T \iff T \vdash^{\Sigma^1_1} \Pi^1_1\text{-Rfn}(S).$$

Walsh's theorem is the 'Π^1_1-version' of the observed phenomena for natural theories.

Example

Let us compare Walsh's theorem

$$S \leq^{\Sigma^1_1}_{\Pi^1_1\text{-Rfn}} T \iff S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T$$

with Steel-Woodin's phenomenon:

$$S \leq_{\mathrm{Con}} T \iff S \subseteq_{\Pi^0_1} T.$$

Note that the above is equivalent to

$$S \leq^{\Sigma^0_1}_{\Pi^0_1\text{-Rfn}} T \iff S \subseteq^{\Sigma^0_1}_{\Pi^0_1} T.$$

I will take Walsh's result as evidence of the following phenomenon:

Phenomenon

The following three coincide for natural theories extending ACA_0:

- Proof-theoretic ordinal comparison.
- Π^1_1-consequence comparison.
- Π^1_1-reflection comparison.

Also, the only way to prove $S <_{\Pi^1_1\text{-Rfn}} T$ for natural S, T is to prove $T \vdash \Pi^1_1\text{-Rfn}(S)$.

Is the consistency comparison linear?

The currently known consistency comparison for "natural" theories actually does more than Π^1_1-reflection comparison, including

1. Forcing and inner models.
2. Ordinal analysis.
3. Sets-as-trees interpretation.
4. Comparisons between subsystems of second-order arithmetic appearing in Simpson's book.
5. Other irregular comparisons (e.g., Krivine's Classical realizability)

It would be plausible to claim the following:

Phenomenon

The consistency comparison between two classical theories extending ACA_0 actually does the Π^1_1-reflection comparison.

Combining with an observation following from Walsh's theorem, we have

Corollary Phenomenon

The consistency comparison for natural theories is prewellordered.

Steel-Woodin's phenomenon, again.

Phenomenon

For two natural S, T extending ZFC,

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Pi^0_1} T \iff S \subseteq_{\Pi^0_\infty} T.$$

How did Steel justify this phenomenon?

The justification

The reason: We want to prove $S \leq_{\mathsf{Con}} T$.

1. Let M be a model of T.

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2. We use forcing / inner model / or their combinations to build a new model $M' \supseteq M$ and $N \subseteq M'$ with the same class of ordinals and $N \vDash S$. This shows $S \leq_{\mathsf{Con}} T$.

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3. Forcing and inner model constructions preserve transitivity and ordinal height, so M' thinks M and N are transitive and $\mathrm{Ord}^M = \mathrm{Ord}^{M'} = \mathrm{Ord}^N$.

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1. Let M be a model of T.
2. We use forcing / inner model / or their combinations to build a new model $M' \supseteq M$ and $N \subseteq M'$ with the same class of ordinals and $N \vDash S$. This shows $S \leq_{\mathrm{Con}} T$.
3. Forcing and inner model constructions preserve transitivity and ordinal height, so M' thinks M and N are transitive and $\mathrm{Ord}^M = \mathrm{Ord}^{M'} = \mathrm{Ord}^N$.
4. By Shoenfield absoluteness, M, M' and N all agree with the Σ^1_2-truth.
5. $M \vDash T$ and $N \vDash S$, so $S \subseteq_{\Sigma^1_2} T$.

The previous justification shows:

Phenomenon

For natural theories S, T extending ZFC,

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Sigma^1_2} T.$$

However, the previous argument does not work for non-traditional (but existing!) consistency comparisons.

Another justification?

We can imagine an alternative justification similar to Case Π^1_1, which is as follows:

1. Isolate an ordinal characteristic $s^1_2(T)$ capturing the Σ^1_2-consequence of T.
2. Prove that the s^1_2-comparison, the Σ^1_2-consequence comparison, and the Σ^1_2-reflection comparison are all agree for natural theories.
3. Then argue that for sufficiently strong natural theories, the consistency comparison and the Σ^1_2-reflection comparison coincide.

Well-orders represent Π^1_1-assertions

Well-orders represent Π^1_1-sentences in the following manner:

Theorem (Kleene)

For a given Π^1_1-sentence ϕ, we can find a recursive linear order α_ϕ such that

1. ACA_0 *proves "α_ϕ is a linear order," and*
2. ACA_0 *proves "α_ϕ is a well-order iff ϕ."*

We need a different object to represent Π^1_2 or Σ^1_2-statements.

Dilators

Girard defined dilators for his Π^1_2-proof theory.

Definition (Girard)

A semidilator is a functor $F\colon \mathrm{LO} \to \mathrm{LO}$ preserving direct limit and pullback.[†]

[†] Alternatively, arbitrary directed union and finite intersection.

Dilators

Girard defined dilators for his Π^1_2-proof theory.

Definition (Girard)

A semidilator is a functor $F\colon \mathrm{LO} \to \mathrm{LO}$ preserving direct limit and pullback.[†] A dilator is a semidilator preserving well-orders.

Notes

1. There is an intermediate notion called predilator.
2. There are other names for semidilators (e.g., Freund's prae-dilator.)
3. In practice, we use different definitions of (semi)dilators.

[†] Alternatively, arbitrary directed union and finite intersection.

Girard's completeness theorem

Let us say a (semi)dilator F is recursive if there is a recursive set coding $F \upharpoonright \mathsf{Nat}$.

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Theorem (Girard)

For a given Π^1_2-sentence ϕ, we can find a recursive predilator D_ϕ such that

1. ACA_0 *proves "D_ϕ is a predilator," and*
2. ACA_0 *proves "D_ϕ is a dilator iff ϕ."*

Pseudodilators

- The dual of Girard's completeness theorem: Every Σ^1_2-statement is equivalent to "D is not a dilator" for some D.

Pseudodilators

- The dual of Girard's completeness theorem: Every Σ^1_2-statement is equivalent to "D is not a dilator" for some D.
- If D is not a dilator, there is the least ordinal α making $D(\alpha)$ illfounded.
  Let us call this ordinal the climax of D (Notation: $\mathrm{Clim}(D)$)

Definition (Aguilera-Pakhomov)

Let T be a Σ^1_2-sound theory. Define

$$s^1_2(T) = \sup\{\mathrm{Clim}(D) \mid D \text{ recursive predilator } \wedge T \vdash \neg\mathrm{Dil}(D)\}.$$

Σ^1_2-analogue of Walsh's characterizations

Theorem (J.)

For recursive Σ^1_2-sound theories S, T extending Σ^1_2-AC_0,

1. $s^1_2(S) \le s^1_2(T) \iff S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T$.
2. $s^1_2(S) \le s^1_2(T) \iff \Sigma^1_2\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{Rfn}(T) \to \Sigma^1_2\text{-}\mathsf{Rfn}(S)$.

Theorem (J.)

For recursive Σ^1_2-sound theories S, T extending Σ^1_2-AC_0,

$$s^1_2(S) < s^1_2(T) \iff T \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{Rfn}(S).$$

Characterizing $s^1_2(T)$

Theorem (J.)

Let T be a $(\Sigma^1_2 \cup \Pi^1_2)$-sound extension of $\Pi^1_1\text{-CA}_0$. Then

$$s^1_2(T) = \min\{\mathrm{Ord}^M \mid M \text{ is a transitive model of } \mathrm{ATR}_0^{\mathrm{set}} + \Sigma^1_2(T)\},$$

where $\Sigma^1_2(T)$ is the set of all Σ^1_2-consequences of T.

Characterizing $s^1_2(T)$

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Let T be a $(\Sigma^1_2 \cup \Pi^1_2)$-sound extension of Π^1_1-CA_0. Then

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where $\Sigma^1_2(T)$ is the set of all Σ^1_2-consequences of T.

Ord^M for a transitive model M of $\mathrm{ATR}^{\mathrm{set}}_0$ is either admissible or limit admissible, so

Corollary

$s^1_2(T)$ for a $(\Sigma^1_2 \cup \Pi^1_2)$-sound extension of Π^1_1-CA_0 is admissible or limit admissible.

Pohler's framework

Theorem (Spector-Gandy)

For every Π^1_1-statement ϕ, there is a Σ_1-sentence ψ in the language of set theory such that

$$\mathbb{N} \vDash \phi \iff L_{\omega_1^{\mathsf{CK}}} \vDash \psi.$$

Thus, they are the same:

- Gauging the proof-theoretic ordinal of T.
- Gauging $\Sigma_1^{L_{\omega_1^{\mathsf{CK}}}}$-consequences of T.

- If T becomes complex, T can see how structures outside of $L_{\omega_1^{\mathsf{CK}}}$ affect $L_{\omega_1^{\mathsf{CK}}}$, so we have to count their effects.
- A good candidate for the 'outside structures' is other admissible sets.
- We work with arithmetical counterparts of admissible sets: Spector classes.

Spector class

Pohlers considered a particular form of $\mathfrak{M}$ by appending a Spector class:

Definition

A collection Γ of subsets of $\mathbb{N}$ is a Spector class if it satisfies the following:

1. Every atomic predicate and function over $\mathfrak{M}$, and their complements are in Γ. (For functions, consider their graph instead.)
2. Γ contains coding scheme for tuples over $\mathfrak{M}$.
3. Γ is closed under $\cap$, $\cup$, $\exists^0$, $\forall^0$, and trivial combinatorial substitutions.‡

‡Trivial combinatorial substitution is a map that is a composition of projection maps and the tuple map.

Definition (continued)

4. Γ has a universal set; That is, for each $n \in \mathbb{N}$ there is an $(n+1)$-ary relation $U \in \Gamma$ such that every n-ary $R \in \Gamma$ is a section of U.
5. Γ has the prewellordering property; That is, for every $P \in \Gamma$ there is a norm $\sigma_P \colon P \to \mathrm{Ord}$ such that the relations
   1. $\vec{m} \le^*_P \vec{n} \iff P(\vec{m}) \land [P(\vec{n}) \to (\sigma(\vec{m}) \le \sigma(\vec{n}))]$, and
   2. $\vec{m} <^*_P \vec{n} \iff P(\vec{m}) \land [P(\vec{n}) \to (\sigma(\vec{m}) < \sigma(\vec{n}))]$

   are both in Γ.

Iterated Spector class

For a collection $\Gamma \subseteq \mathcal{P}(\mathbb{N})$ we can find the the next Spector class

$$\mathsf{SP}(\Gamma) = \bigcap\{\Gamma' \supseteq \Gamma \cup \neg\Gamma \mid \Gamma' \text{ is a Spector class}\}.$$

Definition

For ξ less than the least recursively inaccessible ordinal, define

1. $\mathsf{SP}^0_{\mathbb{N}} = \varnothing$.
2. $\mathsf{SP}^{\xi+1}_{\mathbb{N}}$ is the next Spector class over $\mathsf{SP}^{\xi}_{\mathbb{N}}$.
3. $\mathsf{SP}^{\delta}_{\mathbb{N}} = \bigcup_{\xi<\delta} \mathsf{SP}^{\xi}_{\mathbb{N}}$ if δ is limit.

$\mathsf{SP}^{\delta}_{\mathbb{N}}$ may not be a Spector class when δ is limit.

Generalizing proof-theoretic ordinal

Definition

Let $\mathfrak{M} = (\mathbb{N}; \dots)$ be an expansion of the structure of natural numbers. Suppose that T is an acceptable[§] axiomatization of $\mathfrak{M}$. Define

$$\delta^{\mathfrak{M}}(T) = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is an } \mathfrak{M}\text{-definable linear order such that } T \vdash \mathrm{WO}(\alpha)\}.$$

For $\mathfrak{M} = \mathbb{N}$, $\delta^{\mathfrak{M}}(T) = |T|_{\Pi^1_1}$.

[§] T is sound and proves every true atomic sentence and first-order induction scheme over $\mathfrak{M}$.

Proof-theoretic dilator

Proof-theoretic dilator is the 'least' dilator embedding every recursive dilator such that T knows it is a dilator.

Theorem (Aguilera-Pakhomov)

Let T be a Π^1_2-sound recursive exntesion of ACA_0. Then we can find a recursive dilator $|T|_{\Pi^1_2}$ such that

1. *If $T \vdash$ "D is a dilator" for a recursive predilator D, then D is embedded into $|T|_{\Pi^1_2}$.*
2. *If F is another dilator embedding every recursive D such that $T \vdash$ "D is a dilator", then $|T|_{\Pi^1_2}$ embeds into F.*

$|T|_{\Pi^1_2}$ is unique up to bi-embeddability.

The following theorem tells the proof-theoretic meaning of $|T|_{\Pi^1_2}(\alpha)$ for a recursive α:

Theorem

If α is a recursive well-order and T is a Π^1_2-sound theory extending ACA_0, then

$$|T|_{\Pi^1_2}(\alpha) = |T + \mathsf{WO}(\alpha)|_{\Pi^1_1}.$$

Question

What is the proof-theoretic meaning of $|T|_{\Pi^1_2}(\alpha)$ for a non-recursive α?

Example (Pohlers)

For $\xi \geq 1$ less than the least recursively inaccessible ordinal, we have

$$\delta^{\mathsf{SP}^\xi_{\mathbb{N}}}(\mathsf{ACA}_0 + \mathsf{Th}(\mathbb{N}; X)_{X \in \mathsf{SP}^\xi_{\mathbb{N}}}) = \varepsilon_{\omega^{\mathsf{CK}}_\xi + 1}$$

Here ω^{CK}_ξ is the ξth admissible (or a limit of admissible) ordinal. Let us compare the previous one with the proof-theoretic dilator:

Example (Aguilera-Pakhomov)

$|\mathsf{ACA}_0|_{\Pi^1_2} = \varepsilon^+$, where ε^+ is a dilator such that $\varepsilon^+(\alpha)$ is the epsilon number greater than α.

Is it a coincidence?

Σ^1_2-singleton real

In Pohler's framework, we considered $(\mathbb{N}; A)_{A \in \mathsf{SP}^\xi_\mathbb{N}}$ for ξ less than the least recursively inaccessible ordinal.
We do not need to add every such set, and we can replace them with the ξth iterated hyperjump $\mathsf{HJ}^\xi(0)$, which is a Σ^1_2-singleton real.

Definition

A real R is a Σ^1_2-singleton if there is a Σ^1_2-formula $\phi(X)$ such that

$$\phi(R) \land \forall X, Y[\phi(X) \land \phi(Y) \to X = Y].$$

$\Pi^1_1[R]$ Proof-theoretic ordinal

Definition

Let R be a Σ^1_2-singleton. For a sound theory T proving 'R uniquely exists,' let us define the $\Pi^1_1[R]$ Proof-theoretic ordinal of T by

$$|T|_{\Pi^1_1[R]} = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is an } R\text{-recursive linear order such that } T \vdash \mathsf{WO}(\alpha)\}.$$

More precisely, we use the Σ^1_2-singleton definition of R in place of R to formulate the definition.

Remark

$\delta^{\mathsf{SP}^\xi_{\mathbb{N}}}(T) = |T|_{\Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]}$ for an acceptable T.

Genedendron

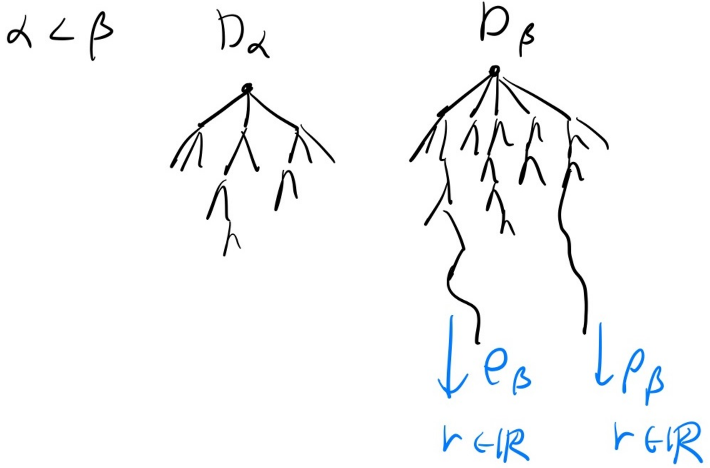
A genedendron is an effective 'passive' way to generate Σ^1_2-singleton reals.

Definition

A genedendron is a pair (D, ϱ) such that

1. D 'generates' a functorial family $\langle D_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ of trees.
2. ϱ generates a functorial family $\langle \varrho_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, and ϱ_α is a function taking an infinite branch of D_α and returning a real.
3. ϱ_α is a constant function if defined.

If every set of immediate successors of D_α is well-ordered, we say (D, ϱ) is locally well-founded.


$\alpha < \beta$
D_α
D_β
e_β
ρ_β
$r \in \mathbb{R}$
$r \in \mathbb{R}$

By functoriality, a genedendron is completely determined by $(D_\omega, \varrho_\omega)$.

Definition

A genedendron (D, ϱ) is recursive if there is a recursive set coding $(D_\omega, \varrho_\omega)$.

Theorem (J., ACA_0)

For every Σ^1_2-singleton real X, there is a recursive genedendron (D, ϱ) generating X. (i.e., the value of ϱ is X.)

Connecting Pohlers' with Girard's

Theorem (J.)

If

- T *be a* Π^1_2*-sound theory extending* ACA_0*,*
- (D, ϱ) *be a recursive locally well-founded genedendron generating* R*.*
- T *proves* (D, ϱ) *is a locally well-founded genedendron.*
- α *is an* R*-recursive well-order such that* D_α *is ill-founded.*

Then

$$|T|_{\Pi^1_2}(\alpha) = |T + \text{“}R \text{ exists”} + \mathrm{WO}(\alpha)|_{\Pi^1_1[R]}.$$

We can find a recursive genedendron (D, ϱ) generating HJ(0) such that $D_{\omega_1^{\mathrm{CK}}}$ illfounded. Hence

Corollary

$$|\mathsf{ACA}_0 + \text{`HJ}(\emptyset)\text{ exists'}|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = |\mathsf{ACA}_0|_{\Pi^1_2}(\omega_1^{\mathrm{CK}}) = \varepsilon_{\omega_1^{\mathrm{CK}}+1}.$$

Steel's observation for projective comparison

Phenomenon

For natural theories S, T extending $\mathsf{ZFC}+\mathsf{PD}$,

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Pi^1_\infty} T.$$

Can we provide a 'conceptual' justification for Steel's phenomenon?

Ptykes

Definition

We define <u>n-semiptykes</u> as follows:

1. 0-semiptyx is a linear order.
2. $(n+1)$-semiptyx is a functor $F\colon n\text{-Semiptyx} \to \mathrm{LO}$ preserving direct limit and pullback.

An <u>n-ptyx</u> is an n-semiptyx P such that

$$\forall Q[Q \in (n-1)\text{-Ptyx} \implies P(Q) \in \mathrm{WO}].$$

n-ptykes capture Π^1_{n+1}-statements, and every Π^1_{n+1}-sound theory admits a <u>proof-theoretic n-ptyx</u> $|T|_{\Pi^1_{n+1}}$.

2-ptyx comparison

Theorem (J., $\mathsf{ACA}_0 + \boldsymbol{\Delta}^1_2$-Det)

We can find a Σ^1_3-definable prewellorder $\trianglelefteq_{\vartheta^2}$ over the collection of countable 2-ptykes.

Its construction follows the proof of the First Periodicity theorem, and uses the cell decomposition of 2-preptykes.

Characterizing Π^1_3-consequence comparison

Theorem (J., $\Sigma^1_3\text{-AC}_0 + \mathbf{\Delta}^1_2\text{-Det}$)

For recursive Π^1_3-sound theories S, T extending $\text{ACA}_0 + \mathbf{\Delta}^1_2\text{-Det}$,

1. $|S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3} \iff S \subseteq^{\Sigma^1_3}_{\Pi^1_3} T$.
2. $|S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3} \iff \text{ACA}_0 \vdash^{\Sigma^1_3} \Pi^1_3\text{-Rfn}(T) \to \Pi^1_3\text{-Rfn}(S)$.

Theorem (J.)

For recursive Π^1_3-sound theories S, T extending $\text{ACA}_0 + \mathbf{\Delta}^1_2\text{-Det}$,

$$|S|_{\Pi^1_3} \triangleleft_{\vartheta^2} |T|_{\Pi^1_3} \iff T \vdash^{\Sigma^1_3} \Pi^1_3\text{-Rfn}(S).$$

3-pseudoptyx comparison

Let P be a 3-pseudoptyx. Then there is a 2-ptyx π such that $P(\pi)$ is ill-founded.
We compare the "$\trianglelefteq_{\vartheta^2}$-least π witnessing "$P(\pi)$ ill-founded."

Definition

Let P and Q be two 3-pseudoptykes. We say $P \trianglelefteq_{\mathsf{Grow}^3} Q$ if

1. π is a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx such that $P(\pi)$ is ill-founded, and
2. ρ is a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx such that $Q(\rho)$ is ill-founded

then $\pi \trianglelefteq_{\vartheta^2} \rho$.

Characterizing Σ^1_4-consequences

For a Σ^1_4-sound theory T, we can define $\|T\|_{\Sigma^1_4}$.

Theorem (J.)

Let S and T be recursive Σ^1_4-sound extension of $\Sigma^1_3\text{-DC}_0 + \mathbf{\Delta}^1_2\text{-Det}$.

1. $\|S\|_{\Sigma^1_4} \trianglelefteq_{\mathrm{Grow}^3} \|T\|_{\Sigma^1_4} \iff S \subseteq^{\Pi^1_4}_{\Sigma^1_4} T$.
2. $\|S\|_{\Sigma^1_4} \trianglelefteq_{\mathrm{Grow}^3} \|T\|_{\Sigma^1_4} \iff \Sigma^1_4\text{-AC}_0 \vdash^{\Pi^1_4} \Sigma^1_4\text{-Rfn}(T) \to \Sigma^1_4\text{-Rfn}(S)$.

Also,

$$\|S\|_{\Sigma^1_4} \triangleleft_{\mathrm{Grow}^3} \|T\|_{\Sigma^1_4} \iff T \vdash^{\Pi^1_4} \Sigma^1_4\text{-Rfn}(S).$$

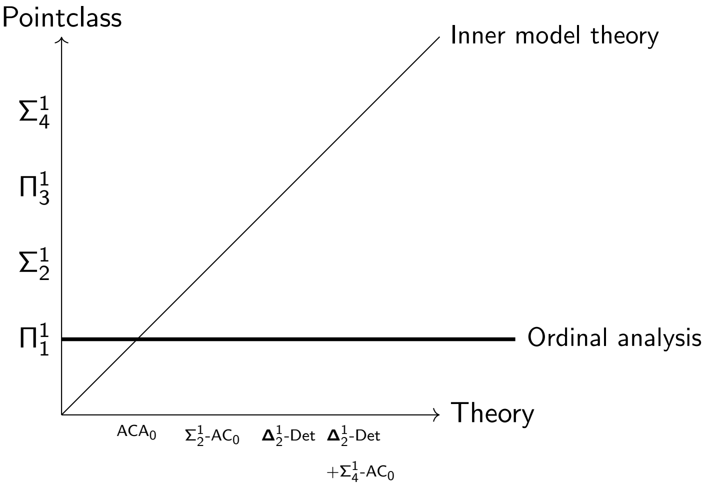
Inner-model-theoretic framework

Inner model theorists use a mouse to gauge the strength of a theory.

> *We should now update the method by which we find the strength of a theory T. This is from now on the least mouse which is not in some wellfounded model of T. (by Andreas Lietz on Mathoverflow.)*

Mice are closely related to (Spector) pointclasses.

The general framework



Alternative ranking on theories

For set theories, it would be 'natural' to use transitive models to compare the strength of theories:

Definition

Let S and T be theories extending $\mathsf{PRS} + \mathsf{Beta}$. We define $S <_\beta T$ iff

> *Every transitive model of T contains a transitive model of S as a member.*

$<_\beta$ is well-founded, so we can rank theories under $<_\beta$.

Ehrenfeucht–Mostowski models

β-rank does not seem to work well for theories at the level of "$0^\sharp$ exists."
The following 'upgraded transitive models' should be more appropriate for ranking theories above "$0^\sharp$ exists":

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Definition

An Ehrenfeucht–Mostowski model (abbr. EM model) is a functor $F\colon \mathsf{LO} \to \mathsf{Mod}_\in$ such that

1. F preserves pullback and direct limit.
2. If α is a well-order, then $F(\alpha)$ is well-founded.

$A^\sharp$ for a set A is an EM model.

2-EM models?

For theories at the level of a single Woodin cardinal[¶], the following notion may be better to rank theories:

[¶] More precisely, "$M_1^\sharp$ exists."

2-EM models?

For theories at the level of a single Woodin cardinal[¶], the following notion may be better to rank theories:

Definition

A 2-EM model is a functor $F\colon \mathrm{PreDil} \to \mathrm{Mod}_\in$ such that

1. F preserves pullback and direct limit.
2. If D is a dilator, then $F(D)$ is well-founded.

I suspect the construction of a measurable dilator yields a 2-EM model.

[¶] More precisely, "$M_1^\sharp$ exists."

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It looks like that n-EM models are "untruncated" version of (M, X) for a transitive $M \vDash \mathsf{ZFC}$ and a projective set X such that "M is X-closed."

n-EM models, and beyond?

We could try to define "n-EM models" in a similar fashion.
It looks like that n-EM models are "untruncated" version of (M, X) for a transitive $M \vDash \mathsf{ZFC}$ and a projective set X such that "M is X-closed."
Its generalization may constitute part of Woodin's Ω-logic.
However, we do not know a 'correct' formulation of a 'higher EM-model' corresponding to (co)inductive pointclass.

Question

Isolate an object representing a (co)inductive set.

Any questions?

Thank you