PROOF THEORY FOR HIGHER POINTCLASSES

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Various logicians observed that the consistency strength comparison is well-ordered and coincides with comparing arithmetical and projective consequences for natural theories. The primary goal of my dissertation is to establish results that justify this observation and to produce results suggesting that various methods for comparing the strength of theories are instances of a single, general framework.

Walsh [102] proved that comparing Π^1_1-consequences modulo true Σ^1_1-sentences is equivalent to comparing proof-theoretic ordinals and comparing the Π^1_1-version of consistency comparison, namely, Π^1_1-reflection comparison. In Chapter 1, I will prove that Σ^1_2-consequence comparison is equivalent to comparing the Σ^1_2-proof-theoretic ordinal $s^1_2(T)$ introduced by Aguilera-Pakhomov [2] and Σ^1_2-reflection comparison. This will be further generalized in Chapter 5 by showing that assuming Projective Determinacy, for a projective pointclass Γ admitting the prewellordering property (i.e., $\Gamma = \Pi^1_3, \Sigma^1_4$, etc.), Γ-consequence comparison modulo true $\check{\Gamma}$-sentences is prewellordered and is equivalent to the Γ-reflection comparison.

In Chapter 3, I will prove how Pohlers' performance ordinal is associated with Girard's Π^1_2-proof theory. Briefly, Pohlers' characteristic ordinal is a generalized proof-theoretic ordinal computed over expansions of $\mathbb{N}$ with definable reals, typically a Σ^1_2-singleton real. Girard's Π^1_2-proof theory computes a proof-theoretic dilator $|T|_{\Pi^1_2}$ encoding the Π^1_2-consequences of the theory. I will prove in this chapter that Pohlers' characteristic ordinals are instances of Girard's proof-theoretic dilator.

Inner model theorists use mice to gauge the strength of theories, which are transitive

models with an additional structure indicating how to iterate the model. Chapters 2 and 4 correspond to the first step towards a larger program of ranking theories via mice. We will examine how to rank theories via their well-founded models or β-models in Chapter 2. In Chapter 4, I will construct a measurable dilator by analyzing Martin's proof of $\mathbf{\Pi}^1_2$-determinacy from a rank-into-rank cardinal, which isolates a large-cardinal object encapsulating $\mathbf{\Pi}^1_2$-determinacy. Its construction is suspected to be related to the construction of higher versions of transitive models, or EM models, introduced by Ressayre [82], corresponding to sharps.

This dissertation makes extensive use of the theory of dilators and ptykes. Hence, I will also provide the development of the basic theory of dilators and ptykes.

BIOGRAPHICAL SKETCH

I was born in Iksan, Jeollabuk-do, South Korea. As an introverted person with few friends in my youth, I solely focused on mathematics. I became interested in logic in my high school years. In March 2011, I took an online course on analysis. During the proof of the equivalence between continuity and sequential continuity, he mentioned that one of the greatest issues of mathematics in the early 20th century was hidden. It took a year to realize he meant the axiom of choice, but the realization was the starting line of my interest in logic. I graduated with a bachelor's from Sungkyunkwan University in 2017 and a master's from Seoul National University in 2021, with a two-year intermediate military service from 2017 to 2019. I started my PhD program in the Fall of 2021 and visited Vienna in the Spring of 2025.

거짓은 참을 이길 수 없다

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INTRODUCTION

"It's no use, Mr. James—— it's turtles all the way down."

From J. R. Ross, *Constraints on Variables in Syntax*

0.1 A Journey to Consistency

Mathematicians, from Weierstrass to Hilbert, in the late 19th and early 20th centuries, were interested in rigorous foundation of mathematics. Weierstrass provided a rigorous definition of limit and continuity in arithmetical language, and Dedekind and Cantor provided a formal definition of real numbers based on what is now called naive set theory. At the same time, Frege established his logicism program. Unfortunately, both Cantor's naive set theory and Frege's framework faced the Russell-type paradoxes, leading to contradictions in their foundational theories. Various mathematicians, like Russell, Brouwer, and Zermelo, suggested solutions to the paradox.

Among the possible resolutions, what is most relevant to my work is Hilbert's program. Hilbert contended a realistic viewpoint for finitary mathematics, so he required his *metamathematics* to be finitary. However, Hilbert, as an instrumentalist, claimed that an ideal mathematics, which can include transfinite reasoning or the excluded middle, can be used as a tool to establish finitary statements. Hilbert wanted to justify the use of ideal mathematics, and he concluded that we should view mathematics as a formal entity, a tool to survey metamathematical truth. According to Zach [109],

> (...) Mathematics itself, however, operates with abstract concepts, e.g., quantifiers, sets, functions, and uses logical inference based on principles

> such as mathematical induction or the principle of the excluded middle. (...) Hilbert's aim was to justify their use. To this end, he pointed out that they can be formalized in axiomatic systems (such as that of Principia or those developed by Hilbert himself), and mathematical propositions and proofs thus turn into formulas and derivations from axioms according to strictly circumscribed rules of derivation. Mathematics, so Hilbert, "becomes an inventory of provable formulas." In this way, the proofs of mathematics are subject to metamathematical, contentual investigation. (Zach [109, p.416]).

As a consequence of formalizing mathematics, the consistency and completeness of mathematics have now become a solid mathematical problem. Ackermann established the consistency of PRA in a 'finitary' manner,[1] and the finitary proof of the consistency of mathematics seemed possible. Nothing but blue skies ahead for Hilbert's program by the 1920s, but Gödel's famous incompleteness theorem showed Hilbert's program flew too close to the sun. Gödel announced in 1930 that every ω-consistent formal theory extending arithmetic is incomplete. Soon after, von Neumann and Gödel independently noticed that no formal theory can prove its own consistency unless it is inconsistent, which is now known as Gödel's second incompleteness theorem.

While Gödel's result demonstrates the failure of Hilbert's quest for consistency, his theorem also unveils the *consistency hierarchy of formal theories*, the hierarchy of formal theories aligned under their consistency strength. Logicians have had a tradition of establishing a hierarchy to resolve the self-referential paradox they confront. As an example, let us recall the resolution of Russell's paradox. We resolve the paradox by restricting the unbounded comprehension to the bounded one, which says we can form a

[1] Zach [108] pointed out that Ackermann's consistency proof of PRA uses transfinite induction up to $\omega^{\omega^{\omega}}$. While it is disputable if Ackermann's proof is completely finistic, Zach [108, §4] stated that "... transfinite induction in the form used by Ackermann was so readily accepted as finitist, not just by Ackermann himself, but also by Hilbert and Bernays."

new set by separation only from a previously given domain set. Pushing this idea further, we arrive at the idea that sets are formed by appropriately collecting previously formed sets. This *iterative conception of sets* [12] arranges sets into a hierarchy of when they are formed. In sum, while we do not have the set of all sets due to Russell's paradox, we have a hierarchy of sets. Similarly, we can understand the unsolvability of the halting problem as the hierarchy of Turing degrees is non-trivial since we can rephrase the unsolvability result as "The Turing degree $\emptyset'$ of the halting problem is not Turing reducible to the Turing degree $\emptyset$ of recursive sets." In some sense, the unsolvability of the halting problem shows we have a hierarchy of degrees of unsolvability. That is, while we have no recursive solution to the halting problem, we have the degree of unsolvability. Along the same lines, Gödel's theorem suggests that while we cannot have a single theory for the mathematical foundation, we have the consistency hierarchy of formal theories. There are various possible ways to align theories according to their strength. However, the second incompleteness theorem, roughly stating $T \nvdash \mathsf{Con}(T)$, suggests the hierarchy under the following relation should be appropriate:

Definition 0.1.1. Let S and T be recursively axiomatizable[2] theories extending a weak first-order arithmetic A.[3] We define the following:

- $S \leq_{\mathsf{Con}} T \iff \mathsf{A} \vdash \mathsf{Con}(T) \to \mathsf{Con}(S)$, and
- $S <_{\mathsf{Con}} T$ iff $S \leq_{\mathsf{Con}} T$ but $T \not\leq_{\mathsf{Con}} S$.

The consistency hierarchy has been studied by various logicians. Not only that, these logicians witnessed that the hierarchy of *natural* theories under $\leq_{\mathsf{Con}}$ is linear. For example, Simpson, Koellner, and Rathjen claimed the following:

[2]We identify theories with their recursive definition. This identification becomes important when we formulate the sentence $\mathsf{Con}(T)$, where we use the recursive definition of T as a proxy.

[3]We take $\mathsf{A} = \mathsf{PRA}$ in this thesis, but A can be far weaker than PRA, like $\mathsf{I}\Delta_0 + \mathsf{Exp}$ or bounded arithmetic S^1_2, since we can arithmetize the syntax and prove incompleteness theorems in such weaker theories. See [13, Chapter 7] for relevant details.

It is striking that a great many foundational theories are linearly ordered by $<_{\mathsf{Con}}$.[4] Of course, it is possible to construct pairs of artificial theories that are incomparable under $<_{\mathsf{Con}}$. However, this is not the case for the "natural" or nonartificial theories which are usually regarded as significant in the foundations of mathematics. (Simpson [90, p.111]).

Remarkably, it turns out that when one restricts to those theories that "arise in nature" the interpretability ordering is quite simple: There are no descending chains and there are no incomparable elements—the interpretability ordering on theories that "arise in nature" is a wellordering. (Koellner [51]).

It is a striking empirical fact that many "natural" theories, i.e. theories which have something like an "idea" to them, are comparable with regard to consistency strength. This has actually been proved in many cases, for theories whose ideas and motivations have nothing at all to do with one another. A plethora of results in proof theory and set theory seems to provide compelling evidence that $\leq_{\mathsf{Con}}$ is a linear ordering on "natural" theories. (Rathjen [78, p.234–235]).

However, the behavior of the consistency hierarchy for arbitrary theories is wild. The following result is folklore (See Hamkins [35] for their proof.)

Theorem 0.1.2. *1. There are theories S, T extending* PA *such that neither $S \leq_{\mathsf{Con}} T$ nor $T \leq_{\mathsf{Con}} S$ hold.*

[4] Simpson defined $S <_{\mathsf{Con}} T$ if and only if $T \vdash \mathsf{Con}(S)$. We will discuss the connection between our $<_{\mathsf{Con}}$ and Simpson's $<_{\mathsf{Con}}$ later.

2. *For theories S, T extending* PA *satisfying $S <_{\mathsf{Con}} T$, we can find an extension U of* PA *such that $S <_{\mathsf{Con}} U <_{\mathsf{Con}} T$.*

3. *There are theories $T_0, T_1, \cdots$ extending* PA *such that $T_0 >_{\mathsf{Con}} T_1 >_{\mathsf{Con}} \cdots$.* □

A similar discrepancy also appears on a more technical side. When we prove $S <_{\mathsf{Con}} T$ for "natural" theories, we prove statements implying $T \vdash \mathsf{Con}(S)$. Furthermore, proving $T \vdash \mathsf{Con}(S)$ seems essentially the only way to establish $S <_{\mathsf{Con}} T$ for natural theories S and T. We can easily see that if $T \vdash \mathsf{Con}(S)$, then $S <_{\mathsf{Con}} T$. However, the converse can fail for arbitrary theories:

Theorem 0.1.3. *There are recursive theories S and T extending* PA *such that $S <_{\mathsf{Con}} T$, but $T \nvdash \mathsf{Con}(S)$.*[5]

Another phenomenon associated with the consistency strength is that for natural theories, consistency strength comparison is "equivalent" to arithmetical consequence comparison. Simpson stated the following:

> Assuming that the language of T_1 is part of the language of T_2, let us write $T_1 \subset T_2$ to mean that the sentences that are theorems of T_1 form a proper subset of the sentences in the language of T_1 which are theorems of T_2. We may think of $T_2 \supset T_1$ as meaning that T_2 is "more powerful" or "stronger" than T_1. In many cases, the $<_{\mathsf{Con}}$ ordering and the $\subset$ ordering coincide. (Simpson [90, p.111–112]).

There are also many theories whose consequence comparison does not coincide with their consistency comparison: ZFC and ZFC + CH have the same consistency strength,

[5] Hamkins' *petulant enumeration* $\mathsf{ZFC}^\bullet$ in [35, §5] is an example: Fix a stronger theory T (say, $T = \mathsf{ZFC} + \mathsf{Con}(\mathsf{ZFC})$). In $\mathsf{ZFC}^\bullet$, we enumerate axioms of ZFC, but once we find a proof of contradiction of T, we immediately add the contradiction to the enumeration. One can see that $\mathsf{ZFC} <_{\mathsf{Con}} \mathsf{ZFC}^\bullet$ since $\mathsf{ZFC}^\bullet$ and T are equiconsistent. However, $\mathsf{ZFC}^\bullet \nvdash \mathsf{Con}(\mathsf{ZFC})$ since if T is consistent, then $\mathsf{ZFC}^\bullet = \mathsf{ZFC}$.

while the latter proves more theorems than the former. However, Simpson made his claim in the context of second-order arithmetic, so we may understand Simpson's claim as "For two natural theories, consistency comparison and comparing projective consequences of theories cohere." Indeed, Woodin [107] and Steel [92] made a more precise observation along this line:

Phenomenon 0.1.4 (Woodin [107], Steel [92])**.** If S and T are natural extensions of ZFC, then

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Pi^0_1} T \iff S \subseteq_{\Pi^0_\omega} T.$$

Of course, this observation fails for arbitrary theories, even if we restrict to Π^0_1-consequence comparisons:

Example 0.1.5. Suppose that ρ is the Rosser sentence over PA, which is Π^0_1. Then $\mathsf{PA} + \rho \leq_{\mathsf{Con}} \mathsf{PA}$, yet $\mathsf{PA} + \rho \not\subseteq_{\Pi^0_1} \mathsf{PA}$.

What makes the gap between the behavior of natural theories and arbitrary theories in the consistency hierarchy? Before stating my opinion, let us talk about the fate of Hilbert's program following Gödel's result.

0.2 From Hilbert's program to Ordinal Analysis

Notwithstanding the incompleteness phenomena, Gentzen attained the next best result for Hilbert's program by showing that we can prove the consistency of Peano arithmetic in a finitary method under a reasonable hypothesis. Gentzen proved in 1934 that, over PRA, the transfinite induction up to ε_0 implies the consistency of PA. That is, we can prove the consistency of PA in a finistic manner with a small cost, namely, well-orderedness of ε_0. Here, we should take caution that we do not apply transfinite induction on ordinals, which

are always well-ordered; Rather, we apply transfinite induction over an *ordinal notation system*, or in other words, a recursive well-order. When we say the transfinite induction up to ε_0 in arithmetic like PRA or fragments of second-order arithmetic, it means the transfinite induction over a recursive linear order defined by mimicking ordinals below ε_0 via their Cantor normal form. The well-orderedness of a linear order imitating ε_0 is not obvious, and it implies the consistency of theories, say, PA. With a conflation of ordinals and recursive well-orders, however, we can say that the consistency strength of PA is represented by the ordinal ε_0. In modern terminology, ε_0 *is a proof-theoretic ordinal of* PA. Gentzen's result initiated a field named *ordinal analysis*, assigning a formal theory an ordinal representing its strength.

Various proof theorists provided ordinal analysis of various theories after Gentzen's milestone in ordinal analysis. The main destination of ordinal analysis is to provide an ordinal analysis of full second-order arithmetic Z_2, which is deemed enough to do real analysis. In 1967, Takeuti [95] provided an ordinal analysis for Π^1_1-CA_0, an impredicative subsystem of Z_2 obtained by restricting its comprehension scheme to Π^1_1-formulas. In the late 1970s and early 1980s, the German school of proof theory refined an ordinal analysis of Π^1_1-CA_0. They also took an ordinal analysis for the theory of iterated inductive definitions and a weak form of set theory named Kripke-Platek set theory, KP. However, the progress towards a larger fragment of comprehension scheme was slow to reach Z_2; After some work towards an extension of KP named KPM and Π_n-reflection by Arai and Rathjen, they independently provided an ordinal analysis of Π^1_2-CA_0 in 1994–1995. In November 2023, Arai [8] announced the full ordinal analysis of Z_2, whose confirmation is ongoing. Pakhomov (unpublished) and Towsner [96] have also been working independently towards ordinal analysis of Z_2.

Ordinal analysis quantifies the strength of theories in ordinals. The proof-theoretic

ordinal $|T|_{\Pi^1_1}$ of a given theory T is a linearly comparable quantity representing the strength of T. However, the proof-theoretic ordinal of a theory does *not* precisely represent the consistency strength of a theory since T and $T + \mathsf{Con}(T)$ have the same proof-theoretic ordinal. More precisely, Kriesel established the following:

Theorem 0.2.1 (Kriesel). *Let T be a recursively axiomatizable, sound*[6] *theory extending ACA_0. If σ is a true Σ^1_1-sentence, then $|T|_{\Pi^1_1} = |T + \sigma|_{\Pi^1_1}$.*

For technical reasons, we shift from first-order arithmetic PA into its corresponding second-order counterpart ACA_0.[7] $\mathsf{Con}(T)$ is a true Π^0_1-sentence for a natural sound theory T extending ACA_0, so T and $T + \mathsf{Con}(T)$ have the same proof-theoretic ordinal for such T. Yet, what does the proof-theoretic ordinal gauge? Walsh [103] answers the question. We need some terminology before stating the result:

Definition 0.2.2. Let Γ be a set of formulas like Σ^m_n or Π^m_n. We define $\breve{\Gamma} = \{\neg\sigma \mid \sigma \in \Gamma\}$. We also define

- $T \vdash^{\breve{\Gamma}} \sigma$ if there is a true sentence τ in $\breve{\Gamma}$ such that $T + \tau \vdash \sigma$.
- $S \subseteq_\Gamma T$ if for every sentence σ in Γ, $S \vdash \sigma$ implies $T \vdash \sigma$.
- $S \subseteq^{\breve{\Gamma}}_\Gamma T$ if for every sentence σ in Γ, $S \vdash^{\breve{\Gamma}} \sigma$ implies $T \vdash^{\breve{\Gamma}} \sigma$.

Also, Γ-$\mathsf{RFN}(T)$ is the following statement:

$$\forall^0 \sigma \in \Gamma [T \vdash \sigma \implies \vDash_\Gamma \sigma],$$

where $\vDash_\Gamma$ is a partial truth predicate for Γ. Γ-$\mathsf{RFN}(T)$ states every T-provable statement in Γ is true.

[6] A theory T in the language of first or second order arithmetic is *sound* if every theorem of T is true; That is, $T \vdash \sigma$ implies $\mathbb{N} \vDash \sigma$.

[7] ACA_0 is a *conservative extension* of PA, that is, every arithmetical theorem of ACA_0 is also a theorem of PA.

Note that Π^0_1-$\mathsf{RFN}(T)$ is equivalent to $\mathsf{Con}(T)$, so Γ-$\mathsf{RFN}(T)$ can be seen as a generalization of $\mathsf{Con}(T)$. Furthermore, if $T \supseteq \mathsf{PRA}$, then $T \vdash^{\Sigma^0_1} \sigma$ if and only if $T \vdash \sigma$ since PRA proves every true Σ^0_1-statements. In particular, for $S, T \supseteq \mathsf{PRA}$, $S \subseteq^{\Sigma^0_1}_{\Pi^0_1} T$ iff $S \subseteq_{\Pi^0_1} T$. This suggests that all notions defined in Definition 0.2.2 are the 'Γ-version' of the Π^0_1-consequence comparison and the consistency assertion.

Theorem 0.2.3 (Walsh [103]). *Suppose that S and T are Π^1_1-sound extensions of ACA_0. Then*

$$S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T \iff |S|_{\Pi^1_1} \leq |T|_{\Pi^1_1}.$$

Furthermore, if both S and T are arithmetically definable, then

$$S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^1_1\text{-}\mathsf{RFN}(T) \to \Pi^1_1\text{-}\mathsf{RFN}(S).$$

Not only that, but Walsh's argument also yields the following:

Theorem 0.2.4. *Suppose that S and T are arithmetically definable Π^1_1-sound extensions of ACA_0. Then*

$$|S|_{\Pi^1_1} < |T|_{\Pi^1_1} \iff T \vdash^{\Sigma^1_1} \Pi^1_1\text{-}\mathsf{RFN}(S).$$

Hence, Walsh's theorem states, modulo true Σ^1_1 sentences, the following are all equivalent:

1. Comparing the Π^1_1-consequences of theories,
2. Comparing the proof-theoretic ordinal of the theories, and
3. A version of consistency comparison of the theories (namely, Π^1_1-reflection comparison over ACA_0 modulo true Σ^1_1-sentences.)

Walsh's equivalence explains the two points: First, it explains how the proof-theoretic ordinal represents the strength of theories, namely, the Π^1_1-reflection strength of theories.

Second, this explains a part of an observed phenomenon that consistency comparison and consequence comparison of natural theories go together—Let us remind that we have observed the following phenomenon for natural theories:

1. The relation $S \leq_{\mathsf{Con}} T$ is prewellordered for natural S and T.
2. $S \leq_{\mathsf{Con}} T$ iff $S \subseteq_{\Pi^0_1} T$ for natural S and T.
3. The only way to establish $S <_{\mathsf{Con}} T$ is to establish $T \vdash \mathsf{Con}(T)$.

By replacing Π^0_1-provability with Π^0_1-reflection with the corresponding Π^1_1-notions, we get a theorem rather than a mere phenomenon.[8]

Let me suggest Walsh's theorem as evidence of the following phenomenon, which is a Π^1_1-version of the observed behavior of consistency comparison:

Phenomenon 0.2.5. For natural theories extending ACA_0, the following three coincide:

1. Comparing the proof-theoretic ordinal of natural theories.
2. Comparing the Π^1_1-consequences of natural theories.
3. Π^1_1-reflection comparison of natural theories.

Moreover, for two natural theories S and T extending ACA_0, $|S|_{\Pi^1_1} < |T|_{\Pi^1_1}$ iff $T \vdash \Pi^1_1\text{-}\mathsf{RFN}(S)$.

[8]The reader may wonder that the Π^0_1-counterpart of $\mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^1_1\text{-}\mathsf{RFN}(T) \to \Pi^1_1\text{-}\mathsf{RFN}(S)$ is $\mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^0_1\text{-}\mathsf{RFN}(T) \to \Pi^0_1\text{-}\mathsf{RFN}(S)$ and not $S \leq_{\mathsf{Con}} T$. However, every true Σ^0_1-sentence is provable in ACA_0, so $\mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^0_1\text{-}\mathsf{RFN}(T) \to \Pi^0_1\text{-}\mathsf{RFN}(S)$ is equivalent to $S \leq_{\mathsf{Con}} T$ once we choose $\mathsf{A} = \mathsf{ACA}_0$.

0.3 On the linearity of the consistency hierarchy

Phenomenon 0.2.5 provides a strong hint on why logicians have observed the linearity and well-foundedness of the consistency hierarchy of theories. To see why, let us observe that when logicians established $S \leq_{\mathsf{Con}} T$, they usually prove something stronger than $S \leq_{\mathsf{Con}} T$, which implies $S \leq_{\Pi^1_1\text{-}\mathsf{RFN}} T$[9] or $S \subseteq_{\Pi^1_1} T$. Here is the list of the consistency comparisons we can consider:[10]

1. For two subsystems of second-order arithmetic S and T, the proof of $S <_{\mathsf{Con}} T$ usually follows from either "T proves an ω- or β-model of S" or "T proves Γ-$\mathsf{RFN}(S)$." Also, the equiconsistency of S and T usually follows from a stronger result, like the Π^1_1-conservativity of T over S.

2. When set theorists prove $S <_{\mathsf{Con}} T$, they typically prove that "T proves there is a transitive model of S." Also, when set theorists prove $S \leq_{\mathsf{Con}} T$, they typically rely on forcing and inner models that respect Σ^1_2-truth.

3. Ordinal analysis-based consistency comparisons also tend to give stronger results than mere consistency comparisons. For example, Rathjen [79] proved that Power Kripke-Platek set theory $\mathsf{KP}(\mathcal{P})$ and $\mathsf{Z}_2 + \{V_\xi \text{ exists} \mid \xi < \mathrm{BH}\}$ are equiconsistent by showing that the two theories prove the same $\Pi^{\mathcal{P}}_2$-sentences, which also implies that the two theories prove the same Π^1_2-consequences.

4. Continuing from the ordinal analysis-based consistency comparison, the independence of Goodstein's theorem over PA can be shown in a way that Goodstein's theorem implies there is no primitive recursive infinite descending se-

[9] $S \leq_{\Pi^1_1\text{-}\mathsf{RFN}} T$ iff $\mathsf{ACA}_0 \vdash \Pi^1_1\text{-}\mathsf{RFN}(T) \to \Pi^1_1\text{-}\mathsf{RFN}(S)$

[10] Note that the following list only mentions classical theories extending RCA_0. For constructive theories, the projective hierarchy may collapse [99], so gauging the strength of theories via projective consequence comparison may not work. For theories at the level of bounded arithmetic, it is even unclear if they are capable of defining Π^1_1- or Σ^1_2-pointclasses, and I do not exclude the possibility that non-linearity consistency comparison may happen for natural theories at the level of bounded arithmetic.

quence over ε_0. Rathjen [80] proved that over RCA_0, Goodstein's theorem implies ε_0 is well-founded, which is equivalent to Π^1_1-$\mathsf{RFN}(\mathsf{ACA}_0)$. Hence, $\mathsf{ACA}_0 <_{\mathsf{Con}} \mathsf{RCA}_0 +$ Goodstein's theorem actually follows from that the latter theory proves Π^1_1-$\mathsf{RFN}(\mathsf{ACA}_0)$.

5. There are 'irregular' consistency comparisons for versions of set theories, like separating Wholeness axioms [43] in a proof-theoretic manner, Krivine's classical realizability [52, 63], or Glazer's equiconsistency result between ZF without a single axiom and higher-order arithmetic.[11] However, [43] shows a Γ-RFN of a weaker theory over a stronger theory for Γ containing every projective pointclass. For the classical realizability, [52, Theorem 30] guarantees it does not alter the Σ^1_2-truth of the ground universe. For Glazer's comparisons, Glazer reported that his equiconsistency proof actually implies Σ^1_2- and Π^1_2-conservativity, and his strict consistency comparison proof implies a strict increase of Σ^1_2- and Π^1_2-theorems.

The previous list indicates the following phenomenon:

Phenomenon 0.3.1. Let S and T be classical natural theories extending RCA_0. When logicians prove S and T are equiconsistent, what they actually prove is (or implies) S and T have the same Π^1_1-consequences. When logicians prove $S <_{\mathsf{Con}} T$, what they actually prove is (or implies) $T \vdash \Pi^1_1$-$\mathsf{RFN}(S)$.

We can consider Phenomenon 0.3.1 as evidence of the assertion "For natural theories, consistency comparison and Π^1_1-reflection comparison coincide," and this is what I assume in this dissertation. However, let me stress that Phenomenon 0.3.1 focuses on the *a posteriori* phenomenon, but does not tell about the *a priori* nature of the consistency comparison. That means, Phenomenon 0.3.1 does not exclude the possibility that

[11] Private communication, but the result appears in [33], and most of the ideas for the proof appear in [32].

in the far future, someone proves $S \leq_{\mathsf{Con}} T$ by not proving Π^1_1-reflection comparison $\Pi^1_1\text{-}\mathsf{RFN}(T) \to \Pi^1_1\text{-}\mathsf{RFN}(S)$, or $S \subseteq_{\Pi^1_1} T$, although there have been no such natural theories. Even with the most skeptical viewpoint on Phenomenon 0.3.1, however, we can still explain why logicians have observed that $\leq_{\mathsf{Con}}$ is well-ordered together with Phenomenon 0.2.5: Phenomenon 0.3.1 says what logicians have done for the consistency comparison is, in fact, the Π^1_1-reflection comparison or its strengthening. By Phenomenon 0.2.5, Π^1_1-reflection comparison for natural theories is well-ordered. In other words, Phenomenon 0.3.1 may have deceived logicians in a way that the properties of the Π^1_1-reflection comparison are the same as those of consistency comparison, which may be unrelated.

0.4 Comparing Σ^1_2 consequences

Focusing back to Phenomenon 0.1.4, Steel provided an informal argument for why the consistency comparison and arithmetical consequence comparison coincide for natural theories extending ZFC: Steel provided the following informal justification for the phenomenon:

> For natural S, T extending ZFC, if $S \leq_{\mathsf{Con}} T$, then for every finite $F \subseteq S, T$ proves "there is an ω-model M_F of F." This implies at once that $S \subseteq_{\Pi^0_\omega} T$: if ϕ is an arithmetic consequence of S, then it is a consequence of some finite $F \subseteq S$, so it is true in M_F, so it is true. This is a partial explanation of arithmetic monotonicity, and it extends to Σ^1_2-monotonicity because, in practice, M_F will be a transitive model containing ω_1, and hence Σ^1_2-correct. (Steel [92, p.161]).

Rephrasing Steel's argument, he stated that when we prove $S \leq_{\mathsf{Con}} T$ for natural S and T, we typically have that T proves there is an ω-model of an arbitrary finite fragment of S. Steel's claimed implication does not hold for arbitrary theories.[12] Moreover, his description does not fit well in the realm of second-order arithmetic since various natural theories in second-order arithmetic have finite axiomatizations. Nevertheless, Steel's description pinpoints why the consistency comparison and Σ^1_2-consequence comparison cohere in the region of ZFC.

To illustrate Steel's idea, let us examine how set theorists work to prove $S \leq_{\mathsf{Con}} T$: Set theorists use forcing and inner models to reduce one theory to another theory, and the Shoenfield absoluteness theorem says none of these changes the validity of Σ^1_2 sentences. More precisely, suppose that M is a model of $T \supseteq \mathsf{ZFC}$, and T proves there is a forcing poset $\mathbb{P}$ such that $\mathbb{P}$ forces $S \supseteq \mathsf{ZFC}$. If G is an $\mathbb{P}$-generic filter over M, then $M[G] \vDash S$ and $\mathrm{Ord}^M = \mathrm{Ord}^{M[G]}$. Shoenfield absoluteness theorem tells that every Σ^1_2-statement is absolute between M and $M[G]$, so every Σ^1_2-theorem σ of S is also valid in M. Since M is arbitrary, we have that $T \vdash \sigma$. The same argument works if we replace forcing with an inner model. We can get Steel's description by appealing to the reflection principle: When we prove $S \leq_{\mathsf{Con}} T$ via forcing, let us work over T and consider a large finite fragment $T' \subseteq T$ sufficient to perform a forcing argument. By reflection, we can find a countable transitive model $M \vDash T'$. Forcing over M gives a generic extension $M[G]$ satisfying an appropriate finite fragment $S' \subseteq S$. By choosing an appropriate T', we can make $M[G]$ a model of a given finite fragment of S. M and $M[G]$ have the same ordinals, so the Shoenfield absoluteness argument works as previously stated.

Examining Steel's argument, it does not merely imply that the consistency comparison and the arithmetical consequence comparison coincide for natural theories. In fact,

[12] $\mathsf{ZFC} + \neg\mathsf{Con}(\mathsf{ZFC}) \leq_{\mathsf{Con}} \mathsf{ZFC}$, but ZFC does not prove there is an ω-model of $T + \neg\mathsf{Con}(\mathsf{ZFC})$ for every finite $T \subseteq \mathsf{ZFC}$.

his argument also applies to Σ^1_2-consequence of theories:

Phenomenon 0.4.1. If S and T are natural extensions of ZFC, then

$$S \leq_{\mathsf{Con}} T \iff S \subseteq_{\Sigma^1_2} T.$$

Note that, however, stronger extensions of ZFC can have more Σ^1_2-consequences than weaker extensions of ZFC: When set theorists prove $S <_{\mathsf{Con}} T$, they prove from T that there is a transitive model of S, or equivalently, a well-founded model of S. If we can prove there is a well-founded model of S, we can also prove there is a countable well-founded model of S. The assertion "S has a countable well-founded model" is a Σ^1_2-assertion, which cannot be proved in S.

Steel's argument has some limitations: First, it only applies to extensions of ZFC since it relies on the reflection principle. Second, although not explicitly appear in Steel's argument, when we justify "A stronger theory proves every finite fragment of a weaker theory has a transitive model," we assume set theorists only use forcing and inner models, which preserves transitive models. However, there *are* (although not popular) constructions establishing equiconsistency results in ZF-level that do not precisely preserve transitive models, like Krivine classical realizability. Moreover, some consistency comparisons like [43] do not follow Steel's argument.

We can provide a better justification by mimicking the argument for Π^1_1. First, we will isolate the ordinal characteristic $s^1_2(T)$ of a theory T capturing the Σ^1_2-consequence of T. Then we establish the Σ^1_2-version of Walsh's characterization, which substantiates a Σ^1_2-analogue of Phenomenon 0.2.5. Chapter 1 will address the very result. Finally, it remains to check that for sufficiently strong[13] natural theories, the consistency comparison and the Σ^1_2-reflection comparison coincide. Steel's justification shows that the consistency

[13] I think "extensions of Σ^1_2-AC_0" suffices based on the statement of Corollary 1.4.6.

comparison via forcing and inner model also does the Σ^1_2-reflection comparison. This line of argument justifies why the Σ^1_2-consequence comparison and the Σ^1_2-reflection comparison are equivalent for the natural theories, and reduces Phenomenon 0.4.1 to justifying the equivalence between consistency comparison and Σ^1_2-reflection comparison for natural theories, which is more plausible since Σ^1_2-$\mathsf{RFN}(T)$ can be seen as a Σ^1_2-version of $\mathsf{Con}(T)$.

The following theorem is the main result in Chapter 1, which is the Σ^1_2-version of Walsh's theorem:

Theorem. *1. (Proposition 1.4.1) Let S and T be Σ^1_2-sound extensions of ACA_0. Then*

$$S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T \iff s^1_2(S) \leq s^1_2(T).$$

2. (Corollary 1.4.6) Let S and T be arithmetically definable Σ^1_2-sound extensions of Σ^1_2-AC_0. Then

$$s^1_2(S) \leq s^1_2(T) \iff \Sigma^1_2\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S).$$

3. (Theorem 1.4.9) For S and T with the same conditions in the previous statement,

$$s^1_2(S) < s^1_2(T) \iff T \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(S).$$

Here $s^1_2(T)$ is a version of proof-theoretic ordinal gauging the complexity of Σ^1_2-consequences of T. Before stating what $s^1_2(T)$ is, we first review how the proof-theoretic ordinal $|T|_{\Pi^1_1}$ is defined. There are various ways to assign an ordinal to a theory[14], but the definition we take is the following:

Definition 0.4.2. Let T be a Π^1_1-sound theory extending a weak arithmetic, say, RCA_0. We define the *proof-theoretic ordinal* $|T|_{\Pi^1_1}$ as follows:

$$|T|_{\Pi^1_1} = \sup\{\mathrm{otp}(\alpha) \mid \alpha \text{ is a recursive linear order such that } T \vdash \mathsf{WO}(\alpha)\}$$

[14] See [78] to see alternative ways to assign an ordinal to theories.

Here $\mathsf{WO}(\alpha)$ is the assertion that a recursive linear order α is well-ordered. By Π^1_1-soundness of T, $T \vdash \mathsf{WO}(\alpha)$ implies α is a well-order. Hence, we can compute its ordertype $\mathrm{otp}(\alpha)$, an ordinal isomorphic to a well-order α. The connection between the proof-theoretic ordinal and Π^1_1-consequences is highlighted by the following theorem, which implies that recursive well-orders capture Π^1_1 sentences:

Theorem 0.4.3 (Kleene normal form theorem). *For every Π^1_1-formula $\phi(X)$ we can effectively find a Turing machine α_X for a linear order with an oracle variable X such that the following holds:*

1. ACA_0 *proves "For every real X, α_X is a linear order."*
2. ACA_0 *proves* $\forall^1 X[\phi(X) \leftrightarrow \mathsf{WO}(\alpha_X)]$. □

To define $s^1_2(T)$, we need a recursive object capturing the ordinal complexity of a Σ^1_2-sentence. Its dual problem is to find a recursive object capturing a Π^1_2-sentence, which turns out to be what Girard's *dilator* accomplishes. There are various definitions of a dilator, and I will examine their definitions and basic properties in Appendixs B and C. Despite that, let me briefly explain what dilators are: We may think of a dilator as a representation of a class ordinal: There is no transitive class isomorphic to $\mathrm{Ord} + \mathrm{Ord}$ or Ord^2, but we can still represent their ordertype. In the case of $\mathrm{Ord} + \mathrm{Ord}$, we can view it as the collection of (i, ξ) for $i = 0, 1$ and $\xi \in \mathrm{Ord}$, and compare them lexicographically. Now observe that the same construction gives not only the ordertype $\mathrm{Ord} + \mathrm{Ord}$, but also that of $X + X$ for every linear order X: That is, $X + X$ is isomorphic to the collection of (i, ξ) for $i = 0, 1$ and $\xi \in X$ equipped with the lexicographic order. The uniform construction $X \mapsto X + X$ is an example of a dilator.

It turns out that dilators are the right objects capturing Π^1_2-statements. Let me introduce some terminology before introducing the reason: A *semidilator* is an autofunctor

over the category of linear orders preserving direct limits and pullbacks, and a *predilator* is a semidilator preserving pointwise domination relation. That is, a semidilator F is a predilator if $f, g\colon X \to Y$ are increasing and $f(x) \leq g(x)$ for every $x \in X$, then $F(f)(z) \leq F(g)(z)$ for every $z \in F(X)$. A *dilator* is a semidilator preserving well-orderedness; That is, a semidilator D is a dilator if $D(X)$ is a well-order for every well-order X. Every dilator is a predilator, and in every application, we will only see predilators and dilators, and not semidilators. (See Appendix B for a more precise definition of semidilator, predilator, and dilator.) In principle, a semidilator is a proper class since it is a functor of a large category domain. However, we can recover the full (semi)dilator from its restriction over the category of natural numbers with strictly increasing maps, so that we can code them as a set. Hence, we can talk about whether a given (semi)dilator is recursive by saying there is a recursive code for the restriction of a (semi)dilator to the category of natural numbers. The following result by Girard implies recursive dilators capture Π^1_2-sentences:

Theorem 0.4.4 (Girard's normal form theorem). *For every Π^1_2 formula $\phi(X)$ we can find a Turing machine D_X with an oracle variable X such that ACA_0 proves the following:*

1. *For every real X, D_X is a predilator.*
2. $\forall^1 X[\mathsf{Dil}(D_X) \leftrightarrow \phi(X)]$.

See Theorem 1.1.1 for more details about Girard's completeness theorem.

Dilators capture Π^1_2-statements, so dually, predilators that are *not* dilators capture Σ^1_2-statements, which we will call a *pseudodilator*. For a pseudodilator D, there is a least ordinal α making $D(\alpha)$ ill-founded, which we call the *climax of D* and denote it

by $\mathsf{Clim}(D)$. We use the climax of a pseudodilator as the ordinal complexity of a Σ^1_2-statement, and formulate the following definition:

Definition 0.4.5 (Aguilera-Pakhomov [2])**.** Let T be a Σ^1_2-sound theory. Its Σ^1_2-*proof-theoretic ordinal* $s^1_2(T)$ is defined as follows:

$$s^1_2(T) = \sup\{\mathsf{Clim}(D) \mid D \text{ is a recursive predilator such that } T \vdash \neg\mathsf{Dil}(D)\}.$$

I will also provide a further characterization of $s^1_2(T)$ in different ways in Chapter 1; As an example, Theorem 1.8.5 implies the following result, which hints at the connection between $s^1_2(T)$ and transitive models.

Theorem. *Let T be a Σ^1_2-sound recursive extension of Π^1_1-CA_0, and let $\Sigma^1_2(T)$ be the set of all Σ^1_2-consequences of T. Then $\mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)$ has a transitive model. Furthermore,*

$$s^1_2(T) = \min\{M \cap \mathrm{Ord} \mid M \textit{ is a transitive model of } \mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)\}.$$

0.5 β-rank comparison

I mentioned in the previous section that when set theorists prove $S <_{\mathsf{Con}} T$, they prove "T proves there is a transitive model of S," and when set theorists prove $S \leq_{\mathsf{Con}} T$, they use forcing and inner models. An interesting aspect of forcing and inner models is that they preserve the transitivity of models. That is, if M is a transitive model of set theory, then its generic extension $M[G]$ via forcing, and a canonical inner model (say, L^M or K^M) computed in M are also transitive. In some sense, set theorists only use transitive models to compare the consistency strength of two theories. Hence, at least for set theorists, comparing the strength of theories should be based on the *β-model* of the theories, which motivates the *β-rank of theories*. In Chapter 2, we examine properties of

the β-rank of the theories, which is joint work with Patrick Lutz, Fedor Pakhomov, and James Walsh.

Let us fix a base theory T_0, which is either a fragment of second-order arithmetic or set theory. For two theories S and T extending T_0, we compare the two theories under the following relation:

$$S <_{\beta}^{T_0} T \iff \text{Every } \beta\text{-model of } T \text{ contains a } \beta\text{-model of } S.$$

For the β-comparison of set theories, we use well-founded models (or transitive models). For the β-comparison of theories in the language of second-order arithmetic, we use β-models, an ω-model that is correct about Π^1_1-statements. These two choices are not random since we can turn a β-model of second-order arithmetic into a well-founded model of set theory via the sets-as-trees interpretation (see Section A.5.) It makes β-comparison of second-order arithmetical theories as a subcase of β-comparison of set theories, so we can focus on β-comparison of set theories.

For T_0 extending $\mathsf{PRS} + \mathsf{Beta}$[15], the β-comparison relation $<_{\beta}^{T_0}$ is well-founded. Hence we can define the β-rank of theories T extending T_0, denoted by $|T|_{\beta}^{T_0}$. We first ask the supremum of β-ranks of theories, which is not entirely trivial since there are continuum many theories and the size of the continuum as an aleph number is independent of ZFC. It turns out that we can determine the supremum of the β-rank of theories without extra assumptions, which is ω_1:

Theorem. 1. *(Section 2.4) The supremum of the* β*-rank of theories extending* $\mathsf{PRS} + \mathsf{Beta}$ *is* ω_1.

2. *(Section 2.5) The supremum of the* β*-rank of theories extending* $\mathsf{PRS} + \mathsf{Beta} + (V = L)$ *is* ω_1^L.

[15] PRS is *primitive recursive set theory* in the language $\{\in\}$, and Beta is Axiom Beta, which is a stronger version of the Mostowski collapsing theorem. See Section 2.1 and Section A.5 for technical details.

The theories we practically use are all recursive, so we may also ask the supremum of the β-rank of recursive theories. It turns out that the supremum is equal to δ^1_2, the supremum of the ordertype of Δ^1_2-definable well-orders:

Theorem (Theorem 2.6.1). *Let $T_0 \supseteq \mathsf{PRS}+\mathsf{Beta}$ be a recursive theory such that $H_{\omega_1} \models T_0$. Then the following are all equal to δ^1_2:*

1. *The supremum of $|T|^{T_0}_\beta$ for finitely axiomatizable $T \supseteq T_0$, if T_0 is finitely axiomatizable.*
2. *The supremum of $|T|^{T_0}_\beta$ for recursively axiomatizable $T \supseteq T_0$,*
3. *The supremum of $|T|^{T_0}_\beta$ for a Σ^1_2-singleton $T \supseteq T_0$.*

We can also compute the β-rank of specific theories. A technique used to establish the supremum of β-rank of theories extending $\mathsf{PRS} + (V = L)$ can be internalized into a weak set theory named KPi,[16], which gives the following result:

Theorem (Theorem 2.7.4). *Let $T \supseteq \mathsf{KPi}$ be a theory such that T proves "L satisfies T." Then*

$$|T|^{\mathsf{PRS}+\mathsf{Beta}}_\beta = \min\{M \cap \mathrm{Ord} \mid M \textit{ is a transitive model of } T\}.$$

If T is compatible with the axiom of countability, then we have

$$|T|^{\mathsf{ATR}^{\mathsf{set}}_0}_\beta = \min\{M \cap \mathrm{Ord} \mid M \textit{ is a transitive model of } T\}.$$

This suggests that comparing the β-rank of theories extending $\mathsf{KPi} + (V = L)$ coincides with comparing the height of transitive models of the theories, which better corresponds with set theorists' practice of comparing the strength of theories. However, comparing the β-ranking and the consistency strength of natural theories also shows some

[16] KPi is KP plus "Every set is contained in an admissible set, i.e., a transitive model of KP"

gap, and β-rank comparison seems not to fit well with theories incompatible with $V = L$ (like, "$0^\sharp$ exists".) We will discuss issues about β-rank comparison in Section 2.8.

0.6 Comparing Pohlers' and Girard's proof theory

Calculating the proof-theoretic ordinal of a theory becomes harder as the target theory becomes stronger. One reason is that the consequences of a theory of higher complexity interact with the proof-theoretic ordinal of the theory. This motivates the necessity of proof-theoretic analysis for the consequences of higher complexity than Π^1_1-formulas, and this is what Girard's Π^1_2-*logic* in part accomplishes. Girard developed the Π^1_2-proof theory to take a finistic control on infinitary objects [27] by separating the 'well-founded part' and the 'indiscernible part' [31]. The 'well-founded part' is represented by dilators. Like ordinal analysis provides proof-theoretic ordinal, dilator-based proof-theoretic analysis yields *proof-theoretic dilator* $|T|_{\Pi^1_2}$ of T characterizing the Π^1_2-consequences of T. $|T|_{\Pi^1_2}$ is a dilator, so we can evaluate $|T|_{\Pi^1_2}(\alpha)$ for an ordinal α. It raises the following question: *Does* $|T|_{\Pi^1_1}(\alpha)$ *encode any proof-theoretic information about* T*?* For a recursive α, the following is known:

Theorem 0.6.1 (Aguilera and Pakhomov, [2, Theorem 9]). *Let* T *be a* Π^1_2*-sound theory extending* ACA_0 *and* α *be a recursive well-order. Then* $|T|_{\Pi^1_2}(\alpha) = |T + \mathsf{WO}(\alpha)|_{\Pi^1_1}$. □

The previous theorem also tells how Π^1_2-consequences interact with Π^1_1-consequences: The proof-theoretic ordinal $|T|_{\Pi^1_1}$ occurs as a 'projection' of a proof-theoretic dilator $|T|_{\Pi^1_2}$ in the sense that $|T|_{\Pi^1_1} = |T|_{\Pi^1_2}(0)$. However, the previous theorem does not provide any proof-theoretic information of $|T|_{\Pi^1_2}(\alpha)$ for a *non-recursive* α. The goal of Chapter 3 is to provide proof-theoretic information of $|T|_{\Pi^1_2}(\alpha)$ for $\alpha < \delta^1_2$, where δ^1_2 is the supremum of ordertypes of Δ^1_2-wellorders (equivalently, the least ordinal satisfying

$L_{\delta_2^1} \prec_{\Sigma_1} L$.) Before describing the main result in Chapter 3, let me illustrate some motivating results and relevant concepts. Let me start with the following Π^1_2-analysis of ACA_0:

Theorem 0.6.2 (Aguilera and Pakhomov [4]). $|\mathsf{ACA}_0|_{\Pi^1_2} = \varepsilon^+$, *where* ε^+ *is a dilator such that* $\varepsilon^+(\alpha)$ *is the least epsilon number greater than* α. □

Meanwhile, Pohlers [75, 76] considered ordinal analysis over structures other than the natural numbers, assigning non-recursive ordinals to represent the strength of a theory relative to the given structure. To address the motivation behind Pohlers' framework, let us examine Pohlers' structural conception on ordinal analysis. Polhers stated [73, p.59] that determining the proof-theoretic ordinal of T is closely related to determining the least model for $\Sigma_1^{L_{\omega_1^{\mathsf{CK}}}}$-sentences provable from T. One related fact about it is the Spector-Gandy theorem, which states that for a Π^1_1-sentence ϕ, there is a Σ_1-sentence ψ in the language of set theory such that

$$\mathbb{N} \vDash \phi \iff \exists \alpha < \omega_1^{\mathsf{CK}} (L_\alpha \vDash \psi).$$

For sufficiently weak theories, predicative cut elimination suffices to catch the proof-theoretic ordinal. However, for a sufficiently strong theory T in the sense that ω-models of T include $L_{\omega_1^{\mathsf{CK}}}$,[17] T can see how outer structures of $L_{\omega_1^{\mathsf{CK}}}$ affects its $\Sigma_1^{L_{\omega_1^{\mathsf{CK}}}}$-consequences. This appears in the form of an ordinal collapsing function in impredicative cut elimination. In particular, ordinal analysis for 'impredicative' theories requires analyzing the behavior of theories over 'outer structures' of $L_{\omega_1^{\mathsf{CK}}}$. We want to analyze how the outer structures affect $L_{\omega_1^{\mathsf{CK}}}$, and we should stratify the outer structures for the analysis. The natural way to stratify the outer structures is employing admissible sets, like L_α for admissible ordinals $\alpha > \omega_1^{\mathsf{CK}}$. Admissible sets are set-theoretic objects, and

[17] KP is an example of such a theory: Suppose that $M \vDash \mathsf{KP}$ is an ω-model (so $\omega = \omega^M$). If M is well-founded, then clearly the transitive collapse of M contains $L_{\omega_1^{\mathsf{CK}}}$. If M is ill-founded, then Ville's lemma implies $L_{\omega_1^{\mathsf{CK}}}$ is contained in a well-founded part of M.

we want to have an arithmetical counterpart to fit them into the arithmetical framework. It turns out that the notion of a *Spector class* is such a counterpart: In a nutshell, a Spector class is a collection of subsets of $\mathbb{N}$ having a recursion-theoretic nature. For a Spector class Γ, its *companion*[18] is an admissible set. (See [66, Ch. 9] or [65, Ch. 6–7] for the details.) Also, the collection of lightface Π^1_1-sets forms a Spector class. By viewing the ordinal analysis of T as a procedure computing how many true Π^1_1-statements T can prove, we can generalize ordinal analysis by replacing Π^1_1 with a Spector class Γ, so the generalized ordinal analysis computes how many facts about Γ are captured by T.

Now, let us illustrate a general framework that Pohlers formulated. Pohlers suggested gauging the performance of a theory T by gauging the gap between the supremum $\delta^{\mathfrak{M}}$ of $\mathfrak{M}$-definable well-orders and the supremum $\delta^{\mathfrak{M}}(T)$ of T-provably well-founded $\mathfrak{M}$-definable well-orders, like the gap between ω_1^{CK} and the proof-theoretic ordinal $|T|_{\Pi^1_1}$ describes the strength of T. Indeed, $\delta^{\mathfrak{M}}(T)$ is the usual proof-theoretic ordinal $|T|_{\Pi^1_1}$ if we take $\mathfrak{M}$ to be the standard structure $\mathbb{N}$ of natural numbers. Informally, larger $\mathfrak{M}$ can be seen as a 'measuring stick' with a coarser scale: On the one hand, calculating $\delta^{\mathfrak{M}}(T)$ for a larger $\mathfrak{M}$ is no harder than for a smaller $\mathfrak{M}$. On the other hand, however, $\delta^{\mathfrak{M}}(T)$ for a large $\mathfrak{M}$ 'ignores' more consequences of lower complexity as $\mathfrak{M}$ grows; Pohlers' No Enhancement theorem [76, Theorem 3.21] tells $\delta^{\mathfrak{M}}(T) = \delta^{\mathfrak{M}}(T+\sigma)$ for a Σ^1_1-statement σ with parameters from $\mathfrak{M}$. Facts about Spector classes (like [66, §.9F]) suggest $\delta^{\mathfrak{M}}(T)$ gauges the degree of T capturing facts about the 'next Spector class' above $\mathfrak{M}$.

The structure $\mathfrak{M}$ can be arbitrary in principle; Most structures we consider are an expansion of $\mathbb{N}$ under certain conditions. In particular, Pohlers focused on the *iterated Spector classes* $\mathsf{SP}^{\xi}_{\mathbb{N}}$ to gauge the performance of iterated inductive definitions $\mathsf{ID}_{<\alpha}$. Notably, we have the following:

[18] Roughly speaking, the companion of Γ for a Spector class Γ is a collection of sets that can be coded by $\Gamma \cap \check{\Gamma}$-relations.

Theorem 0.6.3 (Pohlers [76, Remark 3.35]). *For $\mu \geq 1$ less than the least recursively inaccessible ordinal*[19]*, we have*

$$\delta^{\mathsf{SP}^{\mu}_{\mathbb{N}}}(\mathrm{Th}(\mathsf{SP}^{\mu}_{\mathbb{N}}) + \textit{Mathematical Induction}) = \varepsilon_{\omega^{\mathsf{CK}}_{\mu}+1}.$$

In Chapter 3, I will introduce $\Pi^1_1[R]$-*proof-theoretic ordinal* for a Σ^1_2-singleton R, which turns out to be related to Pohlers' performance ordinal $\delta^{\mathfrak{M}}(T)$. The (Π^1_1-)proof-theoretic ordinal

$$|T|_{\Pi^1_1} = \sup\{\mathrm{otp}(\alpha) \mid \alpha \text{ is a recursive linear order such that } T \vdash \mathsf{WO}(\alpha)\}$$

uses the ordertype of T-provably recursive well-orders to gauge the strength of T. Likewise, $\Pi^1_1[R]$ proof-theoretic ordinal $|T|_{\Pi^1_1[R]}$ uses the ordertype of T-provably R-*recursive* well-orders to gauge the strength of T. (See subsection 3.1.2 for the precise definition of $|T|_{\Pi^1_1[R]}$.) We will see in Theorem 3.A.20 that the iterated Spector class $\mathsf{SP}^{\xi+1}_{\mathbb{N}}$ for ξ less than the least recursively inaccessible ordinal is equivalent to $(\mathbb{N}; R)$ for some Σ^1_2-singleton R, corresponding to the ξth iterate of the hyperjump of $\emptyset$. Under the perspective of Proposition 3.2.8, Pohlers' result can be rephrased as follows:

Theorem 0.6.4. *Let ξ be a successor ordinal less than the least recursively inaccessible. Then*

$$|\mathsf{ACA}_0 + \mathsf{HJ}^{\xi}(\emptyset) \textit{ exists} + \mathrm{Th}(\mathbb{N}; \mathsf{HJ}^{\xi}(\emptyset))|_{\Pi^1_1[\mathsf{HJ}^{\xi}(\emptyset)]} = \varepsilon_{\omega^{\mathsf{CK}}_{\xi}+1}. \qquad \square$$

Here $\mathrm{Th}(\mathbb{N}; \mathsf{HJ}^{\xi}(\emptyset))$ *is the collection of true* first-order *sentences over* $(\mathbb{N}; \mathsf{HJ}^{\xi}(\emptyset))$.

Pohlers' result and the Π^1_2-proof theoretic analysis by Aguilera and Pakhomov hint at the connection between these two. The next theorem, which is the main result of Chapter 3, illustrates how they are systematically entangled:

[19] An ordinal ξ is recursively inaccessible ordinal if L_{ξ} is a model of KP plus "There is a proper class of admissible ordinals."

Theorem (Theorem 3.5.2). *Let T be a Π^1_2-sound theory extending ACA_0 and (D, ϱ) be a recursive* locally well-founded genedendron *generating R. Furthermore, assume that T proves (D, ϱ) is a locally well-founded genedendron. If α is an R-recursive well-order such that D_α is ill-founded, then*

$$|T|_{\Pi^1_2}(\alpha) = |T[R] + \mathsf{WO}(\alpha)|_{\Pi^1_1[R]}.$$

Here $T[R]$ is the theory T + 'R exists.'

The statement of the main theorem uses a new concept named *genedendron*. Informally, we can view a genedendron as a 'passive' description for a given real. For a recursive real, we can determine the real by providing a computer program that computes the real. Such an explicit description is impossible for non-computable reals, but we still hope that we can provide a recursive description of the real. A recursive genedendron comprises a functorial family of trees $\langle D_\alpha \mid \alpha \in \mathrm{Ord} \rangle$, which may or may not have infinite branches. Once we get an infinite branch of a certain tree D_α, we should be able to compute the target real from the branch through a function ρ_α. The real must be independent of the choice of an infinite branch and α. We say a genedendron *generates* R if the value of ϱ_α is R, and the least ordinal α making D_α ill-founded represents the complexity of the real R. We will provide an extended discussion on a genedendron in subsection 3.4.1 and Section 3.4. We will calculate a genedendron (D, ϱ) generating $\mathsf{HJ}(\emptyset)$ such that D_α is ill-founded for a $\mathsf{HJ}(\emptyset)$-recursive well-order of ordertype ω_1^{CK}. Hence we will have

$$|\mathsf{ACA}_0 + \mathsf{HJ}(\emptyset)\ \text{exists}|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = \varepsilon_{\omega_1^{\mathsf{CK}}+1}.$$

Combining it with Proposition 3.1.14, we have

$$|\mathsf{ACA}_0 + \mathsf{HJ}(\emptyset)\ \text{exists} + \mathrm{Th}(\mathbb{N}; \mathsf{HJ}(\emptyset))|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = \varepsilon_{\omega_1^{\mathsf{CK}}+1}.$$

which partially reproduces Pohlers' result. It shows the connection between the proof-theoretic dilator and the proof-theoretic-ordinal for *intermediate pointclass* of the form $\Pi^1_1[R]$ for a Σ^1_2-singleton R.[20]

The notion of genedendron itself looks like an interesting concept that may be worth taking its own interest. One of the reasons is that it provides a canonical way to rank Σ^1_2-singleton reals called Σ^1_2-*altitude* (See Definition 3.4.6.) The Σ^1_2-altitude of arithmetically definable reals is 0, and that of the hyperjump of $\emptyset$ is ω_1^{CK}. Calculating the Σ^1_2-altitude of specific Σ^1_2-singleton reals and characterizing it in terms of a transitive model of set theory (See Conjecture 3.6.3) can be interesting problems in computability theory.

0.7 Homogeneous Suslin tree and Measurable dilator

Stanisław Mazur raised a game-theoretic problem in the *Scottish book* [64, Problem 43], whose following generalization is formulated by Stanisław Ulam:

> Given a set E of reals, Player I and II give in turn the digits 0 or 1. If the resulting real is in E, then Player I wins, and Player II wins otherwise. For which E does one of the players have a winning strategy?

In accord with the standard set-theoretic tradition, we identify a real with an infinite sequence of natural numbers, so we may think of E as a set of infinite binary sequences, which is called a *payoff set*. In 1953, David Gale and Frank M. Stewart [25] studied a generalization of Ulam's problem by considering an infinite game over an arbitrary set

[20] If we line up pointclasses with the prewellordering property under the supremum of the ordertype the pointclass possesses, then $\Pi^1_1[R]$ lies between Π^1_1 and Σ^1_2. It is why we call $\Pi^1_1[R]$ an intermediate pointclass.

A. Gale and Stewart proved that if a payoff set $E \subseteq A^\omega$ is *open or closed*, then either one of the players has a winning strategy in the corresponding game. They also showed from the axiom of choice that there is a payoff set $E \subseteq 2^\omega$ in which no player has a winning strategy.

The axiom of determinacy AD and its extensions have a significant role in descriptive set theory and inner model theory, even though they are incompatible with Choice. One of the reasons is that over $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$, AD implies various regularity properties of the set of reals, like every set of reals is Lebesgue measurable. Another reason is that determinacy axioms provide better consistency bounds for statements whose consistency strength goes beyond that of $\mathsf{AD}^{L(\mathbb{R})}$.[21] (See [11] or [10, §1] for a detailed discussion in this direction.) Although full AD is incompatible with ZFC, we may ask how which fragment of AD is provable in (or compatible with) ZFC by restricting it to a certain class of sets of reals. For a class Γ of sets of reals, Γ-*determinacy* is the assertion that for a payoff set $E \in \Gamma$, the game given by E is *determined* in the sense that either one of the players has a winning strategy. After some partial results from other mathematicians, Martin [57] proved Borel determinacy.

Then what can we say about determinacy for larger classes? It turns out that Γ-determinacy for a larger Γ is closely related to large cardinal axioms. In 1968, Martin [59] proved the $\mathbf{\Pi}^1_1$-determinacy from a measurable cardinal. Later in 1978, Harrington proved that if $\mathbf{\Pi}^1_1$-determinacy holds, then every real has a sharp [36]. Indeed, $\mathbf{\Pi}^1_1$-determinacy and the existence of sharps for reals are equivalent. For a class larger than $\mathbf{\Pi}^1_1$, there was no significant progress for years. But in 1980, Martin [58] proved the $\mathbf{\Pi}^1_2$-determinacy from a rank-into-rank large cardinal called an *iterable cardinal*. A proof of $\mathbf{\Pi}^1_2$-determinacy and projective determinacy (determinacy for $\mathbf{\Pi}^1_n$-sets for every natural

[21] For example, the currently known best upper bound of the consistency strength of $\mathsf{ZFC} + \mathsf{MM}^{++}(\mathfrak{c})$ is obtained by forcing over a model of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} +$ "Θ is regular", which is due to Woodin.

n) from a near-optimal hypothesis appeared in 1989 by Martin and Steel [60]; Namely, we have $\mathbf{\Pi}^1_n$-determinacy from $(n-1)$ many Woodin cardinals and a measurable above. The optimal strength of $\mathbf{\Pi}^1_n$-determinacy requires $M^\sharp_n(x)$, a sharp for a canonical inner model with n many Woodin cardinals. The currently known proofs of $\mathbf{\Pi}^1_n$-determinacy from large cardinals rely on *homogeneous trees* or their technical variations:

Definition 0.7.1. A tree T on $\omega \times Z$ is *homogeneous* if there is a system $\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ of measures such that

1. Each μ_s is a countably additive measure on $T_s = \{t \in Z^{|s|} : \langle s, t\rangle \in T\}$.
2. If $s_0 \subseteq s_1$, then $X \in \mu_{s_0}$ iff $\{t \mid t \restriction \mathrm{lh}(s_0) \in X\} \in \mu_{s_1}$. Informally speaking, μ_{s_1} *projects into* μ_{s_0}.
3. If a real x is in $p[T] = \{x \in \mathbb{R} \mid \forall n < \omega \exists t \in Z^n [\langle x \restriction n, t\rangle \in T]\}$, then the direct limit of the ultrapower $M_{x\restriction n} := V^{Z^n}/\mu_{x\restriction n}$ is well-founded.

A set $A \subseteq \mathbb{R}$ is *κ-homogeneously Suslin* if $A = p[T]$ for some homogeneous tree T over $\omega \times \kappa$.

The proof of Γ-determinacy from large cardinals is usually divided into the following two steps: First, starting from a large cardinal κ, we prove that sets in Γ are κ-homogeneously Suslin. Second, we prove determinacy for κ-homogeneously Suslin sets.

Meanwhile, viewing sets of reals as a projection of a tree over $\omega\times\kappa$ appears frequently in other parts of descriptive set theory. Let us say a set of reals $A \subseteq \mathbb{R}$ is *κ-Suslin* if $A = p[T]$ for some tree T over $\omega \times \kappa$. For example, the proof of the Shoenfield absoluteness theorem relies on the fact that the $\mathbf{\Sigma}^1_2$ set of reals is ω_1-Suslin. But once we examine the proof of ω_1-Suslinness of a $\mathbf{\Sigma}^1_2$-set carefully, we can notice that the resulting tree actually takes the form $D(\omega_1)$ for some predilator D. Furthermore, if the tree projects to a $\Sigma^1_2[R]$

set of reals for some real R, then D can be an R-recursive predilator. That means, an ω_1-tree witnessing the ω_1-Suslinness of a $\Sigma^1_2[R]$-set can be decomposed into an 'effective part' (namely, an R-recursive predilator) and a 'large cardinal part' (namely, ω_1).

We can expect a similar phenomenon for homogeneous Suslin sets: That is, we may decompose a homogeneous tree into an 'effective part' and a 'large cardinal object part.' This is true for a $\mathbf{\Pi}^1_1$ set: Suppose that α_X is a countable linear order with real parameter X. We can define a $\mathbf{\Pi}^1_1$ set by stipulating $A = \{X \in \mathbb{R} \mid \mathsf{WO}(\alpha_X)\}$. Then let us consider a predilator D such that for a well-order γ, $D(\gamma)$ represents a tree trying to find a real X and an embedding $\alpha_X \to \gamma$.[22] Then a κ-homogeneous tree projecting to A takes the form $D(\kappa)$.

We can expect a similar phenomenon for $\mathbf{\Pi}^1_2$ sets of reals: For a countable predilator D_X with real parameter X, we can build a 2-preptyx[23] such that for another dilator E, $P(E)$ tries to find a real X and an embedding $D_X \to P(E)$. We want to have a dilator Ω such that every homogeneous tree for $\mathbf{\Pi}^1_2$-sets takes the form $P(\Omega)$ for an appropriate 2-preptyx P. In the $\mathbf{\Pi}^1_1$-case, we used a measurable cardinal, a cardinal κ equipped with a κ-additive measure. Accordingly, we should use a *measurable dilator* in the $\mathbf{\Pi}^1_2$ case. In Chapter 4, I construct a measurable dilator by reorganizing Martin's proof of $\mathbf{\Pi}^1_2$-determinacy from an iterable rank-into-rank embedding. Roughly speaking, I decompose Martin's proof of $\mathbf{\Pi}^1_2$-determinacy into the following two steps:

1. Starting from a large cardinal assumption, construct a measurable dilator.
2. From a measurable dilator, prove $\mathbf{\Pi}^1_2$-determinacy.

To extract dilator-related information from Martin's proof, we need a tree-like struc-

[22] See Theorem 1.1.1 and Lemma 1.1.4 for more details about an embedding predilator.

[23] 2-ptyx (*sing.* 2-ptykes) is a higher-order generalization of a well-order and dilator, which is an object representing a Π^1_3 formula. See Appendix E for a precise definition of 2-ptykes.

ture of a dilator since Martin defined a measure family by iterating measures induced by a rank-into-rank embedding over a tree. Girard [26, §6] presented a notion of *dendroid*, expressing a dilator as a functorial family of trees. Dendroids themselves are not enough to translate Martin's proof in terms of dilators because a dendroid is a combination of some tree structure with an ordinal, and it is necessary to separate the tree structure from the ordinal in a dendroid. Hence, I will introduce a tree structure named *dendrogram*, which codes a dilator as a single tree. I will iterate measures along a dendrogram to get a measure family of a measurable dilator I construct. In Appendix D, I discuss basic properties of dendrograms and dendroids.

0.8 Comparing Projective consequences

Going back to Steel's Phenomenon 0.4.1, let us observe that Steel also claimed that the consistency comparison and projective consequence comparison go together for theories extending $\mathsf{ZFC} + \mathsf{PD}$:[24]

Phenomenon 0.8.1 (Steel [92])**.** Let S, T be natural theories whose consistency strength is at least that of ACA_0 with Projective Determinacy. Then either

$$S \subseteq_{\Pi^1_\infty} T \text{ or } T \subseteq_{\Pi^1_\infty} S.$$

Also, the projective consequences of sufficiently strong natural theories monotonically increase along their consistency strength.

Aguilera-Pakhomov [5] accordingly proved the following result, which is relevant to the linear comparison phenomenon of Steel's observation:

[24] Steel stated 'natural theories of consistency strength at least ZFC with infinitely many Woodin cardinals,' but I put Projective Determinacy instead since PD is more directly related to projective statements than the existence of infinitely many Woodin cardinals.

Theorem 0.8.2 (Aguilera-Pakhomov [5, Theorem 36]). *Suppose that there are infinitely many Woodin cardinals. Fix a number N. For each $n \geq 1$, let T be ZFC_N[25] plus "There are $(n-1)$ Woodin cardinals." If N is sufficiently large, then the following hold:*

1. *If n is odd, then the set of Π^1_n-sentences endowed with the order*

$$\phi \leq \psi \iff T \vdash^{\Sigma^1_n} \psi \to \phi$$

   is a well-order of ordertype δ^1_n.

2. *If n is even, then the set of Σ^1_n-sentences endowed with the order*

$$\phi \leq \psi \iff T \vdash^{\Pi^1_n} \psi \to \phi$$

   is a well-order of ordertype δ^1_n.

Aguilera and Pakhomov also proved that Π^1_{2n}- and Σ^1_{2n+1}-statements are not linearly comparable even modulo true Σ^1_{2n}- or Π^1_{2n+1}-statements respectively. One interesting aspect of Aguilera-Pakhomov's result is that the levels of projective pointclasses admitting linear comparison coincide with the levels of projective pointclasses admitting the prewellordering and scale property. In particular, in the Π^1_1 case, well-orders have a focal role to prove the prewellordering property of Π^1_1-sets. We may suspect this coincidence happened not just by chance, but the prewellordering property and the linear comparison of the consequences are closely tied. It turns out that mimicking Moschovakis' prewellordering property of $\mathbf{\Pi}^1_{2n+1}$-sets under $\mathbf{\Delta}^1_{2n}$-determinacy produces a $\mathbf{\Delta}^1_{2n+1}$-prewellorder $\trianglelefteq_{\vartheta^{2n}}$ over the collection of countable $2n$-ptykes, which allows us to compare the Π^1_{2n+1}-consequences of theories in the following sense:

[25] ZFC_N is obtained from ZFC by restricting Replacement to Σ_N-formulas.

Theorem (Theorems 5.3.5, 5.3.6 and 5.6.1, Σ^1_{2n+1}-$\mathsf{AC}_0 + \mathbf{\Delta}^1_{2n}$-Det). *Let $n \geq 1$, and S, T be Π^1_{2n+1}-sound extensions of $\mathsf{ACA}_0 + \mathbf{\Delta}^1_{2n}$-Det. Then we have*

$$S \subseteq^{\Sigma^1_{2n+1}}_{\Pi^1_{2n+1}} T \iff |S|_{\Pi^1_{2n+1}} \trianglelefteq_{\vartheta^{2n}} |T|_{\Pi^1_{2n+1}}.$$

Furthermore, if S and T are $\mathsf{Bool}(\Sigma^1_{2n})$-definable, then we have

$$S \subseteq^{\Sigma^1_{2n+1}}_{\Pi^1_{2n+1}} T \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_{2n+1}} \Pi^1_{2n+1}\text{-}\mathsf{RFN}(T) \to \Pi^1_{2n+1}\text{-}\mathsf{RFN}(S).$$

and

$$|S|_{\Pi^1_{2n+1}} \triangleleft_{\vartheta^{2n}} |T|_{\Pi^1_{2n+1}} \iff T \vdash^{\Sigma^1_{2n+1}} \Pi^1_{2n+1}\text{-}\mathsf{RFN}(S).$$

In the case Σ^1_{2n+2} for $n \geq 1$, we need to isolate an object capturing Σ^1_{2n+2}-consequences of a theory via a $(2n+1)$-*pseudoptyx*, and a way to compare two $(2n+1)$-pseudoptykes. In the case Σ^1_2, comparing the climaxes of two pseudodilator works. We do not have an ordinal characteristic representing the magnitude of a $(2n+1)$-pseudoptyx at this point, so we introduce a Π^1_{2n+2}-definable comparison relation $\trianglelefteq_{\mathsf{Grow}^{2n+1}}$ that is a prewellorder over the class of $(2n+1)$-pseudoptykes provably over Σ^1_{2n+2}-$\mathsf{AC}_0 + \mathbf{\Delta}^1_{2n}$-Det $+ \Sigma^1_{2n+1}$-DC_0. The absence of the ordinal characteristic also forces us to compare the *proof-theoretic* $(2n+1)$-*pseudoptykes* of two theories $|T|_{\Sigma^1_{2n+2}}$ via $\trianglelefteq_{\mathsf{Grow}^{2n+1}}$.

Theorem (Theorems 5.5.1, 5.5.3 and 5.6.2, Σ^1_{2n+2}-$\mathsf{AC}_0 + \mathbf{\Delta}^1_{2n}$-Det $+ \Sigma^1_{2n+1}$-DC_0). *Let $n \geq 1$ and S, T be Σ^1_{2n+2}-sound extensions of Σ^1_{2n+2}-$\mathsf{AC}_0 + \mathbf{\Delta}^1_{2n}$-Det $+ \Sigma^1_{2n+1}$-DC_0.*[26] *Then we have*

$$S \subseteq^{\Pi^1_{2n+2}}_{\Sigma^1_{2n+2}} T \iff \|S\|_{\Sigma^1_{2n+2}} \trianglelefteq_{\mathsf{Grow}^{2n+1}} \|T\|_{\Sigma^1_{2n+2}}.$$

Furthermore, if S and T are $\mathsf{Bool}(\Pi^1_{2n+1})$-definable, then we have

$$S \subseteq^{\Pi^1_{2n+2}}_{\Sigma^1_{2n+2}} T \iff$$

$$\Sigma^1_{2n+2}\text{-}\mathsf{AC}_0 + \mathbf{\Delta}^1_{2n}\text{-}\mathsf{Det} + \Sigma^1_{2n+1}\text{-}\mathsf{DC}_0 \vdash^{\Pi^1_{2n+2}} \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(T) \to \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(S).$$

[26] I suspect Σ^1_{2n+1}-DC_0 is redundant; If we can prove $\mathbf{\Delta}^1_{2n}$-Det implies $\mathbf{\Pi}^1_{2n}$-Det over ACA_0, then $\mathbf{\Pi}^1_{2n}$-Det implies Σ^1_{2n+1}-DC_0. See Proposition 5.1.1 for the implication of Σ^1_{2n+1}-DC_0.

and

$$\|S\|_{\Sigma^1_{2n+2}} \lhd_{\mathsf{Grow}^{2n+1}} \|T\|_{\Sigma^1_{2n+2}} \iff T \vdash^{\Pi^1_{2n+2}} \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(S).$$

One of the limitations of my current result is that the comparison relations $\trianglelefteq_{\vartheta^{2n}}$ and $\trianglelefteq_{\mathsf{Grow}^{2n+1}}$ do not indicate which ordinals are associated and compared with the theories. I will address this problem for the Π^1_3-comparison relation $\trianglelefteq_{\vartheta^2}$ in subsection 5.2.3.

0.9 Conclusion

Results in Chapter 1 and Chapter 5, together with Walsh's result [103], provide an informal justification of Steel's observed phenomenon and indicate structural similarity of consequence comparisons. Moreover, these results suggest that comparing Π^1_{2n+1}- or Σ^1_{2n+2}-consequences corresponds to a version of consistency comparison (namely, reflection comparison). This suggests that in my further argument, I could focus on Γ-consequence comparison in place of consistency strength comparison since they are 'equivalent' for $\Gamma = \Pi^1_1, \Sigma^1_2, \Pi^1_3, \Sigma^1_4, \cdots$.

There seems to be no reason to restrict ourselves to Γ for the standard lightface pointclasses $\Pi^1_1, \Sigma^1_2, \Pi^1_3, \Sigma^1_4, \cdots$. The results in Chapter 3 combined with Pohler's framework in [76] show pointclasses lying between Π^1_1 and Σ^1_2, namely, $\Pi^1_1[R]$ for a Σ^1_2-singleton real R, can also be used to gauge the strength of theories. Furthermore, results in Chapter 3 and [76, §3] also indicate that choosing an appropriate pointclass makes isolating the strength of a theory easier; For example, $|\Pi^1_1\text{-}\mathsf{CA}^-_0|_{\Pi^1_1}$[27] involves with ordinal collapsing function, but $|\Pi^1_1\text{-}\mathsf{CA}^-_0|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = \varepsilon_{\omega_1^{\mathsf{CK}}+1}$. This phenomenon seems to extend to the theory of iterated inductive definitions ID_ν (See [72, Appendix 2] or [76, §3.4]), where

[27] $\Pi^1_1\text{-}\mathsf{CA}^-_0$ is the theory of *parameter-free* Π^1_1-comprehension. Its equivalent formulation is ACA_0 + "$\mathsf{HJ}()$ exists."

gauging the performance of ID_ν over $\mathsf{SP}^{\mathbb{N}}_{\nu+1}$[28] yields a predicative result, while computing its proof-theoretic ordinal $|\mathsf{ID}_\nu|_{\Pi^1_1}$ involves ordinal collapsing function. Here, let us observe that $\mathsf{SP}^{\mathbb{N}}_{\nu+1}$ is the 'least' Spector class that ID_ν cannot prove its existence. It looks like the least Spector class whose existence is not provable in the theory admits a predicative performance ordinal. Hence, we speculate that for sufficiently strong theories, we can calculate their strength not by calculating the proof-theoretic ordinal, but by searching for the least Spector class such that the theory cannot prove its existence.

Interestingly, the previous idea aligns with current developments in inner model theory. Steel expressed the idea of using more complicated pointclasses to gauge the strength of theories as follows:

> For example, if S and T are natural theories extending ZFC, and S has consistency strength less than or equal to that of T, then the arithmetic consequences of S are included in those of T. If in addition, S and T have consistency strength at least that of "there are infinitely many Woodin cardinals", then the consequences of S in the language of second order arithmetic are included in those of T. This pattern persists at still higher consistency strengths, with still more logically complicated consequences about reals and sets of reals being brought into a uniform order. This beautiful and suggestive phenomenon has a practical dimension as well: one way to develop the absolute consequences of a strong theory T is to compute a consistency strength lower bound S for T in terms of large cardinal hypotheses, and then work in the theory S. (Steel [93, p2])

Inner model theorists use *mice* to capture the strength of theories. Roughly speaking, a mouse is a transitive model of set theory we can expand through *iteration* so that the

[28] I guess it is equivalent to calculating $|\mathsf{ID}_\nu|_{\Pi^1_1[\mathsf{HJ}^\nu(\emptyset)]}$.

mouse can capture certain information outside of the model. $0^\sharp$ is an example of a mouse, which can be seen as a pair $(L_{\iota^{+L}}, U)$, where ι is the least L-indiscernible, and U is the L-ultrafilter induced by the embedding $j\colon L \to L$ shifting the first ω many L-indiscernibles to the next indiscernible. Iterating U over $L_{\iota^{+L}}$ gives an arbitrarily high section of L, so we can regard $0^\sharp$ as a small transitive model with a machinery U allowing to capture arbitrarily high ordinals. $M_1^\sharp$, a sharp for a Woodin cardinal, is also an example of a mouse, which can capture a real outside of it via *genericity iteration* (See [20] for details). The following explanation by Andreas Lietz explains that the way inner model theorists use mice coincides with the previous speculative way of using Spector classes to catch the strength of theories:

> We should now update the method by which we find the strength of a theory T. This is from now on the least mouse which is not in some wellfounded model of T. In this step, we did not simply relabel our measuring stick as before! For weak theories we could properly measure before, the two notions agree, but for stronger theories this new notion is much better. (Lietz [54])

Inner model theorists have made the deep connection between mice and pointclasses. For example, Steel's conjecture named *Hod Pair Capturing* roughly states "there are unboundedly many universally Baire pointclasses under Wedge reducibility containing a hod pair." A hod pair is a pair of a *hod mouse* and its iteration strategy (See [93] for the details). Also, the following theorem by Woodin describes a way to get a mouse from a given pointclass:

Theorem 0.9.1 (Woodin [93, Theorem 7.2.4]). *Working over* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$*, let* Γ *be a lightface pointclass closed under the real existential quantifier with the scale property, and assume that every set in* $\check{\Gamma}$ *is Suslin. Then for every real* x*, there is a coarse* Γ*-Woodin pair* (M, Σ) *such that* $x \in M$. □

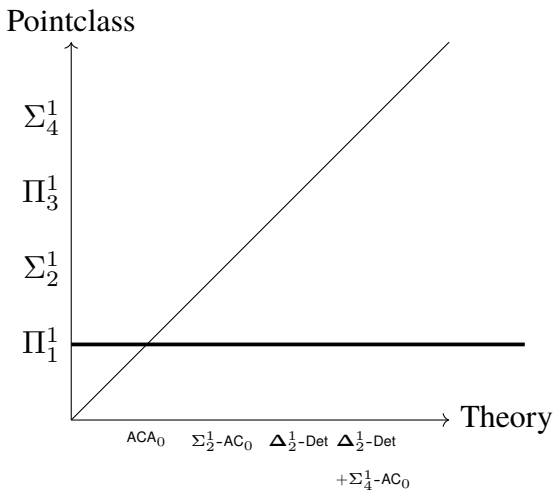


Figure 1: The connection between inner model theory and ordinal analysis. Inner model theory gauges the strength of a theory along the diagonal, and ordinal analysis gauges the strength of a theory along the thick horizontal line. The pointclasses should be lined up along the size of the supremum of the length of definable prewellorders of the pointclass or their Wadge rank.

From the previous explanations, the following could be expected:

1. (Π^1_1-)Ordinal analysis gauges the strength of a theory by measuring the Π^1_1-consequences of the theory. Put differently, ordinal analysis gauges how much the theory can prove facts about $L_{\omega_1^{\mathsf{CK}}}$.

2. Inner model theory gauges the strength of a theory by finding the 'least' pointclass Γ such that the theory does not recognize the existence of a 'Γ-companion,' and uses Γ (or its associated mice) as a certifier of the strength of the theory.

Figure 1 illustrates how ordinal analysis and inner model theory could be related; They might be sections of the single framework, namely, gauging the strength of theories under an appropriate pointclass.

We could also imagine a generalization of the β-rank of theories with certain mice

in place of transitive models; We can think of $0^\sharp$ or other sharps at a low level as an *Ehrenfeucht-Mostowski model (abbr. EM model)*[82], by viewing $0^\sharp$ as a functor taking an ordinal α and returning the least section of L with α many L-indiscernibles. It gives a dilator-like model, and we could mimic the definition of the β-rank with EM models, which could provide a better ranking scheme for theories around at the level of linear iteration, like ZFC with measurable cardinals. The construction of a measurable dilator seems to indicate we may view $M_1^\sharp$ as a 2-ptyx-like EM model, and it might be generalized to $M_n^\sharp$ or other types of mice. These EM-like structures could be tied with Woodin's Ω-provability in a way that a statement is Ω-provable iff the statement is true over every EM-like structure.

We finish this chapter with the alleged association between the computability-theoretic structure of reals and the hierarchy of theories, which is suggested by various results both within and outside this dissertation. One example is Walsh's investigations on the proof-theoretic analogue of Martin's conjecture in degree theory, which is broadly overviewed in [105]. Hereby, we address a summary of [105, §6–8].

While there are various Turing degrees between $\emptyset$ and $\emptyset'$, every known construction for the intermediate degree is deemed 'unnatural'. Sacks asked if there is an 'effective' construction for an intermediate degree between $\emptyset$ and $\emptyset'$, which also uniformly resolves the relativized problem whether there is any degree between A and A' for every real A:

Question 0.9.2 (Sacks)**.** Is there a degree-invariant recursively enumerable operator W such that for every real A, $A <_\mathsf{T} W(A) <_\mathsf{T} A'$?

Sacks' question is generalized by Martin's conjecture describing the behavior of degrees under the axiom of determinacy. Similar to Sacks' question, we consider degree-invariant functions in place of individual degrees to represent relativized constructions.

$\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ proves the *Martin measure* over 2^ω defined by

$$X \in \mu \iff \exists A \in 2^\omega [\{B \in 2^\omega \mid A \leq_\mathsf{T} B\} \subseteq X]$$

is a countably complete ultrafilter. For two degree-invariant functions $f, g \colon 2^\omega \to 2^\omega$, let us define

$$f \leq_\mathsf{M} g \iff \{A \in 2^\omega \mid f(A) \leq_\mathsf{T} g(A)\} \in \mu.$$

$\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ proves $\leq_\mathsf{M}$ linearly orders degree-invariant functions. Martin's conjecture states the following:

Conjecture 0.9.3 (Martin, $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$)**.** 1. Every degree-invariant function from 2^ω to 2^ω is either constant almost everywhere or $<_\mathsf{T}$-above the identity function almost everywhere.

2. $\leq_\mathsf{M}$ restricted to the degree-invariant functions $<_\mathsf{T}$-above the identity function almost everywhere is a prewellorder, and the $\leq_\mathsf{M}$-successor of f is the pointwise Turing jump f' of f.

Here, 'almost everywhere' means almost everywhere under Martin's measure. Martin's conjecture is open, although it is resolved for special cases. (e.g., [55].) Now let us formulate its proof-theoretic analogue by replacing the Turing jump operator with the consistency operator $\phi \mapsto \mathsf{Con}(T + \phi)$ for a fixed sound recursive base theory T. Walsh proved the following proof-theoretic analogue of Part 1 of Martin's conjecture:

Theorem 0.9.4 (Walsh [101]). *Fix a recursive, sound base theory T extending elementary arithmetic* EA*. Suppose that $\mathfrak{g} \colon \mathbb{N} \to \mathbb{N}$ is recursive and*

1. monotone *in the sense that $T \vdash \phi \to \psi$ implies $T \vdash \mathfrak{g}(\phi) \to \mathfrak{g}(\psi)$, and*
2. bounded *in the sense that there is $n \in \mathbb{N}$ such that for every ϕ, $\mathfrak{g}(\phi)$ is Π^0_n.*

A true cone *is a set of formulas of the form* $\{\phi \mid T + \phi \vdash \psi\}$ *for some true sentence* ψ*. Then one of the following holds:*

1. *There is a true cone* $\mathfrak{C}$ *such that for every* $\phi \in \mathfrak{C}$, $T + \phi \vdash \mathfrak{g}(\phi)$.
2. *There is a true cone* $\mathfrak{C}$ *such that for every* $\phi \in \mathfrak{C}$, $T + \phi + \mathfrak{g}(\phi) \vdash \mathsf{Con}(T + \phi)$.

As another example of the connection between computability theory and proof theory, Cai examined the degree structure of Π^0_2-sentences over a fixed theory. (e.g., [15, 14].) Do we have an analogy between computability theory and proof theory at higher levels? Theorem 1.8.5 and Conjecture 3.6.3 suggest there are rooms for degrees of Σ^1_2-singleton reals to appear in Σ^1_2-proof theory. Results in Chapter 1 show Σ^1_2-consequence comparison modulo true Π^1_2-sentences is well-ordered. Interestingly, Π^1_1-singleton reals form a prewellorder under the hyperarithmetical reducibility [37, §V.6]. Similarly, under appropriate determinacy axioms, Σ^1_{2n}-consequence comparison modulo true Π^1_{2n}-sentences is well-ordered, and the class of Π^1_{2n+1}-singleton reals is prewellordered under the Δ^1_{2n+1}-*reducibility*. (cf. [50].) Examining the analogy between the projective consequence comparison of theories and the degree theory of singleton reals under appropriate reducibility relations would be an interesting future problem.

Part I

Proof theory for higher pointclasses

CHAPTER 1

CASE Π^1_1 AND Σ^1_2

> Logical consequences are the scarecrows of fools and the beacons of wise men.
>
> Thomas Huxley, *Science and culture, and other essays*.

In this section, we review Walsh's characterization of the proof-theoretic ordinal [103]. Then we introduce the Σ^1_2-proof-theoretic ordinal, and examine its behavior. We will see that Σ^1_2-proof-theoretic ordinal comparison is also equivalent to Σ^1_2-consequence comparison modulo true Π^1_2-sentence, and they are also equivalent to Σ^1_2-reflection comparison for theories extending Σ^1_2-AC_0. In the last two sections, we state an alternative characterization of the Σ^1_2-proof-theoretic ordinal. Most results in this chapter appear in [39], and the results in the last two sections come from my unpublished results.

1.1 Logical aspects of dilators

In this section, we discuss some notions and facts about dilators that we will use and have not covered in preliminary chapters for dilators.

Proposition B.3.4 shows the sentence 'D is a dilator' is Π^1_2 for a countable dilator D:

$$\mathsf{Dil}(D) \equiv \forall^1 \alpha [\mathsf{WO}(\alpha) \to \mathsf{WO}(D(\alpha))].$$

Similar to Kleene normal form theorem, which says every Π^1_1-formula is equivalent to a formula of the form $\mathsf{WO}(\alpha)$, the Π^1_2-*completeness theorem of dilators* we will introduce says every Π^1_2-formula is equivalent to a formula of the form $\mathsf{Dil}(D)$:

Theorem 1.1.1 (Girard's normal form theorem [29, 16]). *For every Π^1_2 formula $\phi(X)$ we can find a Turing machine D_X with an oracle variable X such that ACA_0 proves the following:*

1. *For every real X, D_X is a predilator.*
2. $\forall^1 X[\mathsf{Dil}(D_X) \leftrightarrow \phi(X)]$.

Sketch of the proof. There are two types of proof: We can manually build the corresponding predilator, or we can use β-logic. I will sketch how to manually build a corresponding predilator for a given Π^1_2-formula since a similar but more technical proof will appear in the proof of Proposition 1.7.4, so the following proof sketch should help the reader understand the latter proof. For a proof using β-logic, see [5, Lemma 7].

Suppose that $\phi(X) \equiv \forall^1 Y \neg\psi(X, Y)$ for some Π^1_1-formula $\psi(X, Y)$. By Kleene's normal form theorem, we can find a primitive recursive linear order $\alpha(X, Y)$ such that $\psi(X, Y)$ holds iff $\alpha(X, Y)$ is well-founded. Then for a given linear order β, we build a primitive recursive tree $T_{\alpha,X}(\beta)$ trying to find a real Y and an embedding $\alpha(X, Y) \to \beta$, with techniques used in [2, Lemma 8] or Lemma 1.1.5. We can see an infinite branch of $T_{\alpha,X}(\beta)$ witnesses the validity of $\exists^1 Y \psi(X, Y)$. Then turn the tree $T_{\alpha,X}$ into a predilator by imposing the Kleene-Brouwer order. □

Thus, we can identify every Π^1_2 sentence with $\mathsf{Dil}(D)$ for some *recursive* predilator D. Similarly, every Σ^1_2-sentence can be identified with $\neg\mathsf{Dil}(D)$ for some recursive predilator D.

We will often quantify over the set of recursive predilators, and so checking the complexity of being a recursive predilator is important. The following proposition shows that the collection of primitive recursive predilators has a low complexity:

Proposition 1.1.2. *The set* RecPreDil *of all codes of recursive predilators is* Π^0_2.

Proof. A recursive object is a recursive predilator if there is a *recursive* set of terms with arity assignment and recursive comparison rules, all of which can be coded by a natural number. Also, we can see that there are countably many arity diagrams, and in fact, we can recursively enumerate all arity diagrams. Thus, a statement 'a natural number codes an arity diagram' is Δ^0_1. By inspecting a definition of a predilator, we can see that a real D predilator is $\Pi^0_2(D)$, so we get the desired result. □

By Π^1_1- and Π^1_2-completeness of recursive well-orders and dilators, we can represent Π^1_1 and Π^1_2 sentences as of the form $\mathsf{WO}(\alpha)$ or $\mathsf{Dil}(D)$ for recursive α and D respectively. We may ask if there are ways to form new well-orders or dilators that correspond to logical connectives, which take formulas and form a new formula. The following constructions will illustrate operators for linear orders and predilators that correspond to logical connectives:

Lemma 1.1.3. *Let D_0, D_1 be predilators of field subsets of $\mathbb{N}$. Then we can construct a predilator $D = D_0 \wedge D_1$ with the following properties:*

1. ACA_0 *proves* $\mathsf{Dil}(D) \leftrightarrow (\mathsf{Dil}(D_0) \wedge \mathsf{Dil}(D_1))$.
2. ACA_0 *proves the following: If D_0 is a dilator, then for every well-order α,* $\mathsf{WO}(D_1(\alpha))$ *iff* $\mathsf{WO}(D(\alpha))$.
3. *If both of D_0 and D_1 are recursive, then so is D.*

Proof. Take $D = D_0 + D_1$. Proving the statements of the lemma is straightforward, so we leave it to the reader. □

Lemma 1.1.4 (Aguilera-Pakhomov [2, Lemma 8], ACA_0). *For linear orders α and β of fields subsets of $\mathbb{N}$, we can effectively construct a predilator $D = (\alpha \to \beta)$ such that*

1. *There is an embedding* $e \colon \beta \to D(\alpha)$.
2. D *is a dilator iff* $\mathsf{WO}(\alpha) \to \mathsf{WO}(\beta)$ *holds.*
3. *If* D *is not a dilator, then* $D(\gamma)$ *is illfounded iff* $\alpha \leq \gamma$ *for any linear order* γ.

Furthermore, if both α *and* β *are recursive, then so is* D. □

The following construction, which gives a 'disjunction' of two predilators, is a variant of [71, Lemma 5.2], which is also stated in [3, Proposition 23].

Lemma 1.1.5 (ACA_0)**.** *Let* D_0 *and* D_1 *be two predilators of field subsets of* $\mathbb{N}$. *Then we can construct a new predilator* $D = D_0 \vee D_1$ *such that the following holds:*

1. D *is a dilator iff either* D_0 *or* D_1 *is a dilator.*
2. *If* α *is a well-order,* D_0 *is a dilator and* $D_1(\alpha)$ *is ill-founded with an infinite decreasing sequence* b*, then there is an embedding* $e^b_\alpha \colon D_0(\alpha) \to D(\alpha)$.
3. *If both of* D_0 *and* D_1 *are recursive, then so is* D.

Proof. For a fixed linear order α, let $T(\alpha)$ be the tree trying to construct decreasing sequences on $D_0(\alpha)$ and $D_1(\alpha)$ simultaneously. More precisely, $T(\alpha)$ is the set of all sequences of the form

$$\langle (x_0, y_0), \cdots, (x_n, y_n) \rangle \in (D_0(\alpha) \times D_1(\alpha))^{<\omega}$$

such that $x_0 >_{D_0(\alpha)} \cdots >_{D_0(\alpha)} x_n$ and $y_0 >_{D_1(\alpha)} \cdots >_{D_1(\alpha)} y_n$. Then define $D(\alpha) = (T(\alpha), <_{\mathsf{KB}})$, but we need to specify the meaning of $<_{\mathsf{KB}}$ to make the point clear that the ordertype of $D(\alpha)$ only depends on D_0, D_1 and the ordertype of α.

Let $c = \langle c_0, \cdots, c_m \rangle$, $c' = \langle c'_0, \cdots, c'_n \rangle$ be members of $T(\alpha)$. Then we say $c <_{\mathsf{KB}} c'$ holds if one of the following holds:

1. If c' is a proper initial segment of c.

2. If it is not the first case, and suppose that $i \leq \min(m, n)$ is the least natural number such that $c_i \neq c'_i$. Furthermore, assume that c_i and c'_i take the following form:

   - $c_i = (s(\vec{\xi}), t(\vec{\eta}))$.
   - $c'_i = (s'(\vec{\xi}'), t'(\vec{\eta}'))$.

   Here s, s' are D_0-terms, t, t' are D_1-terms, and $\vec{\xi}, \vec{\xi}', \vec{\eta}, \vec{\eta}'$ are finite increasing sequences over α. Since D_0 and D_1 are countable, we may view s, s', t, t' as natural numbers. Then we say $c <_{\mathsf{KB}} c'$ when we have

   $$c_i \lhd c'_i \equiv (s, t, \vec{\xi}, \vec{\eta}) < (s', t', \vec{\xi}', \vec{\eta}') \text{ under the lexicographic order over } \mathbb{N} \times \mathbb{N} \times [\alpha]^{<\omega} \times [\alpha]^{<\omega},$$

   where $[\alpha]^{<\omega}$ is the set of finite subsets over α, identified with their increasing enumeration, with the lexicographic order. Here we view D_i-terms as natural numbers.

We can see that $<_{\mathsf{KB}}$ is a strict linear order over $T(\alpha)$. Hence D is a semidilator since the comparison only depends on the terms and the relative order of elements of α.[1] Furthermore, D is recursive if both D_0 and D_1 are recursive.

To see D is a predilator, let us apply Proposition B.3.2: Let $f, g \colon \alpha \to \beta$ to be strictly increasing functions between two linear orders α and β such that $f \leq g$. We can see that for $c = \langle (s_i(\vec{\xi}_i), t_i(\vec{\eta}_i)) \mid i < m \rangle$,

$$D(f)(c) = \langle (s_i(f[\vec{\xi}_i]), t_i(f[\vec{\eta}_i])) \mid i < m \rangle.$$

So by the definition of $<_{\mathsf{KB}}$ over $D(\beta)$, we can see that $D(f)(c) \leq D(g)(c)$ holds.

[1] Alternatively, we can define a support function $\mathrm{supp}_\alpha \colon D(\alpha) \to [\alpha]^{<\omega}$ by $\mathrm{supp}_\alpha(c) = \bigcup_{i<m} \vec{\xi}_i \cup \vec{\eta}_i$, where $c = \langle (s_i(\vec{\xi}_i), t_i(\vec{\eta}_i)) \mid i < m \rangle$.

Now let us claim that for each well-order α,

$$\mathsf{WO}(D(\alpha)) \leftrightarrow \mathsf{WO}(D_0(\alpha)) \vee \mathsf{WO}(D_1(\alpha)).$$

Clearly $D(\alpha)$ is ill-founded if both of $D_0(\alpha)$ and $D_1(\alpha)$ are ill-founded. Conversely, suppose that $D(\alpha)$ is ill-founded, so we have an infinite decreasing sequence

$$c(0) >_{\mathsf{KB}} c(1) >_{\mathsf{KB}} c(2) >_{\mathsf{KB}} \cdots.$$

We claim that the length of $c(i)$ is unbounded, so we can find a decreasing sequence over $D_0(\alpha)$ and $D_1(\alpha)$.

Suppose not, assume that the length of $c(i)$ is stabilized by, say, n. For notational convenience, we may assume that the length of all of $c(i)$ is n, so $c(i) = (c_0(i), \cdots, c_{n-1}(i))$. If there are infinitely many i such that $c_0(i)$ are different, then there must be an infinite sequence $\langle i_k \mid k < \omega \rangle$ such that $c_0(i_0) \rhd c_0(i_1) \rhd \cdots$, which is impossible. Hence $c_0(i)$ is eventually constant. By the same argument, we can see that all of $c_1(i)$, $c_2(i)$, $\cdots$, $c_{n-1}(i)$ are eventually constant, so we get a contradiction.

Hence, if one of D_0 or D_1 is a dilator, then so is D. It remains to construct e^b_α that we promised. It follows from modifying the construction in the proof of [71, Lemma 5.2] Suppose that we are given an infinite decreasing sequence $b = \langle b_n \mid n < \omega \rangle$ over $D_1(\alpha)$.

Now let us enumerate elements of $D_0(\alpha)$ using the lexicographic order $<_{\mathbb{N}\times[\alpha]^{<\omega}}$, which is possible since

$$D_0(\alpha) = \{\langle t, \vec{\xi}\rangle \in \text{field}(D_0) \times [\alpha]^{<\omega} \mid \text{arity}(t) = |\vec{\xi}|\}.$$

Since α is a well-order, $<_{\mathbb{N}\times[\alpha]^{<\omega}}$ is also a well-order. For a notational convenience, we assume that we enumerate elements of $D_0(\alpha)$ by $\{\langle t_\nu, \vec{\xi}_\nu\rangle \mid \nu \in \kappa\}$, where κ is a well-order isomorphic to a subset of $\text{field}(D_0) \times [\alpha]^{<\omega}$ with the lexicographic order.

Define $e^b_\alpha(\langle t_0, \vec{\xi}_0\rangle) = \langle (t_0(\vec{\xi}), b_0)\rangle$. Now for a given $\nu \in \kappa$, consider the two cases:

1. $D_0(\alpha) \vDash t_\nu(\vec{\xi}_\nu) > t_\mu(\vec{\xi}_\mu)$ for all $\mu < \nu$.
2. $D_0(\alpha) \vDash t_\nu(\vec{\xi}_\nu) < t_\mu(\vec{\xi}_\mu)$ for some $\mu < \nu$.

In the first case, set $e^b_\alpha(t_\nu(\vec{\xi}_\nu)) = \langle (t_\nu(\vec{\xi}_\nu), b_0)\rangle$. In the second case, let us pick $\mu' < \nu$ satisfying

$$t_{\mu'}(\vec{\xi}_{\mu'}) = \min_{D_0(\alpha)} \{t_\mu(\vec{\xi}_\mu) \mid \mu < \nu \wedge D_0(\alpha) \vDash t_\nu(\vec{\xi}_\nu) < t_\mu(\vec{\xi}_\mu)\}, \qquad (\spadesuit)$$

where the minimum is computed under the order over $D_0(\alpha)$ induced by a dilator D_0. It is possible since D_0 is a dilator, so $D_0(\alpha)$ is a well-order. Then define

$$e^b_\alpha(t_\nu(\vec{\xi}_\nu)) = e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))^\frown\langle (t_\nu(\vec{\xi}_\nu), b_l)\rangle$$

where l is the length of $e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))$. Then we can inductively show the following holds:

1. If $e^b_\alpha(t_\nu(\vec{\xi}_\nu))$ has length l, then it takes the form

   $$\langle (t_{\nu_0}(\vec{\xi}_{\nu_0}), b_0), (t_{\nu_1}(\vec{\xi}_{\nu_1}), b_1), \cdots, (t_{\nu_{l-1}}(\vec{\xi}_{\nu_{l-1}}), b_{l-1})\rangle$$

   with $\nu_0 < \nu_1 < \cdots < \nu_{l-1} = \nu$.
2. $e^b_\alpha \colon D_0(\alpha) \to D(\alpha)$ is order-preserving.

The first item follows almost immediately from the definition of e^b_α. Let us claim the second by induction on $\nu \in \kappa$: Suppose inductively that $e^b_\alpha \colon \{t_\mu(\vec{\xi}_\mu) \mid \mu < \nu\} \to D(\alpha)$ is order-preserving. Fix $\mu < \nu$, and divide the cases:

1. Suppose that $t_\mu(\vec{\xi}_\mu) > t_\nu(\vec{\xi}_\nu)$. Let $\mu' < \nu$ be the index satisfying ($\spadesuit$), that is, $t_{\mu'}(\vec{\xi}_{\mu'})$ is the $D_0(\alpha)$-least value among $t_\zeta(\vec{\xi}_\zeta)$ greater than $t_\nu(\vec{\xi}_\nu)$ such that $\zeta < \nu$. Then we have

   $$e^b_\alpha(t_\nu(\vec{\xi}_\nu)) = e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))^\frown\langle (t_\nu(\vec{\xi}_\nu), b_l)\rangle <_{D(\alpha)} e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'})) \leq_{D(\alpha)} e^b_\alpha(t_\mu(\vec{\xi}_\mu)),$$

where the last inequality is held by the inductive assumption on e^b_α.

2. Now suppose that $t_\mu(\vec{\xi}_\mu) < t_\nu(\vec{\xi}_\nu)$. If $t_\zeta(\vec{\xi}_\zeta) < t_\nu(\vec{\xi}_\nu)$ holds for all $\zeta < \mu$, then

$$e^b_\alpha(t_\nu(\vec{\xi}_\nu)) = \langle t_\nu(\vec{\xi}_\nu), b_0\rangle <_{D(\alpha)} \langle (t_\zeta(\vec{\xi}_\zeta), b_0), \cdots\rangle = e^b_\alpha(t_\mu(\vec{\xi}_\mu)),$$

for some $\zeta \leq \mu < \nu$. We can see that the above inequality holds since $(t_\zeta(\vec{\xi}_\zeta), b_0) \lhd (t_\nu(\vec{\xi}_\nu), b_0)$, which follows from $\zeta < \nu$ and the definition of the enumeration $\{t_\nu(\vec{\xi}_\nu) \mid \nu \in \kappa\}$.

Now consider the case when $t_\zeta(\vec{\xi}_\zeta) > t_\nu(\vec{\xi}_\nu)$ holds for some $\zeta < \mu$. Choose again $\mu' < \nu$ satisfying (♠). Then we have $e^b_\alpha(t_\mu(\vec{\xi}_\mu)) < e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))$, so we have the following two possible cases:

(a) $e^b_\alpha(t_\mu(\vec{\xi}_\mu)) \supsetneq e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))$, or

(b) There is the least i such that the ith component of $e^b_\alpha(t_\mu(\vec{\xi}_\mu))$ is different from that of $e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))$, and the former is $\lhd$-less than the latter.

In the latter case, we have $e^b_\alpha(t_\mu(\vec{\xi}_\mu)) < e^b_\alpha(t_\nu(\vec{\xi}_\nu))$ since $e^b_\alpha(t_\nu(\vec{\xi}_\nu)) \supseteq e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))$. In the former case, we have

$$e^b_\alpha(t_\mu(\vec{\xi}_\mu)) = e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))^\frown\langle (t_{\mu''}(\vec{\xi}_{\mu''}), b_l), \cdots, (t_\mu(\vec{\xi}_\mu), b_{l'})\rangle$$

for some $\mu' < \mu'' \leq \mu$ and $l \leq l' < \omega$, and we have

$$e^b_\alpha(t_\nu(\vec{\xi}_\nu)) = e^b_\alpha(t_{\mu'}(\vec{\xi}_{\mu'}))^\frown\langle (t_\nu(\vec{\xi}_\nu), b_l)\rangle.$$

Since $\mu'' < \nu$, we have $(t_{\mu''}(\vec{\xi}_{\mu''}), b_l) \lhd (t_\nu(\vec{\xi}_\nu), b_l)$, so $e^b_\alpha(t_\mu(\vec{\xi}_\mu)) < e^b_\alpha(t_\nu(\vec{\xi}_\nu))$.

Thus e^b_α is an order-preserving map, as desired. □

The following construction is a countably infinite variant of Lemma 1.1.5:

Lemma 1.1.6 (ACA_0)**.** *Let $\langle D_n \mid n < \omega \rangle$ be an enumeration of countable predilators. Then we can find a* disjuctive join $D = \bigvee_{n<\omega} D_n$ *of $\langle D_n \mid n < \omega \rangle$ such that for every well-order α,*

$$\mathsf{WO}(D(\alpha)) \iff \exists n < \omega \mathsf{WO}(D_n(\alpha)).$$

Furthermore, if each of D_n and the enumeration $\{D_n \mid n < \omega\}$ are recursive, then so is D.

Proof. First, for each α, define a partial order $P(\alpha)$ as follows: Its domain consists of the following two-dimensional sequence

$$c = \begin{pmatrix} c_0^0 & c_1^0 & \cdots & c_n^0 \\ & c_1^1 & \cdots & c_n^1 \\ & & \ddots & \vdots \\ & & & c_n^n \end{pmatrix}$$

and each ith row of the sequence is $<_{D_n(\alpha)}$-decreasing. The order over $P(\alpha)$ is given by an extension. Then consider $D(\alpha) = (P(\alpha), <_{\mathsf{KB}})$. Then D is a predilator by a similar argument in the proof of Lemma 1.1.5: More formally, we compare $c, d \in P(\alpha)$ as follows. $c <_{\mathsf{KB}} d'$ if and only if either

1. d is a proper initial segment of c, or
2. Suppose that i is the least natural number such that the ith column of c and that of d are different. Then

   $$(c_i^0, \cdots, c_i^i) \lhd (d_i^0, \cdots, d_i^i)$$

   holds, which is given as follows: If $c_i^j = s_i^j(\vec{\xi}_i^j)$ and $d_i^j = t_i^j(\vec{\eta}_i^j)$, then (2) is equivalent to

   $$(\langle s_i^0, \cdots, s_i^i \rangle, \vec{\xi}_i^0 \cup \cdots \cup \vec{\xi}_i^i) < (\langle t_i^0, \cdots, t_i^i \rangle, \vec{\eta}_i^0 \cup \cdots \cup \vec{\eta}_i^i)$$

over $\mathbb{N} \times \alpha^{<\omega}$ under the lexicographic order, where $\langle n_0, \cdots, n_{k-1} \rangle$ is a bijective primitive recursive function coding a finite tuple into a natural number.

Then $<_{\mathsf{KB}}$ is a strict linear order over $P(\alpha)$, and we can see that D is a predilator. Clearly, D is recursive if $\langle D_n \mid n < \omega \rangle$ is recursive.

It is clear that $D(\alpha)$ is ill-founded if all of $D_i(\alpha)$ are ill-founded. Conversely, suppose that $D(\alpha)$ is ill-founded, so we have an infinite decreasing sequence

$$c(0) >_{\mathsf{KB}} c(1) >_{\mathsf{KB}} c(2) >_{\mathsf{KB}} \cdots.$$

By the same argument we provided in the proof of Lemma 1.1.5, we have that the number of columns of $c(i)$ is unbounded. From this, we can extract an infinite decreasing sequence over $D_i(\alpha)$ for every i, so $D_i(\alpha)$ is ill-founded for every i. □

1.2 Reflection rank

In this section, we review and establish facts about the reflection rank.

As we have raised in the paragraph before Theorem 0.1.3, logicians have proved $T \vdash \mathsf{Con}(S)$ for natural S and T is to prove $S <_{\mathsf{Con}} T$. Under the perspective of Phenomenon 0.3.1, logicians have proved $T \vdash \Pi^1_1\text{-}\mathsf{RFN}(S)$ or its strengthening to prove $S <_{\mathsf{Con}} T$. This suggests that the relation $\prec_{\Pi^1_1}$ has a special role in stratifying the consistency strength of natural theories, where

$$S \prec_{\Gamma} T \iff T \vdash \Gamma\text{-}\mathsf{RFN}(S)$$

for Γ-sound r.e. extensions S and T of ACA_0. Indeed, Pakhomov and Walsh [70] proved that $\prec_{\Pi^1_1}$ for Π^1_1-sound r.e. extensions of ACA_0 is well-founded:

Theorem 1.2.1 (Pakhomov-Walsh [70, Theorem 3.2]). *Let* $\Gamma = \Pi^1_n$ *or* $\Gamma = \Sigma^1_{n+1}$ *for* $n \geq 1$. *Then* $\prec_\Gamma$ *is well-founded.* □

Thus, every Π^1_1-sound r.e. extensions of ACA_0 admits a $\prec_{\Pi^1_1}$-rank, which we will call the reflection rank. Pakhomov-Walsh [104] proved that the reflection rank coincides with the proof-theoretic ordinal for Π^1_1-sound r.e. extensions of ACA^+_0:

Theorem 1.2.2 (Pakhomov-Walsh [104]). *Let* T *be a* Π^1_1*-sound r.e. extension of* ACA^+_0. *Then* $\mathrm{rank}_{\Pi^1_1}(T) = |T|_{\Pi^1_1}$.

Pakhomov and Walsh proved Theorem 1.2.1 only for $\Gamma = \Pi^1_1$, but the same proof works for any Σ^1_n or Π^1_n for $n \geq 2$. It allows us to define the Γ-*reflection rank* $\mathrm{rank}_\Gamma(T)$ for Γ-sound r.e. theories T extending ACA_0. We also recall that $\prec_\Gamma$ is defined over the set of Γ-sound r.e. extensions of ACA_0. It is natural to ask about the value and the behavior of the Γ-reflection rank for natural theories. The following lemma says the reflection rank is always less than ω_1^{CK}:

Lemma 1.2.3. *Let* T *be a* Γ*-sound r.e. extension of* ACA_0. *Then* $\mathrm{rank}_\Gamma(T) < \omega_1^{\mathsf{CK}}$.

Proof. Fix a Γ-sound r.e. extension T of ACA_0, and consider the following set:

$$\mathbb{P} = \{S \mid T \vdash \Gamma\text{-}\mathsf{RFN}(S)\},$$

where S is ranged over all *r.e. extension of* ACA_0. That is, we do not assume Γ-soundness of S in $\mathbb{P}$, but the Γ-soundness of S follows from the Γ-soundness of T by Lemma A.6.3.

We can see that $\mathbb{P}$ is a recursive set, and the relation $\prec_\Gamma$ over Γ is a recursive well-founded relation. Hence we can define a $\prec_\Gamma$-rank function ρ over $\mathbb{P}$. Moreover, the supremum of all $\rho(S)$ for $S \in \mathbb{P}$ must be less than ω_1^{CK}. However, we can see that $\rho(S) = \mathrm{rank}_\Gamma(S)$ for all $S \in \mathbb{P}$ by $\prec_\Gamma$-induction, and the supremum of all $\rho(S)$ for $S \in \mathbb{P}$ coincides with $\mathrm{rank}_\Gamma(T)$. Hence $\mathrm{rank}_\Gamma(T) < \omega_1^{\mathsf{CK}}$ □

We will see that $\prec_\Gamma$ does not cohere well with other proof-theoretic characteristics like the Σ^1_2-proof-theoretic ordinal $s^1_2(T)$. Hence, let us define the following variant of $\prec_\Gamma$:

Definition 1.2.4. Let Γ be a complexity class. Define

$$S \prec^{\check{\Gamma}}_{\Gamma} T \iff T \vdash^{\check{\Gamma}} \Gamma\text{-}\mathsf{RFN}(S).$$

The argument provided in the proof of [70, Theorem 3.2] shows $\prec^{\check{\Gamma}}_{\Gamma}$ is well-founded for $\Gamma = \Pi^1_n$ for extensions of Σ^1_n-AC_0. This allows us to define the $\prec^{\check{\Gamma}}_{\Gamma}$-rank for theories extending Σ^1_n-AC_0 for $\Gamma = \Pi^1_n$. Let us provide the proof for $n = 2$:

Proposition 1.2.5 (Σ^1_2-AC_0)**.** $\prec^{\Sigma^1_2}_{\Pi^1_2}$ *for r.e.* Π^1_2*-sound extensions of* Σ^1_2-AC_0 *is well-founded.*

Proof. To prove the above statement from Σ^1_2-AC_0, we instead prove the inconsistency Σ^1_2-AC_0 with the following statement:

$$\theta \equiv \exists^1 \langle T_n \mid n \in \mathbb{N} \rangle \big[\Pi^1_2\text{-}\mathsf{RFN}(T_0) \wedge \forall^0 n \mathsf{Prv}^{\Sigma^1_2}_{T_n}(\Pi^1_2\text{-}\mathsf{RFN}(T_{n+1})) \wedge \forall^0 n (T_n \supseteq \Sigma^1_2\text{-}\mathsf{AC}_0) \big].$$

To prove this, we claim that Σ^1_2-$\mathsf{AC}_0 + \theta$ proves its own consistency. Reasoning over Σ^1_2-$\mathsf{AC}_0 + \theta$, let $\langle T_n \mid n \in \mathbb{N} \rangle$ be a sequence witnessing θ. Then consider the following statement:

$$\tau_0 \equiv \exists^1 \langle S_n \mid n \in \mathbb{N} \rangle \big[S_0 = T_1 \wedge \forall^0 n \mathsf{Prv}^{\Sigma^1_2}_{S_n}(\Pi^1_2\text{-}\mathsf{RFN}(S_{n+1})) \wedge \forall^0 n (S_n \supseteq \Sigma^1_1\text{-}\mathsf{AC}_0) \big].$$

By Σ^1_2-AC_0, τ_0 is equivalent to a Σ^1_2-statement τ. Furthermore, τ is true since $\langle T_{n+1} \mid n \in \mathbb{N} \rangle$ witnesses τ_0. From Σ^1_2-$\mathsf{RFN}(T_0)$ and that τ is true Σ^1_2, we get

$$\mathsf{Con}(T_0 + \tau).$$

By $T_0 \supseteq \Sigma^1_2$-AC_0 and $\mathrm{Pr}_{T_0}(\Sigma^1_2\text{-}\mathsf{RFN}(T_1))$, we have

$$\mathsf{Con}(\Sigma^1_2\text{-}\mathsf{AC}_0 + \Sigma^1_2\text{-}\mathsf{RFN}(T_1) + \tau).$$

However, we can see that Σ^1_2-$\mathsf{RFN}(T_1) + \tau$ implies θ over Σ^1_1-AC_0, so we have $\mathsf{Con}(\Sigma^1_2\text{-}\mathsf{AC}_0 + \theta)$. □

Pakhomov and Walsh proved [70, §5.4] that the rank given by $\prec^{\Sigma^1_1}_{\Pi^1_1}$ (what they called the *robust rank*) coincides with the $\prec_{\Pi^1_1}$-rank for recursive extensions of ACA_0, so we may consider $\prec^{\Sigma^1_1}_{\Pi^1_1}$. In part of this reason, we consider $\prec^{\check{\Gamma}}_{\Gamma}$-rank instead of $\prec_\Gamma$-rank as a generalization of the Π^1_1-reflection rank.

We will prove in Section 1.6 that $\prec^{\Sigma^1_2}_{\Pi^1_2}$ is well-founded for Π^1_2-sound r.e. *extensions of* ACA_0. Although the above proof gives a weaker result, it has the benefit that a similar argument works for the well-foundedness of $\prec^{\Sigma^1_n}_{\Pi^1_n}$ for extensions of Σ^1_n-AC_0. However, the well-foundedness of $\prec^{\Pi^1_2}_{\Sigma^1_2}$ does not follow from the proof similar to that of Proposition 1.2.5 even for theories extending Σ^1_2-AC_0: The main trouble is that θ in Proposition 1.2.5 becomes Σ^1_3 instead of Π^1_2. We will go back to this issue in Section 1.5.

The following theorem says the $\prec^{\Sigma^1_1}_{\Pi^1_1}$-rank is closely connected with a proof-theoretic ordinal:

Theorem 1.2.6 (Pakhomov-Walsh [70, Theorem 5.20]). *Let T be a Π^1_1-sound extension of ACA_0. If $\mathrm{rank}^{\Sigma^1_1}_{\Pi^1_1}(T) = \alpha$ then $|T|_{\Pi^1_1} = \varepsilon_\alpha$.*

Hence, we immediately get the following:

Theorem 1.2.7. *Let S, T be Π^1_1-sound r.e. extensions of ACA_0. Then we have*

$$|S|_{\Pi^1_1} \leq |T|_{\Pi^1_1} \iff \mathrm{rank}^{\Sigma^1_1}_{\Pi^1_1}(S) \leq \mathrm{rank}^{\Sigma^1_1}_{\Pi^1_1}(T).$$ □

1.3 Pseudodilators: A transition to growth and climax

Recursive predilators that are not dilators appear frequently when we handle Σ^1_2-formulas. Thus let us name this concept separately.

Definition 1.3.1. A predilator D is a *pseudodilator* if it is not a dilator. For a pseudodilator D, the *climax of* D is the least ordinal α such that $D(\alpha)$ is not well-founded. We denote the climax of D by $\mathsf{Clim}(D)$.

Pseudodilators can be used to represent ordinals via their climax. For example, we can find a recursive pseudodilator whose climax is ω_1^{CK}, which is impossible to represent via a recursive well-ordering:

Example 1.3.2. Let $\phi(i)$ be a universal Π^1_1-formula with a natural number variable i; That is, for each Π^1_1-sentence σ we can find a natural number n such that $\phi(n)$ and σ are equivalent. By the Kleene normal form theorem, we can find a recursive well-order α_i such that $\phi(i) \leftrightarrow \mathsf{WO}(\alpha_i)$ for every natural number i. Then we can see that the family $\langle (\alpha_i \to \omega^*) \mid i < \omega \rangle$ of dilators is also recursive, where ω^* is ω with the reversed order. Notice that if α_i is well-founded, then $(\alpha_i \to \omega^*)$ is a pseudodilator of climax α_i.

Now define $D_i = (\alpha_i \to \omega^*) \wedge C_{\alpha_i}$, where C_x is a constant predilator with value x, i.e., $C_x(\alpha) = x$ for all α. Then take D as the disjunctive sum of all D_is. Since the enumeration $\langle D_i \mid i < \omega \rangle$ is a recursive enumeration of recursive predilators, D is a recursive predilator. Observe that

1. If α_i is ill-founded, then $D_i(\gamma)$ is ill-founded regardless of γ. Thus D_i is a pseudodilator of climax 0.
2. If α_i is well-founded, then $D_i(\gamma)$ is ill-founded iff $(\alpha_i \to \omega^*)(\gamma)$ is ill-founded, which is equivalent to $\alpha_i \leq \gamma$. Hence D_i is a pseudodilator of climax α_i.

Hence by Lemma 1.1.5 again, $D(\gamma)$ is ill-founded iff $\gamma \geq \alpha_i$ for all i when α_i is well-founded. Hence D is a recursive predilator whose climax is ω_1^{CK}!

However, there is a limit to expressing ordinals in terms of the climax of a recursive pseudodilator: In fact, if D is Σ^1_2-singleton, then its climax must be less than δ^1_2:

Proposition 1.3.3. *Let D be a Σ^1_2-singleton pseudodilator. Then $\mathsf{Clim}(D) < \delta^1_2$. In addition, the supremum of all $\mathsf{Clim}(D)$ for recursive pseudodilators D is δ^1_2.*

Proof. The statement 'D is a pseudodilator' is stated as follows:

$$\exists^1 \alpha[\mathsf{WO}(\alpha) \wedge \neg \mathsf{WO}(D(\alpha))]. \tag{1.1}$$

Since D is Σ^1_2-singleton, $\neg\mathsf{WO}(D(\alpha))$ is also Σ^1_2 because it is equivalent to

$$\exists^1 f \forall^0 n[f(n) \in D(\alpha) \wedge D(\alpha) \vDash f(n) > f(n+1)].$$

Here ϕ is a Σ^1_2-formula witnessing D being a Σ^1_2-singleton, the innermost bracketed formulas are arithmetical, and $\forall n$ is a universal quantifier over the natural numbers. Hence, we can reduce the above formula to a Σ^1_2-formula. Thus (1.1) also holds over $L_{\delta^1_2}$ by Shoenfield absoluteness theorem, Proposition A.3.4, and Proposition A.3.5. This shows $\mathsf{Clim}(D) < \delta^1_2$.

To show δ^1_2 is the supremum of all $\mathsf{Clim}(D)$ for a recursive D, fix an ordinal $\alpha < \delta^1_2$ and its Σ_1-definition $\phi(x)$ over L. We follow an argument through β-preproof presented in [2, Theorem 28]: Consider the β-preproof $P = \{P(\xi) \mid \xi \in \mathrm{Ord}\}$ trying to show the conjunction of the axioms of KPi, a theory KP plus a proper class of admissible ordinals with the following:

1. $V = L$ holds.
2. $\neg\exists x \phi(x)$.

Then P gives a recursive predilator. Since $T = \mathsf{KPi} + (V = L) + \exists x \phi(x)$ has a transitive model (namely $L_{\delta_2^1}$), so P is a pseudodilator. Furthermore, $P(\xi)$ is well-founded for $\xi < \alpha$ since no model of T of height less than α. Hence $\mathsf{Clim}(P) \geq \alpha$. □

It is well-known that every ordinal below ω_1^{CK} has an isomorphic recursive well-order. Similarly, we can ask if every ordinal below δ_2^1 is a climax of a recursive pseudodilator. However, the proof of [2, Theorem 24] implies there is no recursive pseudodilator D such that $\alpha = \mathsf{Clim}(D)$ if α is *parameter-free* Σ_1^1*-reflecting* and is either admissible or a limit of admissibles. The case when α is *neither* an admissible nor a limit admissible is open:

Question 1.3.4. Can we characterize ordinals $\alpha < \delta_2^1$ such that $\alpha = \mathsf{Clim}(D)$ for some recursive pseudodilator D?

Ressayre [81, Proposition III.7] states that "$\alpha = \mathsf{Clim}(D)$ for a recursive pseudodilator D" is equivalent to what he called the β^-*-definability of* α. However, he only proves one direction of the equivalence, namely, every β^--definable ordinal is a climax of a pseudodilator.

One may ask why we do not use semidilators instead of predilators when we define pseudodilators. The reason is that non-pre-semidilators have trivial climaxes, which make them less interesting. The following result follows from the proof of Proposition B.3.5:

Proposition 1.3.5. *Suppose that D is a semidilator that is not a predilator. Then* $\mathsf{Clim}(D) < \omega \cdot 2$. □

The following notion describes how to compare the climaxes of two pseudodilators:

Definition 1.3.6. Let D_0 and D_1 be two pseudodilators. We say D_0 *grows over* D_1 if

$\mathsf{Clim}(D_0) \leq \mathsf{Clim}(D_1)$. Equivalently, D_0 grows over D_1 if the following holds:

$$\mathsf{Grow}(D_0, D_1) \equiv \forall^1 \alpha[[\mathsf{WO}(\alpha) \wedge \mathsf{WO}(D_0(\alpha))] \to \mathsf{WO}(D_1(\alpha))].$$

Note that if D_0 and D_1 are recursive, then $\mathsf{Grow}(D_0, D_1)$ is Π^1_2.

The choice of the terminology 'D_0 grows over D_1' may be awkward because it states $\mathsf{Clim}(D_0) \leq \mathsf{Clim}(D_1)$ and not the reverse. The reason of the current choice is that if $\mathsf{Clim}(D_0) \leq \mathsf{Clim}(D_1)$, then $D_0(\alpha)$ 'diverges' to the ill-founded state earlier than $D_1(\alpha)$. This means we may regard D_0 'grows' faster than D_1, so it diverges sooner.

In the case of a well-order, we can effectively construct its successor. We can ask the same for pseudodilators: That is, for a given pseudodilator D, can we effectively construct a new pseudodilator D' such that $\mathsf{Clim}(D') = \mathsf{Clim}(D) + 1$? We will use the flowerfication $\int D$ of a semidilator D defined in Definition C.4.3 to achieve this. Then we can see the following:

Lemma 1.3.7 (ACA_0)**.** *Let D be a predilator. Then $\int D$ satisfies the following: For a well-order α,*

1. *If $D(\alpha)$ is ill-founded, then so is $(\int D)(\alpha + 1)$.*
2. *If $D(\alpha \restriction i)$ is well-founded for all $i \in \mathrm{field}(\alpha)$, then so is $(\int D)(\alpha)$.*
3. *If D is recursive, then so is $\int D$.*

Proof. Let D be a predilator. Let us recall the definition of $\int D$: It comprises a term $t^{\int}$ of arity $(\mathrm{arity}\, t + 1)$. Let us fix a linear order α, then for given D-terms s, t, and elements $\vec{\xi}, \vec{\eta}, \xi', \eta'$ of α such that $\vec{\xi} < \xi'$ and $\vec{\eta} < \eta'$,

$$\textstyle\int D \vDash s^{\int}(\vec{\xi}, \xi') < t^{\int}(\vec{\eta}, \eta') \iff \text{either } \xi' < \eta' \text{ or } [\xi' = \eta' \text{ and } D(\alpha) \vDash s(\vec{\xi}) < t(\vec{\eta})].$$

On the one hand, we can see that there is an embedding from $D(\alpha)$ to $(\int D)(\alpha+1)$ given by $t(\vec{\xi}) \mapsto t^{\int}(\vec{\xi}, \alpha)$. Therefore if $D(\alpha)$ is ill-founded, then so is $\int D(\alpha+1)$. On the other hand, if $D(\alpha \restriction i)$ is well-founded for every $i \in \mathrm{field}(\alpha)$, then we can find D-terms $s_0, s_1, \cdots$ and ordinals $\vec{\xi}_0, \vec{\xi}_1, \cdots \in \alpha$, $\eta_0, \eta_1, \cdots \in \alpha$ such that $\vec{\xi}_i < \eta_i$ for every i, we have

$$\int D(\alpha) \vDash s_0^{\int}(\vec{\xi}_0, \eta_0) > s_1^{\int}(\vec{\xi}_1, \eta_1) > s_2^{\int}(\vec{\xi}_2, \eta_2) > \cdots.$$

Since α is well-founded, there must be n such that $\eta_n = \eta_{n+1} = \eta_{n+2} = \cdots$. For such n, we get

$$D(\alpha \restriction \eta_n) \vDash s_0(\vec{\xi}_n) > s_1(\vec{\xi}_{n+1}) > s_2(\vec{\xi}_{n+2}) > \cdots,$$

contradicting with that $D(\alpha \restriction \eta_n)$ is well-founded. Clearly, if D is recursive, then so is $\int D$. □

1.3.1 Proof-theoretic pseudodilator

We will define the *proof-theoretic pseudodilator* $\|T\|_{\Sigma^1_2}$ for a given theory T, which is characterized by the following property: For every well-order α,

$$[\forall^0 D \in \mathsf{RecPreDil}(S \vdash \neg\mathsf{Dil}(D) \implies \neg\mathsf{WO}(D(\alpha))) \iff \neg\mathsf{WO}(\|T\|_{\Sigma^1_2}(\alpha)] \quad (1.2)$$

where $\mathsf{RecPreDil}$ is the set of all recursive predilators. One may naively define $\|T\|_{\Sigma^1_2}$ by

$$\|T\|_{\Sigma^1_2} = \bigvee \langle D \mid D \text{ is a recursive predilator such that } T \vdash \neg\mathsf{Dil}(D) \rangle.$$

However, this definition has a fault: $\|T\|_{\Sigma^1_2}$ is not necessarily recursive even if T is since the assertion "D is a recursive predilator" is Π^0_2. We will resolve this issue by modifying our naive definition in the following way:

Definition 1.3.8. Let T be a Σ^1_2-sound extension of ACA_0. A *proof-theoretic pseudodilator* $\|T\|_{\Sigma^1_2}$ is defined as follows: Consider the enumeration of all pairs $\langle D, \pi \rangle$, where D

is a recursive predilator and π is a T-proof for $\neg\mathsf{Dil}(D)$. For a given pair $\langle D, \pi\rangle$, let D_π be a recursive predilator obtained from D by replacing all D-terms t with $\langle \pi, t\rangle$. Then define

$$\|T\|_{\Sigma^1_2} = \bigvee\langle D_\pi \mid D \in \mathsf{RecPreDil} \wedge \pi \text{ is a proof for } \neg\mathsf{Dil}(D) \text{ from } T\rangle.$$

If T is recursive, then the statements 'π is a T-proof of a given statement' and the map $\langle D.\pi\rangle \mapsto D_\pi$ are also recursive. Hence $\|T\|_{\Sigma^1_2}$ is also recursive. Like proof-theoretic dilators, we can still define $\|T\|_{\Sigma^1_2}$ for Σ^1_2-unsound theories; We just do not know if $\|T\|_{\Sigma^1_2}$ is a pseudodilator or not in this case.

We need to check our proof-theoretic pseudodilator meets (1.2):

Lemma 1.3.9 (ACA_0). *Let D be a predilator and T be a Σ^1_2-sound extension of ACA_0. Then for a well-order α, the following two are equivalent:*

1. *For every recursive predilator D such that $T \vdash \neg\mathsf{Dil}(D)$, $D(\alpha)$ is ill-founded, and*
2. *$\|T\|_{\Sigma^1_2}(\alpha)$ is ill-founded.*

Proof. Fix a well-order α. Suppose that for every T-provably recursive pseudodilator D, $D(\alpha)$ is ill-founded, and let π be a T-proof for D being a pseudodilator. Then D and D_π are isomorphic predilators, so $D_\pi(\alpha)$ is also ill-founded. Since D and π are arbitrarily, we have that $\|T\|_{\Sigma^1_2}(\alpha)$ is also ill-founded. The remaining direction can be shown similarly. □

Proof-theoretic pseudodilator of T is associated with the Σ^1_2-reflection of T:

Proposition 1.3.10 (Σ^1_2-AC_0). *Let T be a Σ^1_2-sound extension of ACA_0. Then Σ^1_2-$\mathsf{RFN}(T)$ iff $\neg\mathsf{Dil}(\|T\|_{\Sigma^1_2})$.*

Proof. For one direction, suppose that Σ^1_2-$\mathsf{RFN}(T)$ holds. Then if D is a recursive predilator such that $T \vdash \neg\mathsf{Dil}(D)$, then D is a pseudodilator. That is, if $T \vdash \neg\mathsf{Dil}(D)$, then we can find a well-order α such that $D(\alpha)$ is ill-founded. Then by Σ^1_2-AC_0 applied to the sentence

$$\forall^0 D \in \mathcal{X} \exists^1 \alpha [\mathsf{WO}(\alpha) \wedge \neg\mathsf{WO}(D(\alpha))]$$

for the set $\mathcal{X} = \{D \in \mathsf{RecPreDil} \mid T \vdash \neg\mathsf{Dil}(D)\}$, which can be viewed as a subset of $\mathbb{N}$, we have a set A such that

$$\forall^0 D \in \mathcal{X} [\mathsf{WO}((A)_D) \wedge \neg\mathsf{WO}(D((A)_D))].$$

Now take $\gamma = \sum_{D\in\mathcal{X}} (A)_D$. Then γ is a well-order embedding all $(A)_D$ for $D \in \mathcal{X}$, so $D(\gamma)$ is ill-founded for every recursive D such that $T \vdash \neg\mathsf{Dil}(D)$. Hence by Lemma 1.3.9, $\|T\|_{\Sigma^1_2}$ is also a pseudodilator.

For the remaining direction, suppose that $\|T\|_{\Sigma^1_2}$ is a pseudodilator. By the completeness of dilators, every Σ^1_2-sentence is equivalent to $\neg\mathsf{Dil}(D)$ for some recursive predilator D, and this is a theorem of ACA_0. Thus, we can identify a Σ^1_2-sentence with $\neg\mathsf{Dil}(D)$ for some recursive predilator D.

Now assume that $T \vdash \neg\mathsf{Dil}(D)$. Since $\|T\|_{\Sigma^1_2}$ is a pseudodilator, we can find a well-order α such that $\neg\mathsf{WO}(\|T\|_{\Sigma^1_2}(\alpha))$ holds. But by Lemma 1.3.9, we have $\neg\mathsf{WO}(D(\alpha))$. Hence, D is not a dilator. □

1.3.2 Σ^1_2-proof-theoretic ordinal

Definition 1.3.11. Let T be a Σ^1_2-sound extension of ACA_0. We define $s^1_2(T)$ as follows:

$$s^1_2(T) = \sup\{\mathsf{Clim}(D) \mid D \text{ is a recursive predilator and } T \vdash \neg\mathsf{Dil}(D)\}.$$

The previous definition is due to Aguilera-Pakhomov [2], and they chose s^1_2 because it is intended to mean the *soundness ordinal*. We call $s^1_2(T)$ a proof-theoretic ordinal since its role and behaviors are closer to the Π^1_1-proof-theoretic ordinal $|T|_{\Pi^1_1}$.

We will see in the next section that the Σ^1_2-proof-theoretic ordinal has the role of proof-theoretic ordinal for Σ^1_2-sentences. The following is immediate by Lemma 1.3.9:

Lemma 1.3.12. *Let T be a Σ^1_2-sound extension of ACA_0. Then $\mathsf{Clim}(\|T\|_{\Sigma^1_2}) = s^1_2(T)$.* □

Lemma 1.3.13. *Let T be a Σ^1_2-sound extension of ACA_0. Then $s^1_2(T)$ is the strict supremum of all $\mathsf{Clim}(D)$ for T-provably recursive pseudodilators D. That is, if $T \vdash \neg\mathsf{Dil}(D)$, then $\mathsf{Clim}(D) < s^1_2(T)$.*

Proof. Let D be a recursive pseudodilator such that $T \vdash \neg\mathsf{Dil}(D)$. Then by Lemma 1.3.7, we can construct a new recursive predilator $\int D$. First, we need to show $T \vdash \neg\mathsf{Dil}(\int D)$. Reasoning over T, let α be a well-order such that $\neg\mathsf{WO}(D(\alpha))$. Then by Lemma 1.3.7, $\neg\mathsf{WO}(\int D(\alpha+1))$. Hence $\int D$ is also a pseudodilator. Furthermore, Lemma 1.3.7 ensures $\mathsf{Clim}(\int D) = \mathsf{Clim}(D) + 1$. Hence we get $\mathsf{Clim}(D) < \mathsf{Clim}(\int D) < s^1_2(T)$. □

It is known that the supermum of all $|T|_{\Pi^1_1}$ for Π^1_1-sound r.e. extensions T of ACA_0 is ω^{CK}_1. What about the supremum of $s^1_2(T)$ for Σ^1_2-sound r.e. extensions T of ACA_0? The following theorem provides an answer:

Theorem 1.3.14 (Aguilera-Pakhomov [2, Theorem 33]). *δ^1_2 is the supremum of all $s^1_2(T)$ for Σ^1_2-sound r.e. extension of ACA_0.* □

The following result computes $s^1_2(T)$ for some theories T:

Theorem 1.3.15 (Aguilera-Pakhomov [2, Theorem 36]).

1. $s^1_2(\mathsf{ACA}_0) = s^1_2(\mathsf{KP}) = \omega_1^{\mathsf{CK}}$.

2. $s^1_2(\Pi^1_1\text{-}\mathsf{CA}_0) = \omega_\omega^{\mathsf{CK}}$.

3. $s^1_2(\Pi^1_2\text{-}\mathsf{CA}_0) = \sup_{n<\omega} \sigma_n$, *where* σ_n *is the least ordinal* α *such that there are* $\alpha_0 < \cdots < \alpha_{n-1} = \alpha$ *satisfying* $L_{\alpha_0} \prec_{\Sigma_1} \cdots \prec_{\Sigma_1} L_{\alpha_{n-1}}$.[2] □

Proposition 1.3.16. *Let* BI *stand for Bar Induction schema (or Transfinite Induction for all second-order formulas). Then we have the following:*

1. $s^1_2(\mathsf{ATR}_0 + \mathsf{BI}) = \omega_1^{\mathsf{CK}}$.

2. $s^1_2(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI}) = \omega_\omega^{\mathsf{CK}}$.

Proof. The right-to-left inequality follows from Theorem 1.3.15. To show $s^1_2(\mathsf{ATR}_0 + \mathsf{BI}) \leq \omega_1^{\mathsf{CK}}$, let us consider a β-model M of height ω_1^{CK} (For example, one constructed from the hyperjump of $\emptyset$.) By Theorem VII.2.7. and Lemma VII.2.15 of [89], M satisfies $\mathsf{ATR}_0 + \mathsf{BI}$. If D is a recursive pseudodilator such that $\mathsf{ATR}_0 + \mathsf{BI} \vdash \neg\mathsf{Dil}(D)$, then we get

$$M \vDash \neg\mathsf{Dil}(D).$$

Hence the climax of D must be less than ω_1^{CK}. To prove $s^1_2(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI}) \leq \omega_\omega^{\mathsf{CK}}$, repeat the same argument with the minimal β-model of $\Pi^1_1\text{-}\mathsf{CA}_0$ instead. □

It is open whether $s^1_2(T)$ must be either an admissible ordinal or a limit of admissible ordinals if T is a sound r.e. extension of ACA_0.

[2] The Σ^1_2-proof-theoretic ordinal for $\Pi^1_2\text{-}\mathsf{CA}_0$ presented in [2] is incorrect, and the value presented in this paper is due to Aguilera (Private communication).

1.4 s^1_2, Σ^1_2-consequences, and Σ^1_2-reflection

1.4.1 The first equivalence

The main goal of this section is to provide the Σ^1_2-version of the characterization of Π^1_1-proof theory. Let us state the Σ^1_2 version of the first equivalence stated in [103]:

Proposition 1.4.1. *For Σ^1_2-sound extensions S, T of ACA_0, we have*

$$S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T \iff s^1_2(S) \leq s^1_2(T).$$

Instead of proving the above claim directly, let us prove the following lemma that will immediately imply Proposition 1.4.1. We separate this lemma from the proof of Proposition 1.4.1 for later purposes.

Lemma 1.4.2. *Let T be a Σ^1_2-sound extension of ACA_0, and let D be a recursive predilator. Then*

$$T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D) \iff \mathsf{Clim}(D) < s^1_2(T).$$

Proof. For one direction, assume that $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$. By Π^1_2-completeness of dilators, we have a recursive dilator E such that $T + \mathsf{Dil}(E) \vdash \neg\mathsf{Dil}(D)$. Hence we get

$$T \vdash \neg(\mathsf{Dil}(E) \wedge \mathsf{Dil}(D)).$$

Now let $\hat{D} = D \wedge E$ given by Lemma 1.1.3. Then $\hat{D}$ is a recursive predilator and $T \vdash \neg\mathsf{Dil}(\hat{D})$, so $\mathsf{Clim}(\hat{D}) < s^1_2(T)$. Since E is a dilator, we have for each ordinal α,

$$\mathsf{WO}(\hat{D}(\alpha)) \iff \mathsf{WO}(D(\alpha)).$$

This implies $\mathsf{Clim}(\hat{D}) = \mathsf{Clim}(D)$, which gives the desired inequality.

Let us show the remaining direction. Suppose that $\mathsf{Clim}(D) < s^1_2(T)$. Then we can find a recursive pseudodilator E such that $T \vdash \neg\mathsf{Dil}(E)$ and $\mathsf{Clim}(D) \leq \mathsf{Clim}(E) < s^1_2(T)$. Since $\mathsf{Clim}(D) \leq \mathsf{Clim}(E)$, a Π^1_2-statement

$$\mathsf{Grow}(D, E) \equiv \forall^1\alpha[[\mathsf{WO}(\alpha) \wedge \mathsf{WO}(D(\alpha))] \to \mathsf{WO}(E(\alpha))]$$

is true. Since T proves $\neg\mathsf{Dil}(E)$,

$$T \vdash \exists^1\alpha(\mathsf{WO}(\alpha) \wedge \neg\mathsf{WO}(E(\alpha))).$$

Now let us reason in $T + \mathsf{Grow}(D, E)$: Suppose that α is a well-order such that $\neg\mathsf{WO}(E(\alpha))$. Then by the contrapositive of $\mathsf{Grow}(D, E)$, $D(\alpha)$ is also ill-founded. Hence, D is also not a dilator. That is, we get

$$T + \mathsf{Grow}(D, E) \vdash \neg\mathsf{Dil}(D).$$

Since $\mathsf{Grow}(D, E)$ is a true Π^1_2-statement, we have $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$. □

Proof of Proposition 1.4.1. For one direction, assume that $S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T$. If D is a recursive predilator such that $S \vdash \neg\mathsf{Dil}(D)$, then $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$ by the assumption. Thus, by Lemma 1.4.2, we have $\mathsf{Clim}(D) < s^1_2(T)$. Hence $s^1_2(S) \leq s^1_2(T)$.

For the other direction, assume that $s^1_2(S) \leq s^1_2(T)$ and $S \vdash^{\Pi^1_2} \phi$ for a Σ^1_2-statement ϕ. By Π^1_2-completeness of dilators, we may assume that ϕ takes the form $\neg\mathsf{Dil}(D)$ for some recursive pseudodilator D. Then by Lemma 1.4.2, we have $\mathsf{Clim}(D) < s^1_2(S) \leq s^1_2(T)$. Hence, by Lemma 1.4.2 again, we get $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$. □

Then the following is immediate, which is a Σ^1_2-version of Kreisel's theorem:

Corollary 1.4.3. *Let T be an Σ^1_2-sound extension of ACA_0. If σ is any true Π^1_2 sentence, then $s^1_2(T) = s^1_2(T + \sigma)$.* □

1.4.2 The second equivalence

Walsh [103] also provided a connection between proof-theoretic ordinal and Π^1_1-reflection. The following shows the connection between s^1_2 and Σ^1_2-reflection:

Proposition 1.4.4. *Let S and T be Σ^1_2-sound extensions of ACA_0 such that S is Π^1_2-definable and T is Σ^1_2-definable. Then we have*

$$s^1_2(S) \leq s^1_2(T) \implies \Sigma^1_2\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S). \tag{1.3}$$

Proof. Suppose that $s^1_2(S) \leq s^1_2(T)$, which implies $S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T$. That is, the following sentence becomes true:

$$\theta :\equiv \forall^0\phi \in \Sigma^1_2[\mathsf{Prv}_S(\phi) \to \mathsf{Prv}^{\Pi^1_2}_T(\phi)].$$

Let us first prove the following lemma:

Lemma. $\mathsf{ACA}_0 + \theta \vdash \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S)$.

Proof. Let us reason over $\mathsf{ACA}_0 + \theta$, and suppose that we have $\Sigma^1_2\text{-}\mathsf{RFN}(T)$. If $\mathsf{Prv}_S(\phi)$ holds for a Σ^1_2-sentence ϕ, then by θ, we have $\mathsf{Prv}^{\Pi^1_2}_T(\phi)$. That is, we can find a true Π^1_2-sentence ψ such that $\mathsf{Prv}_T(\psi \to \phi)$. Hence by $\Sigma^1_2\text{-}\mathsf{RFN}(T)$ and since $\psi \to \phi$ is Σ^1_2, $\psi \to \phi$ is true. Since ψ is true, so is ϕ. In sum, we have

$$\forall^0\phi \in \Sigma^1_2[\mathsf{Prv}_S(\phi) \to \mathsf{True}_{\Sigma^1_2}(\phi)]. \qquad \square$$

Now, let us compute the complexity of θ. One can see that the following holds:

1. If a theory T is Γ-definable and $\Gamma \supseteq \Sigma^0_1$, then $\mathsf{Prv}_T(\phi)$ has the complexity of the form $\exists^0\forall^0\Gamma$.
2. Under the same assumption, $\mathsf{Prv}^{\Pi^1_2}_T$ has the complexity of the form $\exists^0\forall^0(\Pi^1_2 \wedge \Gamma)$.

Hence θ has the following complexity:

$$\forall^0[(\exists^0\forall^0\Pi^1_2) \vee (\exists^0\forall^0\Pi^1_2)],$$

which reduces to $\forall^0\exists^0\Pi^1_2$. Furthermore, by Lemma A.6.5, for every Π^1_2-formula $\phi(m)$ we can find a Π^1_2-sentence τ such that

- $\mathsf{ACA}_0 \vdash \exists^0 m\phi(m) \to \tau$, and
- Σ^1_2-$\mathsf{AC}_0 \vdash \tau \to \exists^0 m\phi(m)$.

Hence by using Σ^1_2-AC_0, we can replace θ to an equivalent formula θ' of the form $\forall^0\Pi^1_2$. Then we have

$$\Sigma^1_2\text{-}\mathsf{AC}_0 + \theta' \vdash \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S). \qquad \square$$

Now let us prove the remaining direction:

Proposition 1.4.5. *Suppose that S, T be Σ^1_2-sound extensions of ACA_0 such that both S and T are arithmetically definable. If*

$$\mathsf{ACA}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S), \tag{1.4}$$

then we have $s^1_2(S) \leq s^1_2(T)$. The same holds if we replace all occurrences of ACA_0 with Σ^1_2-AC_0.

Proof. Suppose the contrary that (1.4) holds but $s^1_2(T) < s^1_2(S)$ holds. Then we have a true Π^1_2-sentence τ such that the following holds:

$$\mathsf{ACA}_0 + \tau \vdash \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S), \tag{1.5}$$

We can find a recursive pseudodilator $\hat{D}$ such that $S \vdash \neg\mathsf{Dil}(\hat{D})$ and $s^1_2(T) \leq \mathsf{Clim}(\hat{D}) < s^1_2(S)$. In particular, the following Π^1_2 sentence is true:

$$\theta \equiv \forall^0 D \in \mathsf{RecPreDil}[\mathsf{Prv}_T(\neg\mathsf{Dil}(D)) \to \mathsf{Grow}(D, \hat{D})].$$

Here $\mathsf{RecPreDil}$ is the set of all (codes for) recursive predilators. Since S extends ACA_0, we have

$$S + \theta \vdash \forall^0 D \in \mathsf{RecPreDil}[\mathsf{Prv}_T(\neg\mathsf{Dil}(D)) \to \neg\mathsf{Dil}(D)].$$

Thus, by the Π^1_2-completeness of recursive dilators, we have

$$S + \theta \vdash \Sigma^1_2\text{-}\mathsf{RFN}(T).$$

By applying (1.5), we can derive $S + \theta + \tau \vdash \Sigma^1_2\text{-}\mathsf{RFN}(S)$. Hence, by Lemma A.6.2, we have

$$S + \theta + \tau \vdash \Sigma^1_2\text{-}\mathsf{RFN}(S + \theta + \tau).$$

However, no Σ^1_2-sound arithmetically definable theory can prove its own Σ^1_2-reflection, a contradiction. □

Hence we get the following:

Corollary 1.4.6. *Let S and T be Σ^1_2-sound arithmetically definable extensions of Σ^1_2-AC_0. Then we have*

$$s^1_2(S) \leq s^1_2(T) \iff \Sigma^1_2\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S). \qquad \square$$

1.4.3 The restriction in the second equivalence

The restriction for theories being extensions of Σ^1_2-AC_0 might be annoying at first glance, which is inherited from Proposition 1.4.4. The reader may want to fortify the conclusion (1.3) of Proposition 1.4.4 by

$$s^1_2(S) \leq s^1_2(T) \implies \mathsf{ACA}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S). \tag{1.6}$$

However, the following example shows (1.6) cannot be achieved even if we replace ACA_0-provability with Π^1_1-CA_0-provability for sound recursive extensions of ACA_0:

Example 1.4.7. By Theorem 1.3.15 and Proposition 1.3.16, we have $s^1_2(\Pi^1_1\text{-}\mathsf{CA}_0) = s^1_2(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI})$. Now suppose that we have

$$\Pi^1_1\text{-}\mathsf{CA}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0) \to \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI}).$$

Let θ be a true Π^1_2-sentence satisfying

$$\Pi^1_1\text{-}\mathsf{CA}_0 + \theta + \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0) \vdash \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI}).$$

Then, by Lemma A.6.2, we have

$$\Pi^1_1\text{-}\mathsf{CA}_0 + \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0) + \theta \vdash \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI} + \theta). \tag{1.7}$$

[89, Theorem IX.4.10] says $\Pi^1_1\text{-}\mathsf{CA}_0$ with Σ^1_2-Induction implies $\Pi^1_3\text{-}\mathsf{RFN}(\Sigma^1_2\text{-}\mathsf{AC}_0)$. Hence, by Lemma A.6.2 and the implication $\Sigma^1_2\text{-}\mathsf{AC}_0 \to \Pi^1_1\text{-}\mathsf{CA}_0$, we get

$$\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI} \vdash \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0). \tag{1.8}$$

Combining (1.7) with (1.8), we have

$$\Pi^1_1\text{-}\mathsf{CA}_0 + \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0) + \theta \vdash \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0 + \Sigma^1_2\text{-}\mathsf{RFN}(\Pi^1_1\text{-}\mathsf{CA}_0) + \theta).$$

However, no theory can prove its own Σ^1_2-reflection, a contradiction. In sum, if we let $S = \Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{BI}$ and $T = \Pi^1_1\text{-}\mathsf{CA}_0$, then

$$s^1_2(S) \leq s^1_2(T) \qquad \text{but} \qquad \Pi^1_1\text{-}\mathsf{CA}_0 \nvdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S).$$

Remark 1.4.8. [89, Theorem IX.4.9] states that $\Sigma^1_2\text{-}\mathsf{AC}_0$ is Π^1_3-conservative over $\Pi^1_1\text{-}\mathsf{CA}_0$. Hence, the reader may think

$$\Pi^1_1\text{-}\mathsf{CA}_0 \nvdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S) \implies \Sigma^1_2\text{-}\mathsf{AC}_0 \nvdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S),$$

for recursive S and T, because for a true Π^1_2-sentence τ, the sentence

$$\tau \to [\Sigma^1_2\text{-}\mathsf{RFN}(T) \to \Sigma^1_2\text{-}\mathsf{RFN}(S)] \tag{1.9}$$

looks like a boolean combination of Π^1_2-sentences, which is Π^1_3. However, the complexity of Σ^1_2-$\mathsf{RFN}(S)$ is Π^1_1-CA_0-provably $\forall^0\Sigma^1_2$, at best. Therefore, (1.9) is Π^1_1-CA_0-provably $\Sigma^1_2 \vee \exists^0\Pi^1_2 \vee \forall^0\Sigma^1_2$ at best, so we do not know if (1.9) is Π^1_1-CA_0-provably equivalent to a Π^1_3-sentence.

1.4.4 An equivalent condition for strict comparison

We finish this section by proving the following:

Theorem 1.4.9. *Suppose that S and T are Σ^1_2-sound extensions of ACA_0 such that S is arithmetically definable and $T \supseteq \Sigma^1_2$-AC_0. Then we have*

$$s^1_2(S) < s^1_2(T) \iff T \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(S).$$

Proof. For one direction, suppose that $s^1_2(S) < s^1_2(T)$. We can find a recursive pseudo-dilator D such that $s^1_2(S) \leq \mathsf{Clim}(D) < s^1_2(T)$. By Lemma 1.4.2, there is a true Π^1_2-sentence τ such that $T + \tau \vdash \neg\mathsf{Dil}(D)$. We know that $s^1_2(S) = \mathsf{Clim}(\|S\|_{\Sigma^1_2}) < \mathsf{Clim}(D)$, so $\mathsf{Grow}(\|S\|_{\Sigma^1_2}, D)$ is a true Π^1_2-sentence. We have

$$T + \tau + \mathsf{Grow}(\|S\|_{\Sigma^1_2}, D) \vdash \neg\mathsf{Dil}(\|S\|_{\Sigma^1_2}),$$

and Proposition 1.3.10 with $T \supseteq \Sigma^1_2$-AC_0 implies $T \vdash^{\Pi^1_2} \Sigma^1_2$-$\mathsf{RFN}(S)$.

Conversely, suppose that $T + \tau \vdash \Sigma^1_2$-$\mathsf{RFN}(S)$ for some true Π^1_2-sentence τ. This implies $T + \tau \vdash \neg\mathsf{Dil}(\|S\|_{\Sigma^1_2})$ by Proposition 1.3.10. $s^1_2(T)$ is a strict supremum, so we have

$$s^1_2(S) = \mathsf{Clim}(\|S\|_{\Sigma^1_2}) < s^1_2(T + \tau) = s^1_2(T). \qquad \square$$

Note that the forward implication of Theorem 1.4.9 holds even when we do not require $T \supseteq \Sigma^1_2$-AC_0. See Corollary 1.5.5 for its proof.

1.5 Σ^1_2-Reflection rank

The well-foundedness of $\prec^{\Pi^1_2}_{\Sigma^1_2}$ does not follow from the proof similar to that of Proposition 1.2.5 even for theories extending Σ^1_2-AC_0. The main trouble is that θ in Proposition 1.2.5 becomes Σ^1_3 instead of Π^1_2. To establish the well-foundedness of $\prec^{\Pi^1_2}_{\Sigma^1_2}$, we follow an argument in [102]:

Proposition 1.5.1. *$\prec^{\Pi^1_2}_{\Sigma^1_2}$ is well-founded for Σ^1_2-sound r.e. extensions of Σ^1_2-AC_0.*

Proof. To attack Proposition 1.5.1, it is natural to observe the connection between $\prec^{\Pi^1_2}_{\Sigma^1_2}$ and s^1_2. Suppose not, and let $\langle T_n \mid n < \omega \rangle$ be a $\prec^{\Pi^1_2}_{\Sigma^1_2}$-decreasing sequence of Σ^1_2-sound r.e. extensions of Σ^1_2-AC_0. Then we have, for each n,

$$T_n \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(T_{n+1}).$$

Since $T_n \supseteq \Sigma^1_2$-AC_0, we have $T_n \vdash \neg\mathsf{Dil}(|T_{n+1}|_{\Sigma^1_2})$ for each n by Proposition 1.3.10. Hence by Lemma 1.4.2, we have

$$s^1_2(T_{n+1}) = \mathsf{Clim}(|T_{n+1}|_{\Sigma^1_2}) < s^1_2(T_n)$$

for each n, a contradiction. □

It is natural to ask whether $\prec^{\Pi^1_2}_{\Sigma^1_2}$-rank is related to the Σ^1_2-proof-theoretic ordinal. The subsequent results say it is:

Proposition 1.5.2. *Let S, T be Σ^1_2-sound r.e. extensions of Σ^1_2-AC_0. If $s^1_2(S) \leq s^1_2(T)$, then $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S) \leq \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(T)$.*

Proof. Suppose that $\alpha < \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S)$. We claim the following lemma:

Lemma. *There is a Σ^1_2-sound r.e. extension U of Σ^1_2-AC_0 such that $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(U) = \alpha$ and $S \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(U)$.*

Proof. Let $\mathbb{P}$ be the set of all Σ^1_2-sound r.e. extension of Σ^1_2-AC_0 such that $S \vdash \Sigma^1_2$-$\mathsf{RFN}(U)$, and consider the structure $(\mathbb{P}, \prec^{\Pi^1_2}_{\Sigma^1_2})$. By Proposition 1.5.1, $\prec^{\Pi^1_2}_{\Sigma^1_2}$ over $\mathbb{P}$ is well-founded, so it defines a rank function ρ over $\mathbb{P}$.

We claim that $\rho(U) = \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(U)$ for all $U \in \mathbb{P}$ by $\prec^{\Pi^1_2}_{\Sigma^1_2}$-induction on $\mathbb{P}$: Suppose that $\rho(V) = \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(V)$ holds for all $V \in \mathbb{P}$ such that $V \prec^{\Pi^1_2}_{\Sigma^1_2} U$. Then

$$\rho(U) = \sup\{\rho(V) + 1 \mid V \prec^{\Pi^1_2}_{\Sigma^1_2} U\} = \sup\{\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(V) + 1 \mid V \prec^{\Pi^1_2}_{\Sigma^1_2} U\} = \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(U),$$

proving the claim. By a similar argument, we can see that the supremum of all $\rho(U)$ for $U \in \mathbb{P}$ is $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S)$, which proves the desired claim. □

Going back to the proof of the main proposition, let U be a theory such that $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(U) = \alpha$ and $S \vdash \Sigma^1_2$-$\mathsf{RFN}(U)$. Since $s^1_2(S) \leq s^1_2(T)$, which implies $S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T$, we get $T \vdash^{\Pi^1_2} \Sigma^1_2$-$\mathsf{RFN}(U)$. That is, $U \prec^{\Pi^1_2}_{\Sigma^1_2} T$, which implies $\alpha = \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(U) < \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(T)$. □

Proposition 1.5.3. *Let S, T be Σ^1_2-sound r.e. extensions of Σ^1_2-AC_0. If $s^1_2(S) < s^1_2(T)$, then $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S) < \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(T)$.*

Proof. Let $\hat{D}$ be a recursive pseudodilator such that $s^1_2(S) \leq \mathsf{Clim}(\hat{D}) < s^1_2(T)$. Then by Lemma 1.4.2, we have $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(\hat{D})$.

Now let D be a recursive pseudodilator such that $S \vdash \neg\mathsf{Dil}(D)$. Then we have $\mathsf{Clim}(D) \leq \mathsf{Clim}(\hat{D})$, so the following sentence is true:

$$\theta \equiv \forall^0 D \in \mathsf{RecPreDil}[\mathsf{Prv}_S(\neg\mathsf{Dil}(D)) \to \mathsf{Grow}(D, \hat{D})].$$

Clearly θ is Π^1_2. Furthermore, we get

$$T + \theta \vdash \forall^0 D \in \mathsf{RecPreDil}[\mathsf{Prv}_S(\neg\mathsf{Dil}(D)) \to \neg\mathsf{Dil}(D)].$$

Hence by the Π^1_2-completeness of dilators, we get

$$T + \theta \vdash \Sigma^1_2\text{-}\mathsf{RFN}(S).$$

Since θ is true, $T \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(S)$, which is the definition of $S \prec^{\Pi^1_2}_{\Sigma^1_2} T$. Hence $\mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S) < \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(T)$. □

Combining Proposition 1.5.2 and Proposition 1.5.3, we get the following:

Corollary 1.5.4. *Let S, T be Σ^1_2-sound r.e. extensions of Σ^1_2-AC_0. Then we have*

$$s^1_2(S) \leq s^1_2(T) \iff \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(S) \leq \mathrm{rank}^{\Pi^1_2}_{\Sigma^1_2}(T).$$ □

Note that the only reason we assumed S and T be extensions of Σ^1_2-AC_0 is to impose the $\prec^{\Pi^1_2}_{\Sigma^1_2}$-rank on them. From the proof of Proposition 1.5.3, we can extract the following result:

Corollary 1.5.5. *Let S, T be Σ^1_2-sound r.e. extensions of ACA_0. If $s^1_2(S) < s^1_2(T)$, then $T \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathsf{RFN}(S)$.* □

1.6 A glimpse to Case Π^1_2

Corollary 1.4.3 shows $s^1_2(T)$ is stable under true Π^1_2-statements, and we may ask whether a similar property holds at the Π^1_2-level. The following proposition, which shows adding a true Σ^1_2-true sentence does not change the eventual behavior of the proof-theoretic dilator, gives some hint:

Proposition 1.6.1. *Let T be a Π^1_2-sound theory and let F be a recursive pseudodilator. If α is a well-order such that $F(\alpha)$ is ill-founded, then for every $\beta \geq \alpha$ we have*

$$|T|_{\Pi^1_2}(\beta) = |T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}(\beta).$$

Proof. Clearly, $T \subseteq T + \neg\mathsf{Dil}(F)$, so $|T|_{\Pi^1_2}$ is embeddable into $|T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}$. This shows $|T|_{\Pi^1_2}(\beta) \leq |T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}(\beta)$ for every β.

To show the reverse direction, let E be a recursive dilator such that $T + \neg\mathsf{Dil}(F) \vdash \mathsf{Dil}(E)$. Then we have

$$T \vdash \mathsf{Dil}(F) \vee \mathsf{Dil}(E).$$

Hence $T \vdash \mathsf{Dil}(F \vee E)$. By Lemma 1.1.5, $F \vee E$ is a recursive dilator. Thus $F \vee E$ is embedded into $|T|_{\Pi^1_2}$. Also, $\beta \geq \alpha$ implies $F(\beta)$ is ill-founded, so there is an embedding $E(\beta) \leq (F \vee E)(\beta)$. Therefore

$$E(\beta) \leq (F \vee E)(\beta) \leq |T|_{\Pi^1_2}(\beta). \qquad \square$$

In particular, if F is a recursive pseudodilator, then there is $\alpha < \delta^1_2$ such that $F(\alpha)$ is ill-founded. Thus, we get the following:

Corollary 1.6.2. *Let T be a Π^1_2-sound extension of ACA_0 and let F be a recursive pseudodilator. Then*

$$|T|_{\Pi^1_2}(\delta^1_2) = |T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}(\delta^1_2). \qquad \square$$

By using the eventual value of the proof-theoretic dilator $|T|_{\Pi^1_2}(\delta^1_2)$, we can show that $\prec^{\Sigma^1_2}_{\Pi^1_2}$ is well-founded:

Lemma 1.6.3. *Suppose that S, T be Π^1_2-sound r.e. extensions of ACA_0. If $T \vdash^{\Sigma^1_2} \Pi^1_2\text{-}\mathsf{RFN}(S)$, then we can find $\alpha < \delta^1_2$ such that*

$$\forall \beta \geq \alpha[|S|_{\Pi^1_2}(\beta) < |T|_{\Pi^1_2}(\beta)].$$

In particular, we have $|S|_{\Pi^1_2}(\delta^1_2) < |T|_{\Pi^1_2}(\delta^1_2)$.

Proof. Let us prove the following claim first: If $T \vdash \Pi^1_2\text{-}\mathsf{RFN}(S)$, then we have

$$|S|_{\Pi^1_2} \cdot 2 \leq |T|_{\Pi^1_2}.$$

Since ACA_0 proves Π^1_2-$\mathsf{RFN}(S)$ is equivalent to $\mathsf{Dil}(|S|_{\Pi^1_2})$, $|S|_{\Pi^1_2}$ appears in the enumeration of all T-provable recursive dilators. Furthermore, $T \vdash \Pi^1_2$-$\mathsf{RFN}(S)$ implies every Π^1_2-consequence of S is a theorem of T. This means every S-provable recursive dilator is also a T-recursive dilator. However, no S-provable recursive dilator is equal to $|S|_{\Pi^1_2}$, or S proves its own Π^1_2-reflection that is impossible. Since we defined $|T|_{\Pi^1_2}$ by the sum of all T-provable recursive dilators, we have the desired inequality.

Now let us prove the lemma we stated. If F is a recursive pseudodilator and if $T + \neg\mathsf{Dil}(F)$ proves Π^1_2-$\mathsf{RFN}(S)$, then we have

$$|S|_{\Pi^1_2} \cdot 2 \leq |T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}.$$

If α is a well-order such that $F(\alpha)$ is ill-founded, then by Proposition 1.6.1, we have

$$|S|_{\Pi^1_2}(\beta) \cdot 2 \leq |T + \neg\mathsf{Dil}(F)|_{\Pi^1_2}(\beta) = |T|_{\Pi^1_2}(\beta)$$

for all $\beta \geq \alpha$. Furthermore, since non-zero constant dilators of a natural number value are S-provable dilators, we have $|S|_{\Pi^1_2}(\beta) \geq \omega$ for all β. This shows

$$|S|_{\Pi^1_2}(\beta) < |S|_{\Pi^1_2}(\beta) \cdot 2,$$

proving the desired inequality. □

Now the following is immediate by the definition of $\prec^{\Sigma^1_2}_{\Pi^1_2}$ and the previous lemma:

Corollary 1.6.4. *$\prec^{\Sigma^1_2}_{\Pi^1_2}$ is well-founded for Π^1_2-sound r.e. theories extending ACA_0.* □

We proved in previous sections that $s^1_2(T)$ ranks Σ^1_2-consequences of Σ^1_2-sound theories (modulo Π^1_2-oracle) in a linear way, and it gauges the strength of Σ^1_2-reflection of T if T is an extension of Σ^1_2-AC_0. We may ask whether we can get an ordinal characteristic of T that captures its Π^1_2-consequences modulo Σ^1_2-oracle. However, the following result by Aguilera and Pakhomov says that obtaining such an ordinal characteristic for Π^1_2-consequences is impossible:

Theorem 1.6.5 (Aguilera-Pakhomov [5, §3.1]). *There exists a recursive $T \supseteq \mathsf{ACA}_0$ and Π^1_2-sentences ϕ_0, ϕ_1 such that*

1. *Both of $T + \phi_0$ and $T + \phi_1$ are Π^1_2-sound.*
2. *Neither $T \vdash^{\Sigma^1_2} \phi_0 \to \phi_1$ nor $T \vdash^{\Sigma^1_2} \phi_1 \to \phi_0$ holds.* □

Corollary 1.6.6. *There is no ordinal assignment $o(T)$ for Π^1_2-sound r.e. extensions T of ACA_0 such that*

$$o(S) \leq o(T) \iff S \subseteq^{\Sigma^1_2}_{\Pi^1_2} T.$$

Proof. Let T, ϕ_0, ϕ_1 be a theory and Π^1_2-sentences providede by Theorem 1.6.5. Then neither $T + \phi_0 \vdash^{\Sigma^1_2} \phi_1$ nor $T + \phi_1 \vdash^{\Sigma^1_2} \phi_0$ hold, so $\subseteq^{\Sigma^1_2}_{\Pi^1_2}$ cannot be linear. □

However, the above result does not rule out an ordinal characteristic capturing the $\prec^{\Sigma^1_2}_{\Pi^1_2}$-rank of a theory. We conjecture the eventual value of a proof-theoretic dilator $|T|_{\Pi^1_2}(\delta^1_2)$ should have this role:

Question 1.6.7. Let $\mathrm{rank}^{\Sigma^1_2}_{\Pi^1_2}(T)$ be the $\prec^{\Sigma^1_2}_{\Pi^1_2}$-rank of T for Π^1_2-sound r.e. extension T of ACA_0. Then does the following hold? For two Π^1_2-sound r.e. extensions S, T of ACA_0,

$$\mathrm{rank}^{\Sigma^1_2}_{\Pi^1_2}(S) \leq \mathrm{rank}^{\Sigma^1_2}_{\Pi^1_2}(T) \iff |S|_{\Pi^1_2}(\delta^1_2) \leq |T|_{\Pi^1_2}(\delta^1_2).$$

1.7 Σ^1_2 as a supremum of a Σ^1_2-specifiable ordinals

In this section, we provide an alternative characterization of a Σ^1_2-proof-theoretic ordinal. Roughly, we will prove that $s^1_2(T)$ is the supremum of ordinals whose ordertype is a Σ^1_2-singleton, and T-provably exists modulo true Π^1_2-sentences. We introduce the notion of Σ^1_2-*specifiable well-order* and T-Π^1_2-*provably specifiable well-order* to express the previous informal explanation:

Definition 1.7.1. Let $\phi(X)$ be a formula in a pointclass Γ. A well-order α is Γ-*specifiable by* $\phi(X)$ $\forall^1 X[\phi(X) \to \mathsf{WO}(X)]$ and $\forall^1 X, Y[\phi(X) \wedge \phi(Y) \to X \cong Y]$ are true.

Furthermore, if $T \vdash^{\Pi^1_2} \exists^1 X \phi(X)$ holds for a theory T, we say α *is* T-Π^1_2-*provably* Γ-*specifiable by* $\phi(X)$. We say α is T-Π^1_2-*provably* Γ-*specifiable* if there is a formula $\phi(X) \in \Gamma$ T-Π^1_2-provably Γ-specifying α.

A well-order being Γ-specifiable is a weaker assertion than it being Γ-singleton since different isomorphic well-orders can also satisfy ϕ for a Γ-specifying formula $\phi(X)$. A T-Π^1_2-provably Γ-specifiable well-order is a Γ-specifiable well-order whose existence is provable in T with true Π^1_2-sentences. We will only consider the case $\Gamma = \Pi^1_1$ and Σ^1_2 in this section. The following theorem is the main result of this section:

Theorem 1.7.2. *Let T be a r.e. Σ^1_2-sound extension of $\mathsf{ATR}_0 + \Pi^1_1$-TI_0. Then*

$$\begin{aligned} s^1_2(T) &= \sup\{|\alpha| \mid \alpha \textit{ is } T\textit{-provably } \Pi^1_1\textit{-specifiable}\} \\ &= \sup\{|\alpha| \mid \alpha \textit{ is } T\textit{-provably } \Sigma^1_2\textit{-specifiable}\}. \end{aligned}$$

To prove the left-to-right, we claim that $\mathsf{Clim}(D)$ is T-provably Π^1_1-specifiable for a T-provably recursive pseudodilator D.

Proposition 1.7.3. *Let T be a Σ^1_2-sound r.e. extension of $\mathsf{ATR}_0 + \Pi^1_1$-TI_0. If D is a recursive pseudodilator such that $T \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$, then $\mathsf{Clim}(D)$ is T-Π^1_2-provably Π^1_1-specifiable.*

Proof. We can specify $\mathsf{Clim}(D)$ by the Π^1_1-formula $\phi(X)$, which is the conjunction of the following claims:

1. X is a well-order.

2. For every function $F\colon \mathbb{N} \to D(X)$, if $D(X) \vDash F(n) > F(n+1)$ for every n, then each $x \in X$ is less than an element of $\mathrm{supp}^D_X(F(n))$ for some $n \in \mathbb{N}$.

The second claim states 'For every infinite descending sequence F over $D(X)$, the union of supports $\bigcup_{n\in\mathbb{N}} \mathrm{supp}^D_X(F(n))$ is cofinal in X.' We first claim that any two well-orders satisfying ϕ are isomorphic: Suppose not, let $X < Y$ be two well-orders satisfying ϕ. We may assume that $X = Y \restriction i$ for some $i \in \mathrm{field}(Y)$. Now let F be an infinite descending sequence over $D(Y \restriction i)$, which is clearly an infinite descending sequence over $D(Y)$. Then by the assumption, $\bigcup_{n\in\mathbb{N}} \mathrm{supp}^D_{Y\restriction i}(F(n))$ is cofinal in Y, contradicting with that the union is bounded by i in Y. By the same argument, we can also show that $\phi(X)$ holds for every well-order X such that $X \cong \mathsf{Clim}(D)$.

Now we claim that T with the true Π^1_2-sentences proves $\exists^1 X \phi(X)$. Let us reason over T with the true Π^1_2-statements. From $\neg\mathsf{Dil}(D)$, we have a well-order X such that $\neg\mathsf{WO}(D(X))$. We are done if X satisfies $\phi(X)$. Otherwise, observe that Π^1_1-TI_0 is equivalent to the assertion 'For every Σ^1_1-formula $\psi(x)$ and a well-order A, if $\exists^0 i \in \mathrm{field}(A)\psi(i)$, then there is the A-least i satisfying $\psi(i)$.' We apply this statement to $A = X$ and $\psi(n) \equiv \neg\mathsf{WO}(D(X \restriction n))$, so we get $i \in \mathrm{field}(X)$ that is X-minimal satisfying $\neg\mathsf{WO}(D(X \restriction i))$. Then we can show that $\phi(X \restriction i)$ by following the previous argument for $\forall^1 X\big[\phi(X) \to X \cong \mathsf{Clim}(X)\big]$. □

By Proposition 1.7.3, we have

$$\begin{aligned} s^1_2(T) &\leq \sup\{|\alpha| \mid \alpha \text{ is } T\text{-provably } \Pi^1_1\text{-specifiable}\} \\ &\leq \sup\{|\alpha| \mid \alpha \text{ is } T\text{-provably } \Sigma^1_2\text{-specifiable}\} \end{aligned}$$

for a Σ^1_2-sound r.e. extension T of $\mathsf{ATR}_0 + \Pi^1_1$-TI_0. To prove the remaining direction, we construct a predilator whose climax is above a given Σ^1_2-specifiable well-order.

Proposition 1.7.4. *Let T be a Σ^1_2-sound r.e. extension of ATR_0. If α be a T-Π^1_2-provably Σ^1_2-specifiable well-order, then we can find a recursive pseudodilator D such that $|\alpha| \leq \mathsf{Clim}(D) < s^1_2(T)$.*

Proof. Suppose that $\beta(X, Y)$ is a primitive recursive linear order with real parameters X, Y such that the formula $\exists^1 Y \mathsf{WO}(\beta(X, Y))$ specifies α in T with the true Π^1_2-formulas. Let us describe the main idea of the proof before providing the details: Fix a linear order γ. We will construct a tree $\tau(\gamma)$, functorial in γ, trying to build α and an embedding $\alpha \to \gamma$. To ensure the real we are constructing is α, the tree also attempts to build a real Y and an embedding $\beta(\alpha, Y) \to \gamma$. The resulting construction will give a recursive pseudodilator D whose climax is at least α, and the statement

$$\theta \equiv \forall^1 X \big[(\exists^1 Y \mathsf{WO}(\beta(X, Y))) \to \mathsf{WO}(X)\big]$$

will be a true Π^1_2-sentence. We will show that $T + \theta \vdash^{\Pi^1_2} \neg\mathsf{Dil}(D)$, so $\mathsf{Clim}(D) < s^1_2(T)$.

For a linear order γ over $\mathbb{N}$, $\tau(\gamma)$ will be a tree over $\omega \times \omega \times \gamma \times \gamma$. $(u, v, \vec{\xi}, \vec{\eta})$ is in $\tau(\gamma)$ if and only if the following holds: u, v, $\vec{\xi}$, $\vec{\eta}$ have the same length l and for every $i, j < l$,

1. u is a finite sequence of binary tuples, and $\langle i, j\rangle \in u$ if and only if $\gamma \vDash \xi_i < \xi_j$.
2. If u and v are long enough to determine $i, j \in \mathrm{field}(\beta(\underline{u}, \underline{v}))$, then $\beta(\underline{u}, \underline{v}) \vDash i < j$ if and only if $\gamma \vDash \eta_i < \eta_j$.

Informally, u encodes a fragment of α, v encodes a witnessing real Y, $\vec{\xi}$ encodes an embedding $\alpha \to \gamma$, and $\vec{\eta}$ encodes $\beta(\alpha, Y) \to \gamma$. Note that $\beta(X, Y)$ is primitive recursive, so determining $i \in \mathrm{field}(\beta(X, Y))$ or $\beta(X, Y) \vDash i < j$ only requires a finite fragment of X and Y. We can turn $\tau(\gamma)$ into a linear order by imposing the Kleene-Brouwer order lexicographically, that is, we compare two quadruples $(u, v, \vec{\xi}, \vec{\eta})$ in the lexicographical

order, and compare each component under the Kleene-Brouwer order. Hence, we can view $\tau(\gamma)$ as a linear order.

Claim (ACA_0)**.** Suppose that γ is a well-order. Then $\tau(\gamma)$ has an infinite decreasing sequence if and only if there is a linear order α, a real Y, and embeddings $\alpha \to \gamma$ and $\beta(\alpha, Y) \to \gamma$.

Proof. Let us prove the forward direction first. Note that $\tau(\gamma)$ has an infinite decreasing sequence under the given linear order if and only if $\tau(\gamma)$ has an infinite branch. Thus let us suppose that there is an infinite branch $\langle (u_n, v_n, \vec{\xi}_n, \vec{\eta}_n) \mid n \in \mathbb{N} \rangle$ over $\tau(\gamma)$. Take $\alpha = \bigcup_n u_n$ and $y = \bigcup_n v_n$. Then the map $i \mapsto (i\text{th component of } \bigcup_n \vec{\xi}_n)$ is an relation-preserving map from α to γ, so α is a well-order. Moreover, $i \mapsto (i\text{th component of } \bigcup_n \vec{\eta}_n)$ is a map from $\beta(\alpha, Y)$ to γ, so $\beta(\alpha, Y)$ is well-founded, proving $\exists y \mathsf{WO}(\beta(\alpha, Y))$. The converse is clear. □

We can easily turn τ into a semidilator D by taking

1. $D(\gamma) = \tau(\gamma)$ equipped with the Kleene-Brouwer order.
2. For an increasing function $f : \gamma \to \delta$, $D(f)$ is a map sending $(u, v, \vec{\xi}, \vec{\eta})$ into $(u, v, f^{"}\vec{\xi}, f^{"}\vec{\eta})$, where $f^{"}\vec{\xi}$ is a componentwise application of f to the finite sequence $\vec{\xi}$.
3. $\mathrm{supp}^D_\gamma((u, v, \vec{\xi}, \vec{\eta})) = \bigcup \vec{\xi} \cup \bigcup \vec{\eta}$.

It is easy to check that D is a recursive predilator. Now let us work over $T + \theta$. We claim that D is not a dilator as promised.

Claim. $T + \theta$ proves D is not a dilator.

Proof. By the assumption on $\beta(X,Y)$, T proves $\exists^1\alpha\exists^1 Y\mathsf{WO}(\beta(\alpha,Y))$. Let us fix α and Y satisfying $\mathsf{WO}(\beta(\alpha,Y))$. Since the formula $\exists^1 Y\mathsf{WO}(\beta(X,Y))$ specifies a well-order by the assumption, we have $\mathsf{WO}(\alpha)$. If we take $\gamma = \alpha + \beta(\alpha,Y)$, then α, y and the natural embeddings $\alpha, \beta(\alpha,Y) \to \gamma$ form an infinite branch of $\tau(\gamma)$. Hence $D(\gamma)$ is ill-founded. □

It shows $\mathsf{Clim}(D) < s^1_2(T+\theta) = s^1_2(T)$, where the last equality holds since θ is a true Π^1_2-sentence. Hence we have $|\alpha| \leq \mathsf{Clim}(D) < s^1_2(T)$. □

1.8 s^1_2 and the height of a transitive model

In this section, we investigate the connection between the height of transitive models and the Σ^1_2-proof-theoretic ordinal. We will only consider a specific type of theories we will call *competent theories*.

1.8.1 Competent theories

In this subsection, we define competent theories. Before the definition, let us introduce the following auxiliary terminology:

Definition 1.8.1. Let Γ be Π^1_1 or Σ^1_2. A real A is T-Π^1_2-*provably* Γ-*singleton*, if there is a formula $\phi(X) \in \Gamma$ such that

1. The Π^1_2-assertion $\forall^1 X, Y[\phi(X) \wedge \phi(Y) \to X = Y]$ is true.
2. $T \vdash^{\Pi^1_2} \exists^1 X \phi(X)$.

Definition 1.8.2. A Σ^1_2-sound theory T is *competent* if for every T-provably recursive pseudodilator D, there is a T-Π^1_2-provably Σ^1_2-singleton well-order α such that $\neg\mathsf{WO}(D(\alpha))$.

Roughly speaking, a theory T is competent if T proves "D is a pseudodilator," then T can find a Σ^1_2-singleton well-order witnessing D being a pseudodilator, modulo Π^1_2-truth. Note that by the definition of $s^1_2(T)$, which is the supremum of $\mathsf{Clim}(D)$ for a T-provably recursive pseudodilator D, T being competent is equivalent to the claim that the set of ordertypes of T-Π^1_2-provably Σ^1_2-singleton well-orders is cofinal in $s^1_2(T)$.

We should ask which theories are competent, and here is one criterion working for sufficiently strong theories:

Proposition 1.8.3. *Let T be a $\Sigma^1_2 \cup \Pi^1_2$-sound extension of Π^1_1-CA_0. Then T is competent.*

Proof. In fact, we can prove a stronger claim: For every T-provably recursive pseudodilator D, there is a T-Π^1_2-provably Π^1_1-singleton well-order isomorphic to $\mathsf{Clim}(D)$. Let D be a recursive pseudodilator such that $T \vdash \neg\mathsf{Dil}(D)$. By Proposition 1.7.3, we can find a Π^1_1-formula $\phi(X)$ T-Π^1_2-provably specifying the ordertype of $\mathsf{Clim}(D)$. $\phi(X)$ itself does not define a well-order, so we appeal to the Π^1_1-uniformization to get a defining formula. More precisely, Π^1_1-CA_0 proves we can find another Π^1_1-formula $\hat{\phi}(x)$ such that

1. ACA_0 proves $\forall^1 X(\hat{\phi}(X) \to \phi(X))$,
2. ATR_0 proves $\forall^1 X, Y(\hat{\phi}(X) \wedge \hat{\phi}(Y) \to X = Y)$, and
3. Π^1_1-CA_0 proves $(\exists^1 X \phi(X)) \to (\exists^1 X \hat{\phi}(X))$.

Then $\hat{\phi}(X)$ defines a well-order isomorphic to $\mathsf{Clim}(D)$, which verifies the third condition of Definition 1.8.1 for $\hat{\phi}$. The other conditions are straightforward, but we need

the Π^1_2-soundness of T to guarantee the truth of the characterizing properties of $\hat{\phi}$, like $\forall^1 X[\hat{\phi}(X) \to \phi(X)]$. □

The following criterion works for weak theories like ATR_0 or ATR_0 plus the n-th iterate of the hyperjump of $\emptyset$ exists:

Proposition 1.8.4. *Let T be a Σ^1_2-sound extension of ATR_0 such that*

$$s^1_2(T) = \sup\{\omega_1^A \mid A \text{ is a } T\text{-provably } \Sigma^1_2\text{-singleton real}\}.$$

Then T is competent.

Proof. Suppose that $\alpha < s^1_2(T)$, so $\alpha < \omega_1^A$ for some T-provably Σ^1_2-singleton real A. Then α is isomorphic to an A-recursive real, and let $\phi(n, A)$ be a Σ^0_1-formula defining an ordertype of α with parameter A, i.e., $\alpha \cong \{n \mid \phi(n, A)\}$. Now let us identify α with an A-recursive well-order. Let us fix a Σ^1_2-formula $\psi(A)$ witnessing A being a Σ^1_2-singleton over T. Then α is a T-provably Σ^1_2-singleton by the following formula:

$$\chi(X) \equiv \exists^1 A[\psi(A) \wedge \forall^0 n[n \in X \leftrightarrow \phi(n, A)]]. \tag{1.10}$$

This shows the set of ordertypes of T-provably Σ^1_2-singleton well-orders below $s^1_2(T)$ is cofinal in $s^1_2(T)$, so T is competent. □

Note that *every* Σ^1_2-sound r.e. theory $T \supseteq \mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{TI}_0$ satisfies

$$s^1_2(T) \geq \sup\{\omega_1^A \mid a \text{ is } T\text{-provably } \Sigma^1_2\text{-singleton}\}$$

since every ordinal $\alpha < \omega_1^A$ for a T-provably Σ^1_2-singleton A is isomorphic to a Σ^1_2-singleton well-order (defined by (1.10)) and T-provably exists, so $\alpha < s^1_2(T)$ by Theorem 1.7.2. I do not know any T making the inequality strict.

1.8.2 The main result

Theorem 1.8.5. *Let T be a competent Σ^1_2-sound recursive extension of ATR_0, and let $\Sigma^1_2(T)$ be the set of all Σ^1_2-consequences of T. Then $\mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)$ has a transitive model. Furthermore,*

$$s^1_2(T) = \min\{M \cap \mathrm{Ord} \mid M \textit{ is a transitive model of } \mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)\}.$$

Let us see the left-to-right inequality first by showing the following: If M is a transitive model of $\mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)$, then $M \cap \mathrm{Ord} \geq s^1_2(T)$. Suppose that D is a recursive pseudodilator such that $T \vdash \neg\mathsf{Dil}(D)$, then $M \vDash \neg\mathsf{Dil}(D)$ for a transitive M such that $M \vDash \Sigma^1_2(T)$. Since M satisfies the axiom Beta, there is an ordinal $\alpha \in M$ such that $M \vDash D(\alpha)$ is illfounded. Hence $\mathsf{Clim}(D) < M \cap \mathrm{Ord}$. It holds for every recursive D with $T \vdash \neg\mathsf{Dil}(D)$, so we have $s^1_2(T) \leq M \cap \mathrm{Ord}$.

Hence, it suffices to prove that there is a transitive model of $\mathsf{ATR}^{\mathsf{set}}_0 + \Sigma^1_2(T)$ with height $\leq s^1_2(T)$. N will be constructed by considering the collapse of specific types of suitable trees. The reader should refer to Section A.5 for details about suitable trees and the sets-as-trees interpretation of set theory, whose details will be employed in the latter proof.

Definition 1.8.6. We say a Σ^1_2-formula $\phi(X)$ with its all free variables displayed *defines a T-Π^1_2-provably Σ^1_2-singleton suitable tree* if it satisfies the following:

1. $T \vdash^{\Pi^1_2} \exists^1 X \phi(X)$ and $\forall^1 X[\phi(X) \to X \text{ is a suitable tree}]$.
2. $\forall^1 X, Y[\phi(X) \wedge \phi(Y) \to (X = Y)]$ is true.

We say a real $\mathtt{T}$ is a *T-Π^1_2-provably Σ^1_2-singleton suitable tree* if there is a Σ^1_2-formula $\phi(X)$ that defines a T-Π^1_2-provably Σ^1_2-singleton suitable tree such that $\phi(\mathtt{T})$. Since the

word "T-Π^1_2-provably Σ^1_2-singleton suitable tree" is too long to use frequently, we also call it *T-provably Σ^1_2-suitable tree*.

The collection of all T-provably Σ^1_2-suitable trees forms a countable set. Hence, the following set is also countable:

$$N = \{|\mathtt{T}| : \mathtt{T} \text{ is a } T\text{-provably } \Sigma^1_2\text{-suitable tree}\}.$$

We claim that $N = (N, \in)$ is a desired transitive model.

Lemma 1.8.7. *N is transitive.*

Proof. Suppose that $\mathtt{T}$ is T-provably Σ^1_2-suitable. The elements of $|\mathtt{T}|$ takes the form $|\mathtt{T} \upharpoonright \langle n\rangle|$ for some natural number n, and we can see that $\mathtt{T} \upharpoonright \langle n\rangle$ is also a T-provably Σ^1_2-suitable: More precisely, if $\phi(X)$ is a Σ^1_2-formula witnessing $\mathtt{T}$ is T-Π^1_2-provably Σ^1_2-singleton suitable tree, then the formula

$$\psi(Y) \equiv \exists^1 X[\phi(X) \wedge Y = X \upharpoonright \langle n\rangle]$$

witnesses $\mathtt{T} \upharpoonright \langle n\rangle$ is T-Π^1_2-provably Σ^1_2-singleton suitable tree. □

Lemma 1.8.8. *Every ordinal in N is less than $s^1_2(T)$.*

Proof. Let $\mathtt{T}$ be a T-provably Σ^1_2-suitable tree coding an ordinal, and let $\phi(X)$ be a Σ^1_2-formula witnessing $\mathtt{T}$. Then $\mathtt{T}$ equipped with the Kleene-Brouwer order is a T-Π^1_2-provably Σ^1_2-singleton well-order, whose ordertype is at least that of $|\mathtt{T}|$. By Proposition 1.7.4, the ordertype of $(\mathtt{T}, <_{\mathsf{KB}})$ is bounded by some climax of a T-provably recursive pseudodilator D, so we have

$$|\mathtt{T}| \leq \mathrm{ordertype}(\mathtt{T}, <_{\mathsf{KB}}) \leq \mathsf{Clim}(D) < s^1_2(T).$$ □

The proof of [89, Lemma VII.3.20] shows N is a model of $\mathsf{ATR}_0^{\mathsf{set}}$. However, we need to verify that new trees witnessing the axioms are also definable, so let us check each axiom by hand:

Lemma 1.8.9. *N is closed under unordered pairs.*

Proof. For two suitable trees $\mathtt{T}_0$ and $\mathtt{T}_1$, consider the new tree $\mathtt{T}_2 = \mathtt{T}_0 \oplus \mathtt{T}_1$ given by

$$\mathtt{T}_2 = \{\langle\rangle\} \cup \{\langle 0\rangle^\frown u \mid u \in \mathtt{T}_0\} \cup \{\langle 1\rangle^\frown u \mid u \in \mathtt{T}_1\}.$$

It satisfies $|\mathtt{T}_2| = \{|\mathtt{T}_0|, |\mathtt{T}_1|\}$. Now we claim that if $\mathtt{T}_0$ and $\mathtt{T}_1$ are T-provably Σ^1_2-suitable, then so is $\mathtt{T}_2$: Suppose that ϕ_0 and ϕ_1 are Σ^1_2-formulas witnessing $\mathtt{T}_0$ and $\mathtt{T}_1$ are T-provably Σ^1_2-suitable. Consider

$$\phi_2(X) \equiv \exists^1 X_0 \exists^1 X_1 [\phi_0(X_0) \wedge \phi_1(X_1) \wedge X = X_0 \oplus X_1].$$

Then it is clear that ϕ_2 witnesses $\mathtt{T}_2$ is T-provably Σ^1_2-suitable. □

Lemma 1.8.10. *N is closed under set differences.*

Proof. We need a 'partial isomorphism map' between two trees to define a tree coding a set difference. Let $\mathtt{T}_0$ and $\mathtt{T}_1$ be suitable trees, and let R be the unique set such that $\mathsf{Iso}(R, \mathtt{T}_0 \oplus \mathtt{T}_1,)$. (See Proposition A.5.5 for the definition of Iso.) Now consider

$$\mathtt{T}_2 = \{\langle\rangle\} \cup \{u \in \mathtt{T}_0 \mid \neg\exists^0 m[\langle m\rangle \in \mathtt{T}_1 \wedge (\langle 0, u(0)\rangle, \langle 1, m\rangle) \in R]\}$$

Let us claim $|\mathtt{T}_2| = |\mathtt{T}_0| \setminus |\mathtt{T}_1|$: $|\mathtt{T}'| \in |\mathtt{T}_0|$ iff $\mathtt{T}' =^* \mathtt{T}_0 \upharpoonright \langle n\rangle$ for some $n \in \mathrm{immd}(\mathtt{T}_0)$. Similarly $|\mathtt{T}'| \in |\mathtt{T}_1|$ iff $\mathtt{T}' =^* \mathtt{T}_1 \upharpoonright \langle m\rangle$ for some $m \in \mathrm{immd}(\mathtt{T}_1)$. Hence $|\mathtt{T}'| \in |\mathtt{T}_0| \cap |\mathtt{T}_1|$ iff there are $n \in \mathrm{immd}(\mathtt{T}_0)$ and $m \in \mathrm{immd}(\mathtt{T}_1)$ such that $\mathtt{T}' =^* \mathtt{T}_0 \upharpoonright \langle n\rangle =^* \mathtt{T}_1 \upharpoonright \langle m\rangle$, so we have $\langle n\rangle \in \mathtt{T}_0$, $\langle m\rangle \in \mathtt{T}_1$, and $(\langle 0, n\rangle, \langle 1, m\rangle) \in R$. From this, we get

$$|\mathtt{T}'| \in |\mathtt{T}_0| \setminus |\mathtt{T}_1| \iff \exists^0 n \in \mathrm{immd}(\mathtt{T}_0)(\mathtt{T}' =^* \mathtt{T}_0 \upharpoonright \langle n\rangle) \wedge \neg[\exists^0 m(\langle 0, n\rangle, \langle 1, m\rangle) \in R],$$

and we can see that the latter condition is equivalent to $|\mathtt{T}'| \in |\mathtt{T}_2|$.

Now suppose that ϕ_0 and ϕ_1 are Σ^1_2-formulas witnessing $\mathtt{T}_0$ and $\mathtt{T}_1$ are T-provably Σ^1_2-suitable. Then consider

$$\begin{aligned}\phi_2(X) \equiv \exists^1 X_0 \exists^1 X_1 \exists^1 R[&\phi_0(X_0) \wedge \phi_1(X_1) \wedge \mathsf{Iso}(R, X_0 \oplus X_1) \\ &\wedge X = \{\langle\rangle\} \cup \{u \in X_0 \mid \neg\exists^0 m[\langle m\rangle \in X_1 \wedge (\langle 0, n\rangle, \langle 1, m\rangle) \in R]\}].\end{aligned}$$

Then ϕ_2 is a Σ^1_2-formula witnessing $\mathtt{T}_2$ being a Σ^1_2-singleton suitable tree, and $T \vdash^{\Pi^1_2}$ $\exists^1 X \phi_2(X)$. $\square$

The cases for the Cartesian product and union are not too hard but tedious, so I leave these cases to the reader. However, the tree representations for them are worth mentioning, so let us include their description: For a suitable tree $\mathtt{T}_0$, the following tree represents $\bigcup |\mathtt{T}_0|$:

$$\mathtt{T}_1 = \{\langle\rangle\} \cup \left\{\langle\langle m, n\rangle\rangle^\frown u \;\middle|\; \langle m, n\rangle^\frown u \in \mathtt{T}_0\right\}. \tag{1.11}$$

Let us define $\mathsf{Union}(\mathtt{T}_0)$ by $\mathtt{T}_1$ in (1.11). Clearly, $\mathsf{Union}(\mathtt{T}_0)$ is arithmetically definable from $\mathtt{T}_0$. Also, for two suitable trees $\mathtt{T}_0$ and $\mathtt{T}_1$,

$$\mathsf{prod}(\mathtt{T}_0, \mathtt{T}_1) = \{\langle\rangle\} \cup \left\{\langle\langle n_0, n_1\rangle\rangle^\frown u \;\middle|\; \forall i < 2(\langle n_i\rangle \in t_i),\ u \in \mathsf{op}(\mathtt{T}_0 \restriction n_0, \mathtt{T}_1 \restriction n_1)\right\},$$

where $\mathsf{op}(\mathtt{S}_0, \mathtt{S}_1)$ is the tree coding the ordered pair $\langle|\mathtt{S}_0|, |\mathtt{S}_1|\rangle = \{\{|\mathtt{S}_0|, |\mathtt{S}_0|\}, \{|\mathtt{S}_0|, |\mathtt{S}_1|\}\}$:

$$\mathsf{op}(\mathtt{S}_0, \mathtt{S}_1) = (\mathtt{S}_0 \oplus \mathtt{S}_0) \oplus (\mathtt{S}_0 \oplus \mathtt{S}_1).$$

It is also clear that $\mathsf{prod}(\mathtt{T}_0, \mathtt{T}_1)$ is arithmetically definable from $\mathtt{T}_0$ and $\mathtt{T}_1$.

Lemma 1.8.11. *N is closed under $a \mapsto \in\restriction a$.*

Proof. Let $\mathtt{T}_0$ be a suitable tree and R be a set satisfying $\mathsf{Iso}(R, \mathtt{T}_0)$. Let us first define the following subsidiary set:

$$R' = \{(n, m) \in \mathbb{N}^2 \mid \exists^0 k[(\langle n\rangle, \langle m, k\rangle) \in R]\}.$$

Note that $(n, m) \in R'$ iff $|\mathtt{T}_0 \restriction \langle n \rangle| \in |\mathtt{T}_0 \restriction \langle m \rangle|$. Hence the following tree codes $\in\restriction |\mathtt{T}_0|$:

$$\mathtt{T}_1 = \{\langle\rangle\} \cup \{\langle\langle n, m\rangle, u\rangle \mid (n, m) \in R' \wedge u \in \mathsf{op}(\mathtt{T}_0 \restriction \langle n \rangle, \mathtt{T}_0 \restriction \langle m \rangle)\}.$$

It is straightforward to see that if $\mathtt{T}_0$ is T-provably Σ^1_2-suitable, then so is $\mathtt{T}_1$. □

Lemma 1.8.12. *N is closed under $a \mapsto \operatorname{dom} a$.*

Proof. Note that

$$\operatorname{dom} a = \{x \in \bigcup\bigcup a \mid \{x\} \in \bigcup a\}. \tag{1.12}$$

Now let $\mathtt{T}_0$ be a suitable tree. We shall provide a suitable tree $\mathtt{T}_1$ satisfying $|\mathtt{T}_1| = \operatorname{dom} |\mathtt{T}_0|$. By (1.12), $\mathtt{T}_1$ should be a subtree of $\mathsf{Union}^2(\mathtt{T}_0) := \mathsf{Union}(\mathsf{Union}(\mathtt{T}_0))$. To express the condition $\{x\} \in \bigcup a$ of (1.12), we need a new tree $\mathtt{T}_2$ representing $\{\{x\} \mid x \in \bigcup\bigcup |\mathtt{T}_0|\}$ and R such that $\mathsf{Iso}(R, \mathtt{T}_2 \oplus \mathsf{Union}(\mathtt{T}_0))$.

For $\mathtt{T}_2$, consider

$$\mathtt{T}_2 = \{\langle\rangle\} \cup \{\langle n\rangle^\frown u \mid \langle n\rangle \in \mathsf{Union}^2(\mathtt{T}_0),\ u = \langle\rangle, \langle 0\rangle\} \cup \{\langle n, 0\rangle^\frown u \mid \langle n\rangle^\frown u \in \mathsf{Union}^2(\mathtt{T}_0)\}.$$

That is, $\mathtt{T}_2$ is a tree obtained from $\mathsf{Union}(\mathtt{T}_0)$ by inserting a new node between $\langle n\rangle$ and $\langle n\rangle^\frown u$ for $u \neq \langle\rangle$. $\mathtt{T}_2$ is arithmetically definable from $\mathtt{T}_0$, and proving $|\mathtt{T}_2| = \{\{x\} \mid x \in |\mathsf{Union}(\mathtt{T}_0)|\}$ is easy.

Now fix R such that $\mathsf{Iso}(R, \mathtt{T}_2 \oplus \mathsf{Union}(\mathtt{T}_0))$. Then consider

$$\mathtt{T}_1 = \{\langle\rangle\} \cup \{u \in \mathsf{Union}^2(\mathtt{T}_0) \mid \exists^0 m \in \operatorname{immd}(\mathsf{Union}(\mathtt{T}_0))\ (\langle 0, u(0)\rangle, \langle 1, m\rangle) \in R\}$$

Observe that $(\langle 0, u(0)\rangle, \langle 1, m\rangle) \in R$ is equivalent to

$$\{|\mathsf{Union}^2(\mathtt{T}_0) \restriction \langle u(0)\rangle|\} = |\mathtt{T}_2 \restriction \langle u(0)\rangle| = |\mathsf{Union}(\mathtt{T}_0) \restriction m|.$$

The first equality holds by the definition of $\mathtt{T}_2$, and the second equality follows from the choice of R. Hence, we have $|\mathtt{T}_1| = \mathrm{dom}\,|\mathtt{T}_0|$. It is easy to see that if $\mathtt{T}_0$ is T-provably Σ^1_2-suitable, then so is $\mathtt{T}_1$. □

Lemma 1.8.13. *N is closed under $a, b \mapsto \{x \mid \exists y \in a(x = b^{"}\{y\})\}$.*

Proof. Fix two suitable trees $\mathtt{T}_0$ and $\mathtt{T}_1$ that correspond to a and b respectively.

First, let us find how to express $b^{"}\{y\}$ by trees. Note that $b^{"}\{y\} = \{z \mid \langle y, z\rangle \in b\}$, so we need to express $\langle y, z\rangle \in b$ by trees. For this purpose, let us fix R such that

$$\mathsf{Iso}(R, \mathsf{prod}(\mathsf{Union}^2(\mathtt{T}_1), \mathsf{Union}^2(\mathtt{T}_1)) \oplus \mathtt{T}_1).$$

The choice of $\mathsf{Union}^2(\mathtt{T}_1)$ comes from the fact that if $\langle x, y\rangle \in b$, then $x, y \in \bigcup\bigcup b$.

Now fix $n \in \mathrm{immd}(\mathsf{Union}^2(\mathtt{T}_1))$, which will be thought as y in the sense that $y = |\mathtt{T}_1 \restriction \langle n\rangle|$, consider the tree $\mathsf{sect}_X(\mathtt{T}_1, n)$ given by

$$\mathsf{sect}_R(\mathtt{T}_1, n) = \{\langle\rangle\} \cup \{u \in \mathtt{T}_1 \mid \exists^0 m[(\langle n, u(0)\rangle, m) \in R]\}.$$

We can see that

$$(\langle n, u(0)\rangle, m) \in R \iff \mathsf{op}(\mathtt{T}_\cup \restriction n, \mathtt{T}_\cup \restriction u(0)) =^* \mathtt{T}_1 \restriction m,$$

where $\mathtt{T}_\cup = \mathsf{Union}^2(\mathtt{T}_1)$. Furthermore, $\mathsf{sect}_R(\mathtt{T}_1, n)$ is arithmetically definable from R and $\mathtt{T}_1$, and R is hyperarithmetically definable from $\mathtt{T}_1$ by Proposition A.5.5. Hence $\mathsf{sect}_R(\mathtt{T}_1, n)$ is hyperarithmetically definable from $\mathtt{T}_1$. It is straightforward to see that

$$|\mathsf{sect}_R(\mathtt{T}_1, n)| = |\mathtt{T}_1|^{"}\{|\mathsf{Union}^2(\mathtt{T}_1) \restriction \langle n\rangle|\}. \tag{1.13}$$

Then consider the tree $\mathtt{T}'$ that codes the set $\{|\mathtt{T}_1|^{"}\{y\} \mid y \in \bigcup\bigcup |\mathtt{T}_1|\}$:

$$\mathtt{T}' = \{\langle\rangle\} \cup \{\langle n\rangle^\frown u \mid \langle n\rangle \in \mathsf{Union}^2(\mathtt{T}_1) \wedge u \in \mathsf{sect}_R(\mathtt{T}_1, n)\}.$$

$\mathtt{T}'$ is hyperarithmetically definable from $\mathtt{T}_1$. Now fix R' such that $\mathsf{Iso}(\mathtt{T}_0 \oplus \mathtt{T}', R')$ Then let us consider the following tree:

$$\mathtt{T}_2 = \{\langle\rangle\} \cup \left\{\langle n\rangle^\frown u \;\middle|\; \langle n\rangle \in \mathtt{T}_0 \wedge \exists m \in \mathrm{immd}(\mathtt{T}')\big[(\langle n\rangle, \langle m\rangle) \in R' \wedge \langle m\rangle^\frown u \in \mathtt{T}'\big]\right\}.$$

$\mathtt{T}_2$ is hyperarithmetically definable from $\mathtt{T}_0$ and $\mathtt{T}_1$. Hence if both of $\mathtt{T}_0$ and $\mathtt{T}_1$ are T-provably Σ^1_2-suitable, then so is $\mathtt{T}_2$.

To finish the proof, let us claim $|\mathtt{T}_2| = \{x \mid \exists y \in |\mathtt{T}_0|(x = |\mathtt{T}_1|^{"}\{y\})\}$. By the definition of $|\mathtt{T}_2|$,

$$x \in |\mathtt{T}_2| \iff \exists^0 n \in \mathrm{immd}(\mathtt{T}_0)\exists^0 m \in \mathrm{immd}(\mathtt{T}')\big[(\langle n\rangle, \langle m\rangle) \in R' \wedge x = |\mathtt{T}' \restriction \langle m\rangle|\big].$$

Also, $(\langle n\rangle, \langle m\rangle) \in R$ implies

$$|\mathtt{T}_0 \restriction \langle n\rangle| = |\mathtt{T}' \restriction \langle m\rangle| \in |\mathtt{T}_0|.$$

For a given $x \in |\mathtt{T}_2|$, fix such n and m. By the definition of $\mathtt{T}'$, we have $x = |\mathsf{sect}_R(\mathtt{T}_1, m)|$. So by (1.13),

$$x = |\mathsf{sect}_R(\mathtt{T}_1, m)| = |\mathtt{T}_1|^{"}\{\mathtt{T}' \restriction \langle m\rangle\}.$$

It shows $x \in \{|\mathtt{T}_1|^{"}\{y\} \mid y \in a\}$, proving one direction of the inclusion. Repeating the previous argument in a reverse direction gives a proof for the other inclusion, finishing the proof. □

We can also prove that N is also closed under the following operators:

- $a \mapsto a^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in a\}$.
- $a, b \mapsto \{\langle y, \langle x, z\rangle \mid \langle y, x\rangle \in a \wedge z \in b\}$.
- $a, b \mapsto \{\langle y, \langle z, x\rangle \mid \langle y, x\rangle \in a \wedge z \in b\}$.

Its proof follows from straightforward, but tedious, manipulations. So I leave the details to the reader.

In conclusion, N is closed under rudimentary operators, or according to Mathias' terminology [61], N is a transitive model of *Gandy-Jensen set theory* GJ. By [61, Remark 2.87], we have

Corollary 1.8.14. *N satisfies Δ_0-Separation.* □

Verifying the axiom of infinity over N is straightforward since the canonical tree $\mathbb{N}^*$ coding the set of natural numbers is arithmetically definable, so $\mathbb{N} = |\mathbb{N}^*| \in N$. By transitivity of N, N thinks $\mathbb{N}$ witnesses the axiom of infinity.

Lemma 1.8.15. *N is closed under the transitive closure of a set: That is, if $a \in N$, then*

$$\mathrm{trcl}(a) := \bigcup\{\underbrace{\bigcup\cdots\bigcup}_{n \text{ times}} a \mid n \in \mathbb{N}\} \in N.$$

Proof. Let us recall that a finite sequence $u \in \mathbb{N}^{<\omega}$ can be viewed as a natural number, and let us subsume this identification. For a suitable tree $\mathtt{T}_0$, consider

$$\mathtt{T}_1 = \{\langle u\rangle^\frown v \mid u \in \mathtt{T}_0 \wedge u^\frown v \in \mathtt{T}_0\}.$$

Clearly, $\mathtt{T}_1$ is arithmetically definable from $\mathtt{T}_0$. To see $|\mathtt{T}_1| = \mathrm{trcl}\,|\mathtt{T}_0|$, observe that the following holds:

$$\underbrace{\bigcup\cdots\bigcup}_{n \text{ times}} |\mathtt{T}_0| = \{|\mathtt{T}_0 \restriction u| : u \in \mathtt{T}_0 \wedge \mathrm{length}(u) = n\}.$$ □

Proposition 1.8.16. *N is a model of the Axiom of Countability.*

Proof. The axiom of countability states 'every set is contained in a countable transitive set.' By Lemma 1.8.15, it suffices to show that N thinks every set is countable. The main

idea is that every set in $|\mathtt{T}|$ for a suitable $\mathtt{T}$ is indexed by an element of $\mathrm{immd}(\mathtt{T})$. We will describe a way to extract a bijection from the indexing, and the resulting function is of low definitional complexity.

For a suitable tree $\mathtt{T}_0$, let $\kappa \leq \omega$ be a cardinal such that $|\mathrm{immd}(\mathtt{T}_0)| = \kappa$. κ is either finite or countable, and always an initial segment of ω. Then we can find the unique arithmetically definable increasing enumeration $f \colon \kappa \to \mathrm{immd}(\mathtt{T}_0)$. Now consider

$$\mathtt{T}_1 = \{\langle\rangle\} \cup \{\langle n \rangle^\frown u \mid n \in \kappa \wedge u \in \mathsf{op}(n^*, \mathtt{T}_0 \restriction \langle f(n) \rangle)\},$$

where n^* is the canonical suitable tree coding n, so $|n^*| = n$. It is easy to see that $\mathtt{T}_1$ is arithmetically definable from $\mathtt{T}_0$ and $|\mathtt{T}_1|$ is an onto function from κ to $|\mathtt{T}_0|$. Hence if $\mathtt{T}_0$ is T-provably Σ^1_2-suitable, then so is $\mathtt{T}_1$. By the transitivity of N, we can see that N satisfies 'Every set is an image of an initial section of $\mathbb{N}$.' □

Proving N satisfies the Axiom Beta is the hardest part of this subsection.

Proposition 1.8.17. *N is a model of the Axiom Beta.*

Proof. For Axiom Beta, we prove the following: If $\mathtt{T}$ is a T-provably Σ^1_2-suitable, then we can find a new T-provably Σ^1_2-suitable tree $\mathtt{F}$ such that if N thinks $|\mathtt{T}|$ is a regular relation, then N thinks $|\mathtt{F}|$ is the collapsing function for $|\mathtt{T}|$.

To do this, let us analyze the proof of [89, Lemma VII.3.19]. The proof goes as follows: Working over $\mathsf{ATR}^{\mathsf{set}}_0$, let X be the set of all triples of $\mathtt{T}$. Why do we choose triples? Observe that for a suitable tree $\mathtt{T}$, its immediate successors of the root 'encodes' elements of $|\mathtt{T}|$. Repeating this observation, we can think of grandchildren of the root, which have length 2, 'encodes' elements of elements of $|\mathtt{T}|$, that is, elements of $\bigcup |\mathtt{T}|$. Thus, we understand X as nodes of $\mathtt{T}$ encoding elements of $\bigcup \bigcup |\mathtt{T}|$.

Now let $\mathtt{T}_1$ be a tree consisting of $\langle\rangle$ and all $\langle u\rangle^\frown v$ such that $u \in X$ and $v \in \mathtt{T} \restriction u$. Then we have $|\mathtt{T}_1| = \bigcup\bigcup\bigcup |\mathtt{T}| = \mathrm{field}\,|\mathtt{T}|$. By Proposition A.5.5, we can take the unique R_0 such that $\mathrm{Iso}(R_0, \mathtt{T})$, which is hyperarithmetically definable from $\mathtt{T}$ since R_0 can be defined from the arithmetical transfinite recursion along a Kleene-Brouwer order over $\mathtt{T}$. The reader may not know what R_0 precisely is, but should remember that R_0 decides $|\mathtt{T} \restriction u| = |\mathtt{T} \restriction v|$ for $u, v \in \mathtt{T}$: That is $(u, v) \in R_0$ iff $|\mathtt{T} \restriction u| = |\mathtt{T} \restriction v|$. Here is where we need the well-foundedness of $\mathtt{T}$ since otherwise R_0 may not exist.

We will recover a regular relation R isomorphic to $|\mathtt{T}|$ as follows: R is the set of all $(u, v) \in X \times X$ such that we have some $i, j, k, l, m \in \mathbb{N}$ such that

1. $(u, \langle k, i, l\rangle), (v, \langle k, j, m\rangle) \in R_0$,
2. $(u, v) \notin R_0$.
3. $(\langle k, i\rangle, \langle k, j\rangle) \notin R_0$.

The first condition is a subsidiary condition to get k, which equalizes the first component of u and v. The second condition says u and v do not code the same set, and the third condition says the second components are different; Otherwise, the set k codes may be of the form $\{\{x\}\} = \{\{x\}, \{x\}\}$. Thus, the last two conditions are to avoid x satisfying $(x, x) \in R$.

From the previous paragraph, we can see that R is hyperarithmetically definable from $\mathtt{T}$. Then consider the tree $\mathtt{T}_2$ and $\mathtt{F}$ defined by the following: Take

$$\mathtt{T}_2 = \{\langle\rangle\} \cup \{\langle u_0, \cdots, u_{k-1}\rangle \mid \forall i < k (u_i, u_{i+1}) \in R\}.$$

$|\mathtt{T}_2|$ will be the image of the transitive collapse of $|\mathtt{T}|$, and $\mathtt{T}_2$ is arithmetically definable from R. Then take

$$\mathtt{F} = \{\langle\rangle\} \cup \{\langle u\rangle^\frown v \mid u \in X \wedge v \in (\tau_2 \restriction \langle u\rangle \oplus \tau \restriction u) \oplus (\tau \restriction u \oplus \tau \restriction u)\}.$$

It is clear that $\mathtt{F}$ is also arithmetically definable from R, and it is straightforward that $\mathtt{F}$ is a suitable tree satisfying the desired property.

Now let us analyze the above proof to see why the above proof ensures $\mathtt{F}$ is T-provably Σ^1_2-suitable: The following Σ^1_2-formula

$$\phi_{\mathtt{F}}(X) \equiv \exists^1 Y(\phi_{\mathtt{T}}(Y) \wedge \chi(X, Y)),$$

witnesses $\mathtt{F}$ is T-provably Σ^1_2-suitable, where $\phi_{\mathtt{T}}(X)$ is a Σ^1_2-formula witnessing $\mathtt{T}$ being a Σ^1_2-singleton, and $\chi(X, Y)$ is a hyperarithmetical formula describing the construction of $\mathtt{F}$ from $\mathtt{T}$. The construction uniquely determines $\mathtt{F}$ from $\mathtt{T}$, which shows the uniqueness of X satisfying $\phi_{\mathtt{F}}(X)$. We can also see the construction of $\mathtt{F}$ carries over the theory $T + \psi + \mathsf{WF}(\mathtt{T})$, where ψ is a true Π^1_2-formula such that $T + \psi$ proves the existence of $\mathtt{T}$, and $\mathsf{WF}(\mathtt{T})$ is a true Π^1_2-sentence given by $\forall^1 X(\phi_{\mathtt{T}}(X) \to \mathsf{WF}(X))$. That is, we can prove $\exists^1 X \phi_{\mathtt{F}}(X)$ from $T + \psi + \mathsf{WF}(\mathtt{T})$. In sum, $\mathtt{F}$ is a T-Π^1_2-provably Σ^1_2-singleton suitable tree. □

Finally, let us prove that the competence of T ensures the height of N is precisely $s^1_2(T)$, which is necessary to guarantee that N satisfies every Σ^1_2-consequence of T.

Proposition 1.8.18. *If T is competent, then $N \cap \mathrm{Ord} \supseteq s^1_2(T)$*

Proof. Suppose that $\alpha < s^1_2(T)$. Since T is competent, there is a Σ^1_2-formula $\phi(X)$ defining a well-order of ordertype at least α and $T \vdash^{\Pi^1_2} \exists^1 X \phi(X)$. Now for a well-order A such that $\mathrm{field}(A) \subseteq \mathbb{N}$, consider the tree

$$\mathtt{T}_A = \{\langle \xi_0, \cdots, \xi_{n-1} \rangle \mid \xi_0, \cdots, \xi_{n-1} \in \mathrm{field}(A) \wedge A \vDash \xi_0 > \cdots > \xi_{n-1}\}.$$

One can see that $\mathtt{T}_A$ is a suitable tree if A is a well-order, and $|\mathtt{T}_A| \cong A$. Furthermore, $\mathtt{T}_A$ is arithmetically definable with parameter A. If $\psi(\mathtt{T}, A)$ is an arithmetical formula

defining $\mathtt{T} = \mathtt{T}_A$ from A, the following formula defines a tree $\mathtt{T}$ such that $|\mathtt{T}| \geq \alpha$ given by ϕ:

$$\chi(X) \equiv \exists^1 Y \phi(Y) \wedge \psi(X, Y).$$

It is clear that $T \vdash^{\Pi^1_2} \exists^1 X \chi(X)$, so $\alpha \leq |\mathtt{T}| \in N$, so $\alpha \in N$. □

Now the following is immediate:

Corollary 1.8.19. *If T is competent, then N satisfies every Σ^1_2-statement provable over T.*

Proof. Let D be a recursive pseudodilator such that $T \vdash \neg\mathsf{Dil}(D)$, so $\mathsf{Clim}(D) < s^1_2(T)$ and $M \vDash \neg\mathsf{Dil}(D)$. Observe that $D \in N$ since D is recursive. Since $s^1_2(T) \subseteq N$ by Proposition 1.8.18, N contains $\mathsf{Clim}(D)$, so $N \vDash \neg\mathsf{Dil}(D)$. □

1.9 Final remarks

Let me finish this chapter with some remarks and questions. The contents in this section slightly diverge from [39] as the comments in this chapter reflect more recent results, which I did not know when I wrote [39]. The following remark includes content in later chapters, so the reader may read this section after reading other chapters.

In this chapter, I defined a proof-theoretic pseudodilator $\|T\|_{\Sigma^1_2}$ as an infinite disjunction. This definition is simple to provide, but does not allow any concrete calculation, unlike the definition of a proof-theoretic dilator. This leaves an open question of whether we can provide a more abstract characterization of a proof-theoretic pseudodilator.

The results in Chapter 3 suggest there is a dual relationship between cut-elimination and completeness theorem, and the cut-elimination is closely related to proof-theoretic

Π^1_n-objects, like proof-theoretic n-ptyx. It might be tempting to argue that the completeness theorem gives a proof-theoretic pseudodilator as follows: Apply the β-completeness theorem to the empty sequent. This gives a β-preproof that becomes ill-founded for some α, so turning it into a pseudodilator gives a proof-theoretic pseudodilator. I am skeptical of this tentative construction since the resulting β-preproof gives a genedendron generating a β-model of the target theory, say T. Hence, the resulting pseudodilator has its climax at the least height of a β-model of T, which may differ from Theorem 1.8.5. Using the β-completeness theorem to calculate the proof-theoretic pseudodilator may be promising, but we may need an extra step for the calculation.

Regarding the reflection rank, the proof for Proposition 1.5.1 relies on the connection between Σ^1_2-proof-theoretic ordinal $s^1_2(T)$ and Σ^1_2-reflection principle for extensions of Σ^1_2-AC_0. The argument breaks down even at the level of Π^1_1-CA_0 as we examined in Example 1.4.7. However, it does not rule out the possibility of well-foundedness of $\prec^{\Pi^1_2}_{\Sigma^1_2}$ for extensions of ACA_0:

Question 1.9.1. Can we prove $\prec^{\Pi^1_2}_{\Sigma^1_2}$ is well-founded for Σ^1_2-sound r.e. extensions of ACA_0?

Lastly, Theorem 1.8.5 gives a partial answer to the question raised in Aguilera-Pakhomov [2, Question 39], asking whether $s^1_2(T)$ for Σ^1_2-sound recursive extension of ACA_0 is either admissible or limit admissible. Theorem 1.8.5 implies that $s^1_2(T)$ for a *competent* T must be either admissible or limit admissible, due to the following argument informed from Noah Schweber:

Proposition 1.9.2. *If M is a transitive model of $\mathsf{ATR}^{\mathsf{set}}_0$, then Ord^M is either admissible or limit admissible.*

Proof. Suppose the contrary that there is a transitive model M of $\mathsf{ATR}^{\mathsf{set}}_0$ whose height

is neither admissible nor limit admissible. Then there is $\alpha < \mathrm{Ord}^M$ that is largest among admissible or limit admissible below Ord^M. By the axiom of countability in M, there is a real $X \in M$ coding the ordertype of α.

Now let $\beta > \alpha$ be the next admissible above α, so $\alpha < \mathrm{Ord}^M < \beta \leq \omega_1^X$, where ω_1^X is the least X-admissible ordinal. ω_1^X is also equal to the supremum of the ordertypes of all X-recursive well-orders, so there is an X-recursive well-order Y whose ordertype is $\geq \mathrm{Ord}^M$. Clearly $Y \in M$, but Axiom Beta in M implies a contradiction. □

Hence, a counterexample for Aguilera-Pakhomov's question should fail to be competent. Furthermore, Proposition 1.8.3 implies

Corollary 1.9.3. *Suppose that T is a $(\Sigma^1_2 \cup \Pi^1_2)$-sound recursive extension of Π^1_1-CA_0. Then $s^1_2(T)$ is either admissible or limit admissible.* □

This suggests that $s^1_2(T)$ for a 'natural' $T \supseteq \Pi^1_1$-CA_0 should be either admissible or limit admissible. Also, a counterexample for Aguilera-Pakhomov's question extending Π^1_1-CA_0 should be Π^1_2-unsound.

CHAPTER 2

β-RANKING OF THEORIES

> Either my piece is a work of the highest rank,
> or it is not a work of the highest rank.
>
> Ludwig Wittgenstein, in his letter to Russell in 1920.
> From *Wittgenstein in Cambridge: Letters and Documents 1911–1951*.

In this section, we discuss the β-rank of theories and provide their properties. Most contents of this chapter appear in the joint work [45] with Patrick Lutz, Fedor Pakhomov, and James Walsh. It appears in my thesis with their permission. I would also like to thank Theodore Slaman for suggesting the main idea of the proof of Theorem 2.4.5.

2.1 Preliminaries

In this section, we review facts we need in this chapter. The initial draft of [45] also mentioned the sets-as-trees interpretation in this section. I moved it to Section A.5 since Section 1.8 also needs facts about sets-as-trees interpretation. For the same reason, some basic facts about δ^1_2 is also moved to Section A.3

Lemma 2.1.1 ([9, Theorem V.7.8]). *Let $\sigma(\alpha)$ be the least σ such that $L_\sigma \prec_{\Sigma_1} L_\alpha$. Then*

$$L_{\sigma(\alpha)} = \{x \in L \mid x \text{ is } \Sigma_1\text{-definable over } L_\alpha\}.$$

Similarly, we have

$$L_{\delta^1_2} = \{x \in L \mid x \text{ is } \Sigma_1\text{-definable over } L\}. \qquad \square$$

Lemma 2.1.2 ([17, Lemma II.5.3], KP). *For a limit ordinal α, let $\hat{\sigma}(\alpha)$ be the least ordinal σ such that $L_\sigma \prec L_\alpha$. Then*

$$L_{\hat{\sigma}(\alpha)} = \{x \in L_\alpha \mid x \text{ is definable in } L_\alpha \text{ without parameters}\}. \qquad \square$$

Lemma 2.1.3. *$\alpha \mapsto \hat{\sigma}(\alpha)$ is monotone, that is, $\alpha \leq \beta \to \hat{\sigma}(\alpha) \leq \hat{\sigma}(\beta)$.*

Proof. By Lemma 2.1.1, it suffices to show that if $a \in L_{\sigma(\alpha)}$, then a is Σ_1-definable over L_β. To see this, let $\phi(x) \equiv \exists y \psi(x, y)$ be a Σ_1-formula defining a over L_α, where ψ is a bounded formula. We claim that the formula

$$\phi'(x) \equiv \exists y[\psi(x, y) \wedge \forall (x', y') <_L (x, y) \neg\psi(x', y')]$$

is a formula defining a over L_β, which is equivalent to a Σ_1-formula. From $L_\alpha \vDash \phi(a)$, we have $b \in L_\alpha$ satisfying $\psi(a, b)$. Hence if (a', b') is the $<_L$-least pair satisfying $\psi(a, b)$, then $(a', b') \leq_L (a, b) <_L L_\alpha$. However, it is known that L_α forms an initial segment under $<_L$, so $(a', b') \in L_\alpha$. This means $L_\alpha \vDash \psi(a', b')$, so $L_\alpha \vDash \phi(a')$. Since ϕ defines an element over L_α, $a = a'$.

The previous argument says two things: We have $\phi'(a)$, and an element x satisfying $\phi'(x)$ is unique, as desired. $\square$

2.2 β-models and β-ranks of theories

In this section, we will begin to prove some basic facts about β-ranks of theories. We begin by reminding the reader of the definition of a β-model.

Definition 2.2.1. An infinite model M of set theory is a *β-model* if it is well-founded. An ω-model M of second-order arithmetic is a *β-model* if for every Σ^1_1 sentence φ with parameters from M, $M \vDash \varphi$ if and only if φ is true.

One might be tempted to define "β-model of set theory" as an ω-model of set theory (in the sense that $\omega^M = \omega$) that is Π^1_1-correct. However, to define β-ranks of theories, it is important that we restrict our attention to well-founded structures. We will see in Example 2.2.3 that there is a Π^1_1-correct ill-founded ω-model of ZFC. The current definition of a β-model of set theory also coheres well with Girard's notion of β-models appearing in [29, 31] where he used transitive models as β-models of set theory. Some materials about second-order set theory (e.g., [106]) used the term β-model to denote a model of second-order set theory whose class well-orders are genuinely well-founded. However, we also do not follow this convention.

The following lemma shows the connection between two different notions of β-models. (See Section A.5 for the definition of $\mathfrak{i}$.)

Lemma 2.2.2. *Let $M \vDash \mathsf{ATR}_0^{\mathsf{set}}$ be an ω-model of set theory (i.e., $\omega^M = \omega$). Then the following are equivalent:*

1. *M is well-founded.*
2. *$\mathfrak{i}(M)$ is Π^1_1-correct.*

That is, M is a β-model (in the sense of set theory) if and only if $\mathfrak{i}(M)$ is a β-model (in the sense of second-order arithmetic).

Proof. For (1) implies (2), suppose that $M \vDash \mathsf{PRS} + \mathsf{Beta}$ is a well-founded model. By the Kleene normal form theorem, every Π^1_1-sentence takes the form "α is well-founded" for a recursive linear order α. Now fix a recursive well-order α, and assume that M thinks α is well-founded. We claim that α is a genuine well-order.

Working inside M, we know that α is well-founded. Hence by Beta, we can find an ordinal ξ and an isomorphism $f \colon \alpha \to \xi$. Arguing externally, a map f is still an

isomorphism between α and ξ since this assertion is Δ_0, and well-founded models are correct about Δ_0-statements. Since M is well-founded, so is ξ. This shows α is well-founded.

Conversely, assume that $M \vDash \mathsf{ATR}_0^{\mathsf{set}}$ is an ill-founded ω-model of set theory, so there are $a_0, a_1, \cdots$ such that

$$M \vDash a_0 \ni a_1 \ni a_2 \ni \cdots.$$

(Note that the sequence $\langle a_n \mid n < \omega \rangle$ itself need not be a member of M.) Working inside M, let a be the transitive closure of a_0 and fix a bijection $f : \omega \to a$. Still working inside M, define a tree T over $\omega = \omega^M$ as follows:

$$T = \{\langle n_0, \cdots, n_k \rangle \in \omega^{<\omega} \mid f(n_0) \ni f(n_1) \ni \cdots \ni f(n_k)\}.$$

We may view T as a subset of $\omega = \omega^M$ with appropriate coding, so the statement 'T is well-founded' is a Π^1_1 statement with a real parameter T. T is externally ill-founded since if we let $n_k = f^{-1}(a_k)$, then $\langle n_0, n_1, n_2 \cdots \rangle$ is an infinite branch of T. However, M thinks there is no infinite descending $\in$-sequence of sets, so M thinks T is well-founded. This means M cannot be well-founded; Otherwise, T is externally well-founded. $\square$

The converse direction of the above proof uses the existence of a bijection between ω and a. The following example shows that this was necessary: We will see there is an ill-founded model of ZFC that is correct about Π^1_1-sentences.

Example 2.2.3. Suppose that we have $0^\sharp$, and consider the iteration M of $0^\sharp$ along $-\omega$.[1] If $0^\sharp$ exists, then every real in L is parameter-free definable in L, thus also in M. Now for a given Π^1_1-statement $\phi(X)$ with a real $X \in M$, $M \vDash \phi(X)$ iff $L \vDash \phi(X)$ iff $\phi(X)$. The first follows from the fact that M and L are elementarily equivalent, and the second follows from Shoenfield absoluteness. Hence M is Π^1_1-correct but ill-founded.

[1] In terms of EM blueprint, M is a model satisfying $0^\sharp$ as a theory whose set of indiscernibles is isomorphic to $-\omega$.

Lemma 2.2.4. *The following relations are well-founded:*

1. *The restriction of the membership relation to well-founded models of set theory, i.e.,* $\{(N, M) \mid N \in M,\ N$ *and* M *are well-founded* $\mathcal{L}_\in$ *structures*$\}$.
2. *The restriction of the coded membership relation to* β*-models of second-order arithmetic, i.e.,* $\{(N, M) \mid N \dot{\in} M,\ N$ *and* M *are* $\mathcal{L}_2$ β*-models*$\}$.

Proof. The first part is clear. To see the second part, it suffices to note that if a countable β-model M encodes a β-model N, then $\omega_1^M > \omega_1^N$. For details, see [69, 56]. □

Lemma 2.2.5. *Let* M *be a* β*-model and* $N \in M$ *be such that* $M \vDash$ *"*N *is a* β*-model". Then* N *is again a* β*-model.* □

Definition 2.2.6. Let T_0 be an extension of ATR_0 or $\mathsf{PRS} + \mathsf{Beta}$. Let us define the relation $<_\beta^{T_0}$ for theories extending T_0 as follows:

> For $S, T \supseteq T_0$, $S <_\beta^{T_0} T$ if for every β-model M of T, there is a β-model $N \in M$ such that $N \vDash S$.

Lemma 2.2.7. *Let* T_0 *be an extension of* ATR_0 *or* $\mathsf{PRS} + \mathsf{Beta}$. *Then* $<_\beta^{T_0}$ *is well-founded for theories extending* T_0.

Proof. Suppose that there is a sequence $\langle T_i \mid 1 \leq i < \omega \rangle$ of theories extending T_0 such that $T_1 >_\beta^{T_0} T_2 >_\beta^{T_0} \cdots$. By Dependent Choice, we can choose β-models M_i such that $M_i \vDash T_i$ and $M_{i+1} \in M_i$ for all $1 \leq i < \omega$. Hence, we get an $\in$-decreasing sequence of β-models, a contradiction. □

Definition 2.2.8. For a β-model M of arithmetic or set theory, let $\beta\text{-rank}(M)$ be the rank of M given by the membership relation between β-models. For T_0 extending either ATR_0 or $\mathsf{PRS} + \mathsf{Beta}$ and $T \supseteq T_0$, let $|T|_\beta^{T_0}$ be the rank of T given by $<_\beta^{T_0}$.

So far, we have used well-founded models to define the β-rank of a set theory. We may use transitive models instead:

Definition 2.2.9. Let T_0 be an extension of $\mathsf{PRS} + \mathsf{Beta}$. Let us define the relation $<^{T_0}_{\mathsf{trans}}$ for theories extending T_0 as follows:

> For $S, T \supseteq T_0$, $S <^{T_0}_{\mathsf{trans}} T$ if for every transitive model M of T, there is a transitive model $N \in M$ such that $N \vDash S$.

It turns out that using transitive models instead of well-founded models does not change anything:

Lemma 2.2.10. *Let T_0 be an extension of* $\mathsf{PRS}+\mathsf{Beta}$*, and $S, T \supseteq T_0$. Then the following are equivalent:*

1. $S <^{T_0}_{\beta} T$
2. $S <^{T_0}_{\mathsf{trans}} T$

Proof. Suppose that $S <^{T_0}_{\beta} T$ holds, and let M be a transitive model of T. Then M contains a well-founded model N of S. Since $M \vDash \mathsf{Beta}$, M contains a transitive collapse of N, which is a transitive model of S. Hence we have $S <^{T_0}_{\mathsf{trans}} T$.

Conversely, assume that $S <^{T_0}_{\mathsf{trans}} T$, and let M be a well-founded model of T. Let $\pi : M \to \pi"[M]$ be the transitive collapse of M, then $\pi"[M]$ contains a transitive model N of S. We can see that $\pi^{-1}(N) \in M$ is a well-founded model of S. □

Corollary 2.2.11. *Let T_0 be an extension of* $\mathsf{PRS} + \mathsf{Beta}$*. Then both of $<^{T_0}_{\beta}$ and $<^{T_0}_{\mathsf{trans}}$ define the same rank function.*

Likewise, the notion of β-rank for second-order arithmetic and the notion of β-rank for set theory are not too different: For $T_0 \supseteq \mathsf{ATR}_0$ and $T \supseteq T_0$, we will see that $|T|_\beta^{T_0} = |T^{\mathsf{set}}|_\beta^{T_0^{\mathsf{set}}}$. This follows from the following lemma:

Lemma 2.2.12. *Let T_0 be an extension of ATR_0, and $S, T \supseteq T_0$. Then the following are equivalent:*

1. $S <_\beta^{T_0} T$.
2. $S^{\mathsf{set}} <_\beta^{T_0^{\mathsf{set}}} T^{\mathsf{set}}$.

Proof. Suppose that we have $S <_\beta^{T_0} T$, and let $A \vDash T^{\mathsf{set}}$ be a well-founded model. By taking the transitive collapse if necessary, we can assume that A is transitive.

It is clear that $\mathfrak{i}(A) \vDash T$, so it contains a β-model N of S, which can be viewed as a subset of $\mathbb{N}$. Clearly N^* is also a member of $\mathfrak{i}(A)$, so $[N^*] \in \mathfrak{t}(\mathfrak{i}(A)) \cong A$, so $N \in A$ since $[t] \mapsto |t|$ for a well-founded tree t is an isomorphism from $\mathfrak{t}(\mathfrak{i}(A))$ to A.

Thus A encodes a β-model $N \vDash S$. By Lemma A.5.14, the construction of $\mathfrak{t}(N)$ can go over A. Also, since N is a model of S, $\mathfrak{t}(N)$ is a model of $S^{\mathsf{set}} = \mathsf{ATR}_0^{\mathsf{set}} + \{\phi^{\mathfrak{i}} \mid S \vdash \phi\}$: If $S \vdash \phi$, then $N \vDash \phi$, which is equivalent to $N \vDash (\phi^{\mathfrak{i}})^{\mathfrak{t}}$ by Lemma A.5.8, so we have $\mathfrak{t}(N) \vDash \phi^{\mathfrak{i}}$

Conversely, assume that we have $S^{\mathsf{set}} <_\beta^{T_0^{\mathsf{set}}} T^{\mathsf{set}}$, and let $M \vDash T$ be a β-model. By taking a transitive collapse if necessary, assume that $\mathbb{N}^M = \mathbb{N}$. Then $\mathfrak{t}(M) \vDash T^{\mathsf{set}}$, and so there is a transitive model B of S^{set} such that $B \in \mathfrak{t}(M)$. The construction of $\mathfrak{i}(B)$ also goes over $\mathfrak{t}(M)$, so $\mathfrak{i}(B) \in \mathfrak{t}(M)$. Furthermore, $\mathfrak{i}(B)$ is a model of S: Suppose that $S \vdash \phi$, so $B \vDash \phi^{\mathfrak{i}}$. Then again, the definition of $\mathfrak{i}(B)$ gives $\mathfrak{i}(B) \vDash \phi$. Since $\mathfrak{i}(B)$ can be coded as a subset of $\mathbb{N} = \mathbb{N}^M$, we have $\mathfrak{i}(B) \mathrel{\dot\in} M$. $\square$

2.3 All Ranks are Countable

The first main theorem is that every theory's β-rank is countable. We emphasize that this theorem concerns all deductively closed sets of formulas, not merely those that are, e.g., recursively axiomatized. Moreover, this theorem applies both to $\mathcal{L}_2$ theories and to $\mathcal{L}_\in$ theories. For an upper bound, let us rely on the following subsidiary notion for the rank of transitive models:

Definition 2.3.1. For two transitive models M, N of T_0, let us consider the relation $\in$, which is well-founded. Hence $\in$ defines the rank function over the collection of all transitive models of T_0. Let us define

$$\beta\text{-rank}_{T_0}(M) = \sup\{\beta\text{-rank}_{T_0}(N) + 1 \mid N \in M \wedge N \text{ is transitive model of } T_0\}.$$

Lemma 2.3.2. *For every $T \supseteq T_0$, we have*

$$|T|_\beta^{T_0} \leq \min\{\beta\text{-rank}_{T_0}(M) \mid M \vDash T \textit{ is transitive}\}.$$

Proof. Let us take $\alpha = \beta\text{-rank}_{T_0}(T)$ and fix a transitive model M of T. We want to show that $\alpha \leq \beta\text{-rank}_{T_0}(M)$ by induction on α.

Suppose that $\gamma < \alpha$, so we have a theory $U <_\beta^{T_0} T$ such that $\gamma = \beta\text{-rank}_{T_0}(U)$. From $U <_\beta^{T_0} T$, we can see there is a transitive model $N \in M$ of U. By the inductive hypothesis, we have $\gamma \leq \beta\text{-rank}_{T_0}(N)$. From $N \in M$, we get $\beta\text{-rank}_{T_0}(M) \geq \gamma + 1$. Since it holds for every $\gamma < \alpha$, we have $\alpha \leq \beta\text{-rank}_{T_0}(M)$, as desired. □

Theorem 2.3.3. *For every $T \supseteq T_0$ with a β-model, $|T|_\beta^{T_0} < \omega_1$.*

Proof. By Lemma 2.2.10 and Lemma 2.2.12, it suffices to show that for every $T_0 \supseteq \mathsf{PRS} + \mathsf{Beta}$ and $T \supseteq T_0$ with a β-model, $|T|_\beta^{T_0} < \omega_1$.

Suppose that M is a β-model of T. By taking its Skolem hull if necessary (may not be definable in M), we may assume that M is countable. Furthermore, we may assume that M is transitive by considering its transitive collapse. Then $|T|_\beta^{T_0}$ is less than or equal to $\beta\text{-rank}_{T_0}(M)$. There are at most countably many transitive models in M, so $\beta\text{-rank}_{T_0}(M) < \omega_1$. □

2.4 Ranks are cofinal for theories without $V = L$

In this section, we prove that the β-rank of theories extending various T_0 is cofinal in ω_1. Every proof of the cofinality in this section relies on the following theorem:

Theorem 2.4.1. *Suppose that $T_0 \supseteq \mathsf{PRS} + \mathsf{Beta}$ satisfies the following* real encoding property*: For every real X, there is $T \supseteq T_0$ such that T has a countable transitive model and every transitive model of T contains X. Then the supremum of $|T|_\beta^{T_0}$ for $T \supseteq T_0$ is ω_1.*

Proof. We prove by induction on $\alpha < \omega_1$ that there is $T \supseteq T_0$ of β-rank at least α. Let us inductively assume that each $\xi < \alpha$ has a theory whose β-rank is at least ξ. Let $\{T_n \mid n \in \mathbb{N}\}$ be the collection of extensions of T_0 such that for each $\xi < \alpha$ we can find n such that $|T_n|_\beta^{T_0} \geq \xi$. Let us choose $\{X_n \mid n \in \mathbb{N}\}$ such that X_n codes a transitive model of T_n, and X be a real coding $\{X_n \mid n \in \mathbb{N}\}$. Suppose that T is an extension of T_0 such that every transitive model of T contains X. If $M \vDash T$, then M contains a transitive model of T_n for each $n \in \mathbb{N}$, so $T_n <_\beta^{T_0} T$ for each n. Hence

$$|T|_\beta^{T_0} \geq \sup_{n<\omega}(|T_n|_\beta^{T_0} + 1) \geq \alpha.$$ □

As an example, we can see that $T_0 = \mathsf{ZFC}$ satisfies the real encoding property if

every real is contained in a transitive model of ZFC, via coding a real into the continuum pattern as presented in the following example:

Example 2.4.2. Suppose that for every real X, we can find a countable transitive model of ZFC containing X.[2] Let us consider the theory

$$T_X := \mathsf{ZFC} + \{2^{\aleph_n} = \aleph_{n+1} \mid n \notin X\} + \{2^{\aleph_n} = \aleph_{n+2} \mid n \in X\}.$$

Every transitive model of T_X contains X since T_X can decode X from the continuum pattern. We claim that T_X has a transitive model: Let M be a countable transitive model of ZFC containing X. By taking $L[X]^M$ if necessary, we may assume that M satisfies GCH. Working inside M, we consider Easton forcing $\mathbb{P}$ whose continuum pattern codes X. Let G be a $\mathbb{P}$-generic filter in M, and consider $M[G]$, which is a desired model.

The above coding scheme — coding a real into a continuum pattern — does not work for theories extending $\mathsf{ATR}_0^{\mathsf{set}}$, which contains the axiom of countability. To resolve this issue, we encode a real into the constructibility degree instead of the continuum pattern. We first clarify what the constructibility degree is:

Definition 2.4.3. For two reals X and Y, we say $X \leq_L Y$ if $L[X] \subseteq L[Y]$. We say $X \equiv_L Y$ if $L[X] = L[Y]$. An $\equiv_L$-equivalence class is called a *constructibility degree*.

Note that $X \leq_L Y$ iff $X \in L_{\omega_1}[Y]$ since every real in $L[Y]$ is in $L_{\omega_1}[Y]$, and $L[X] \subseteq L[Y]$ iff $X \in L[Y]$. The following theorem by M. Groszek [34, Corollary 14] says we can force in a way that the constructibility degree encodes a fixed subset of ω_1:

Theorem 2.4.4 (Groszek). *Let $X \subseteq \omega_1$, and suppose that we have a maximum real a in the constructibility degree. Then there is a forcing $\mathbb{P}$ preserving ω_1 such that for a $\mathbb{P}$-generic filter G over V, the only constructibility degrees over a in $V[G]$ are precisely*

$$\{d_\gamma \mid \gamma < \omega_1\} \cup \{a_\gamma, b_\gamma \mid \gamma \in X\}$$

[2] This hypothesis holds when there is an inaccessible cardinal: For an inaccessible κ and a real X, consider a countable Skolem hull of V_κ containing X.

satisfying the following:

1. *For $\gamma < \beta$, $d_\gamma <_L d_\beta$.*
2. *When $\gamma \in X$, $d_\gamma <_L a_\gamma, b_\gamma <_L d_{\gamma+1}$, and a_γ and b_γ are $\leq_L$-incompatible.* □

Let $\mathsf{Z}_2^{\mathsf{set}}$ be the theory ZFC^- plus "Every set is countable." We can see that $\mathsf{Z}_2^{\mathsf{set}}$ is precisely the sets-as-trees interpretation of full second-order arithmetic.

Theorem 2.4.5. *$\mathsf{Z}_2^{\mathsf{set}}$ has the real encoding property.*

Proof. For a given real X, consider the theory T_X comprising the following axioms:

1. Axioms of $\mathsf{Z}_2^{\mathsf{set}}$.
2. There is a real Y such that the degree of constructibility above Y forms a well-founded partial order of height $\omega + \omega$, and moreover
   (a) Each level of the constructibility degrees above Y has at most two constructibility degrees.
   (b) For every $n < \omega$, there is only one constructibility degree of level n above Y.
   (c) For each (meta-)natural number n, the $(\omega + h_n + n + 1)$-th level of the construcbility degree over Y has two incompatible degrees, where h_n is the number of elements in $\{m < n \mid m \in X\}$.
   (d) For (meta-)natural m not of the form $h_n + n + 1$, the $(\omega + m)$-th level of the construcbility degree over Y has only one degree.

T_X requires that the degree of constructibility codes $\{\omega + n \mid n \in X\}$ above the top constructiblity degree. We will construct a model of T_X by forcing over $L[X]$ by applying

Theorem 2.4.4. $L[X]$ has X as the top in the constructibility degree, so the reader may wonder why we do not simply add a constructibility degree pattern coding X instead of padding the ω-sequence of degrees first. However, the constructibility degrees over $L[X]$ may have a tail of constructibility degrees isomorphic to $-\omega$. In this case, the generic extension $L[X][G]$ may not find where the top real in the ground model is. To avoid losing the starting point, we add the ω-sequence of reals and then code the real after the ω-sequence, so that the generic extension can read off the real from the last limit point of the constructibility degree.

Every transitive model of T_X contains X since T_X can define X from the constructibility degrees. Hence it remains to show that T_X has a countable transitive model. Let X be a real, and consider the inner model $L[X]$, which thinks there is a maximum real in the constructibility degree (namely X). Working in $L[X]$, let G be a $\mathbb{P}$-generic filter over $L[X]$ for the forcing poset $\mathbb{P}$ given by Theorem 2.4.4 coding the set $\{\omega+n \mid n \in X\}$ into the constructibility degree. Then we can see that $M_0 = (H_{\omega_1})^{L[X][G]}$ is a model of T_X.

However, we proved the existence of M_0 in $L[X][G]$ and not in V, and M_0 may not be countable. To solve both issues, let us work in $L[X][G]$ and consider a countable elementary submodel $M_1 \prec M_0$. We can encode M_1 as a real, which we can treat as a well-founded model of T_X. Thus $L[X][G]$ satisfies the assertion "There is a countable well-founded model of T_X," which is a Σ^1_2-assertion with a parameter X. By Shoenfield absoluteness, the assertion also holds in V, so V thinks T_X has a countable well-founded model. Collapsing a countable well-founded model of T_X gives a desired model. □

Corollary 2.4.6.

1. $|T|_\beta^{\mathsf{ATR}_0^{\mathsf{set}}}$ *for* $T \supseteq \mathsf{ATR}_0^{\mathsf{set}}$ *is cofinal in* ω_1.
2. $|T|_\beta^{\mathsf{ATR}_0}$ *for* $T \supseteq \mathsf{ATR}_0$ *is cofinal in* ω_1.

Proof. The first claim follows from that $|T|_\beta^{\mathsf{ATR}_0} \geq |T|_\beta^{\mathsf{Z}_2}$ for $T \supseteq \mathsf{Z}_2^{\mathsf{set}}$. The second claim follows from Lemma 2.2.12. □

2.5 Ranks are cofinal for theories with $V = L$

John Steel raised a question whether the β-rank of theories extending $\mathsf{ZFC} + (V = L)$ is cofinal in ω_1. It turns out that a β-rank of a theory containing $V = L$ is bounded by ω_1^L. First, we show the lemma asserting that the β-rank of a transitive model is bounded by its height:

Lemma 2.5.1. *Let $T_0 \supseteq \mathsf{PRS} + \mathsf{Beta}$ and M be a transitive model of T_0. Then $\beta\text{-rank}_{T_0}(M) \leq \mathrm{Ord} \cap M$.*

Proof. We prove it by induction on M: Suppose that the desired inequality holds for every transitive model $N \in M$ of T_0. Then

$$\begin{aligned}\beta\text{-rank}_{T_0}(M) &\leq \{(\mathrm{Ord} \cap N) + 1 \mid N \in M \wedge N \text{ is a transitive model of } T_0\} \\ &\leq \mathrm{Ord} \cap M,\end{aligned}$$

where the last inequality holds since $\mathrm{Ord} \cap N \in M$ and M is closed under successor ordinals. □

Proposition 2.5.2. *Let T be a theory extending $\mathsf{PRS} + \mathsf{Beta} + (V = L)$ and having a transitive model. Then $|T|_\beta^{\mathsf{PRS}+\mathsf{Beta}} < \omega_1^L$.*

Proof. Suppose that M is a transitive model of T. Then $M = L_\alpha$ for some ordinal α. Now let us find $\alpha' < \omega_1^L$ such that $L_{\alpha'} \prec L_\alpha$ by computing a transitive collapse of a Skolem hull of L_α in L. By Lemma 2.5.1, we have $|T|_\beta^{\mathsf{PRS}+\mathsf{Beta}} \leq \alpha' < \omega_1^L$. □

However, we can also show that the β-rank of theories extending $\mathsf{PRS}+\mathsf{Beta}+(V=L)$ is cofinal in ω_1^L. We will suggest two proofs for this claim; One is easier, and the other is more convoluted, but the idea of the other proof will be employed to calculate the β-rank of specific theories.

The following lemmas are modifications of exercises in Devlin [17, Exercise II.3.B–3.D.]. We include their proofs for completeness. The following lemma follows from Tarski undefinability argument.

Lemma 2.5.3. *Let α be a limit ordinal, and fix a recursive enumeration $\{\sigma_m \mid m<\omega\}$ of all sentences in the language $\{\in\}$. Then there is no formula $\varphi(x)$ with a single free variable such that $L_\alpha \vDash \sigma_m \iff L_\alpha \vDash \varphi(\underline{m})$. ($\underline{m}$ is a term denoting the natural number m.)*

Before starting the next lemma, let us define a preliminary notion: Let $\mathsf{KP}\ell$ be the combination of primitive recursive set theory $\mathsf{PRS}\omega$ plus "every set is a member of an admissible set." It is known that $\mathsf{KP}\ell$ proves Beta. Also, if α is a limit of admissible ordinals, then $L_\alpha \vDash \mathsf{KP}\ell$. Especially, if α is a limit of admissibles and is *locally countable*, that is, L_α satisfies the axiom of countability, then $L_\alpha \vDash \mathsf{ATR}_0^{\mathsf{set}}$.

Lemma 2.5.4. *The set $A=\{\mathrm{Th}(L_\alpha) \mid \alpha<\omega_1^L \wedge L_\alpha \vDash \mathsf{ATR}_0^{\mathsf{set}}\}$ is uncountable in L.*

Proof. Let us work in L throughout the whole proof. A is clearly not empty: If $L_\alpha \prec L_{\omega_1^L}$ and α is countable, then $\mathrm{Th}(L_\alpha)\in A$. Now suppose that A is countable, and let $T\subseteq\omega\times\omega$ be the $<_L$-least set such that $A=\{T"\{n\} \mid n<\omega\}$. We also have $T\in L_{\omega_1^L}$ since $T\subseteq\omega\times\omega$, and so $A\in L_{\omega_1}$. Then T is definable by the formula

$$(T \text{ is the } <_L\text{-least set such that } A=\{T"\{n\} \mid n<\omega\})^{L_{\omega_1^L}},$$

so in particular, T is definable over $L_{\omega_1^L}$ with a definable parameter $A\in L_{\omega_1^L}$ (A is definable using the equation in the statement of the lemma), so T is parameter-free definable

in $L_{\omega_1^L}$.

Fix a recursive enumeration $\{\sigma_m \mid m < \omega\}$ of sentences in the language $\{\in\}$, and pick n_0 such that $T"\{n_0\} = \mathrm{Th}(L_{\omega_1^L})$. (Such n_0 exists since we have a countable α such that $L_\alpha \prec L_{\omega_1^L}$, and such α satisfies $L_\alpha \vDash \mathsf{ATR}_0^{\mathsf{set}}$.) Then we have

$$L_{\omega_1^L} \vDash \sigma_m \iff L_{\omega_1^L} \vDash [\sigma_m \in T"\{n_0\}].$$

T is definable over $L_{\omega_1^L}$, so we have a contradiction with Lemma 2.5.3. □

Proposition 2.5.5. *The supremum of* β*-rank of theories extending* $\mathsf{PRS}+\mathsf{Beta}+(V=L)$ *is exactly* ω_1^L.

Proof. Let A be the set we defined in Lemma 2.5.4. Say α is *decent* if both (i) $\mathrm{Th}(L_\alpha) \in A$ and (ii) for all γ, if $L_\alpha \equiv L_\gamma$, then $\alpha \leq \gamma$.

First, we make an observation: If every β-model of T encodes a β-model of U, then every well-founded model of the set interpretation of T encodes a well-founded model of the set interpretation of U. We let $\{\gamma_\xi \mid \xi < \omega_1^L\}$ be an enumeration of all of the decent ordinals. These are cofinal in ω_1 by the previous Lemma. Therefore it suffices to show that whenever $\eta < \xi$, $\mathrm{Th}(L_{\gamma_\eta}) <_\beta \mathrm{Th}(L_{\gamma_\xi})$. Note that these theories will be distinguished by the definition of a decent ordinal.

So consider some well-founded model M for $\mathrm{Th}(L_{\gamma_\xi})$. We take the transitive collapse L_δ of M. We infer that $\delta \geq \gamma_\xi$ since γ_ξ is decent. Now let $\eta < \xi$. We infer that $L_{\gamma_\eta} \in L_\delta$. Obviously, $L_{\gamma_\eta} \vDash \mathrm{Th}(L_{\gamma_\eta})$. We take the uncollapse map we see that M contains a well-founded of $\mathrm{Th}(L_{\gamma_\eta})$. This shows the desired inequality, namely, $\mathrm{Th}(L_{\gamma_\eta}) <_\beta \mathrm{Th}(L_{\gamma_\xi})$. □

2.5.1 The second proof for the cofinality of rank for theories extending $(V = L)$

In this subsection, we give a second proof for that the supremum of the β-rank of theories extending $\mathsf{PRS} + \mathsf{Beta} + (V = L)$ is ω_1^L. Although this proof is more convoluted, its relativization will give a β-rank of extensions of KPi.

Lemma 2.5.6 (KPi)**.** *The class of all locally countable limit admissibles is unbounded below ω_1^L. If the axiom of countability holds and $V = L$, then the class is unbounded in* Ord.

Proof. Let $\alpha_0 < \omega_1^L$ be any ordinal. Now let us recursively choose the least countable admissible ordinal $\alpha_{n+1} > \alpha_n$ such that $L_{\alpha_{n+1}}$ contains an onto map from ω to α_n. Then consider $\alpha_\omega = \sup_n \alpha_n$. □

Lemma 2.5.7 (KPi)**.** *Let A be the collection of all ordinals α such that α is a limit of admissibles, locally countable, and every element of L_α is definable in L_α. Then A is unbounded in ω_1^L. If we assume the axiom of countability in the background theory, then A is a proper class.*

Proof. A is not empty since $\omega_\omega^{\mathsf{CK}} \in A$: In fact, each ω_n^{CK} is Σ_1-definable in $L_{\omega_\omega^{\mathsf{CK}}}$, and every ordinal below ω_n^{CK} is Σ_1-definable in $L_{\omega_n^{\mathsf{CK}}}$. The conclusion follows since every element of $L_{\omega_\omega^{\mathsf{CK}}}$ is ordinal definable in $L_{\omega_\omega^{\mathsf{CK}}}$.

Now suppose that A is bounded, and let λ be an upper bound for A. We claim that if $\gamma > \lambda$ is a limit of admissibles, then L_γ is a model of ZF^-, which will yield the contradiction: Because then we would be able apply Levy reflection in the least such limit of admissibles L_γ to obtain even smaller $L_{\gamma'} \in L_\gamma$ that is also a limit of admissibles and contains λ, contradicting the minimality of the choice of γ.

Let us follow the notation in Lemma 2.1.2. First, observe that $\hat{\sigma}(\alpha) \in A$ if α is limit: If $a \in L_{\hat{\sigma}(\alpha)}$, then a is defined by some formula $\phi(x)$ in L_α by Lemma 2.1.2. Since the statement 'a is defined by $\phi(x)$' is first-order expressible in L_α with a parameter in $L_{\hat{\sigma}(\alpha)} \prec L_\alpha$, the same statement holds in $L_{\hat{\sigma}(\alpha)}$. Hence $\phi(x)$ defines a in $L_{\hat{\sigma}(\alpha)}$, which means every element of $L_{\hat{\sigma}(\alpha)}$ is parameter-free definable in $L_{\hat{\sigma}(\alpha)}$.

Then we claim that if $L_\sigma \prec L_\alpha$ for $\sigma < \alpha$, then L_σ satisfies (first-order) Full Collection: Suppose that we have

$$L_\sigma \vDash \forall x \in a \exists y \phi(x, y, p)$$

for some $a, p \in L_\sigma$. Then we have

$$L_\alpha \vDash \exists b \forall x \in a \exists y \in b \phi(x, y, p)$$

witnessed by $b = L_\sigma$. By elementarity, we have $L_\sigma \vDash \exists b \forall x \in a \exists y \in b \phi(x, y, p)$, as desired.

Hence if A has an upper bound λ, then for every locally countable limit admissible $\gamma > \lambda$, we have $\hat{\sigma}(\gamma) \leq \lambda < \gamma$ and $L_{\hat{\sigma}(\gamma)} \prec L_\gamma$, so $L_{\hat{\sigma}(\gamma)}$ satisfies ZF^-. Hence $L_\gamma \vDash \mathsf{ZF}^-$ too. □

Lemma 2.5.8 (PRS). *Let A be the collection defined in Lemma 2.5.7. If $\alpha, \beta \in A$, $\alpha \neq \beta$, then* $\mathrm{Th}(L_\alpha) \neq \mathrm{Th}(L_\beta)$.

Proof. Suppose not, let us consider the case $\alpha < \beta$, $\alpha, \beta \in A$ but $\mathrm{Th}(L_\alpha) = \mathrm{Th}(L_\beta)$. Then α is definable in L_β, so $\mathrm{Th}(L_\alpha) = \mathrm{Th}(L_\beta)$ is also definable in L_β. It contradicts with Lemma 2.5.3. □

Proposition 2.5.9. *The supremum of* $\beta\text{-rank}_{\mathsf{ATR}_0^{\mathsf{set}}}(T)$ *for every* $T \supseteq \mathsf{ATR}_0^{\mathsf{set}}$ *is* ω_1^L.

Proof. By Lemma 2.5.7, A is cofinal in ω_1^L. Also, $\mathrm{Th}(L_\alpha)$ is an extension of $\mathsf{ATR}_0^{\mathsf{set}}$. We claim that for $\alpha < \gamma$ such that $\alpha, \gamma \in A$, $\mathrm{Th}(L_\alpha) <_\beta \mathrm{Th}(L_\gamma)$.

Let $M \vDash \mathrm{Th}(L_\gamma)$ be a transitive model. Then $M \vDash \mathsf{ATR}_0^{\mathsf{set}} + (V = L)$, so $M = L_\delta$ for some δ. We claim that $\delta \geq \gamma$: Suppose the contrary that $\delta < \gamma$ holds. Let $\phi(x)$ be a formula defining $\delta_0 := \delta$ in L_γ. Since $\mathrm{Th}(L_\gamma) = \mathrm{Th}(L_{\delta_0})$, the following sentence is in $\mathrm{Th}(L_\gamma)$ for each sentence $\psi \in \mathrm{Th}(L_\gamma)$:

$$\theta_\psi \equiv \forall x(\phi(x) \to \psi^{L_x}).$$

Now let $\delta_1 < \delta_0$ be the unique ordinal satisfying $L_{\delta_0} \vDash \phi(\delta_1)$. Since $L_{\delta_0} \vDash \theta_\psi$ for each sentence $\psi \in \mathrm{Th}(L_\gamma)$, L_{δ_0} and L_{δ_1} are elementarily equivalent. Repeating the same argument, we can derive an infinite decreasing sequence $\delta_0 > \delta_1 > \delta_2 > \cdots$ satisfying $\mathrm{Th}(L_\gamma) = \mathrm{Th}(L_{\delta_0}) = \mathrm{Th}(L_{\delta_1}) = \cdots$, a contradiction.

Hence $\delta \geq \gamma$, so $L_\alpha \in M = L_\delta$. Hence $\mathrm{Th}(L_\alpha) <_\beta \mathrm{Th}(L_\gamma)$. □

2.6 Ranks for effectively definable theories

In this section, we will determine the upper bound of the β-ranks of recursively axiomatized theories.

Theorem 2.6.1. *Let $T_0 \supseteq \mathsf{PRS} + \mathsf{Beta}$ be a recursive theory such that $H_{\omega_1} \vDash T_0$. Then the following are all equal to δ_2^1:*

1. *The supremum of $|T|_\beta^{T_0}$ for finitely axiomatizable $T \supseteq T_0$, if T_0 is finitely axiomatizable.*
2. *The supremum of $|T|_\beta^{T_0}$ for recursively axiomatizable $T \supseteq T_0$,*
3. *The supremum of $|T|_\beta^{T_0}$ for a Σ_2^1-singleton $T \supseteq T_0$.*

For an upper bound, let us rely on the rank of transitive models:

Lemma 2.6.2. *Let $T \in L_{\delta_2^1}$ be a theory with a transitive model. Then T has a transitive model in $L_{\delta_2^1}$.*

Proof. The statement 'T has a transitive model' is $\Sigma_1(T)$ and $T \in H_{\omega_1}$. Hence H_{ω_1} thinks there is a transitive model for T since $H_{\omega_1} \prec_{\Sigma_1} V$. Working over H_{ω_1}, the statement 'T has a transitive model' is equivalent to a Σ_2^1-statement without parameters by the following argument: Since $T \in L_{\delta_2^1}$, there is a Σ_1-formula $\theta(x)$ defining T over L. Then the statement

$$\exists X[(L \vDash \theta(X)) \wedge \exists M(M \text{ is a transitive model of } X)] \tag{2.1}$$

is a valid Σ_1-sentence over H_{ω_1}, which is equivalent to a Σ_2^1-statement. By the Shoenfield absoluteness, (2.1) also holds over L. Since $L_{\delta_2^1} \prec_{\Sigma_1} L$, (2.1) also holds over $L_{\delta_2^1}$, so T has a transitive model in $L_{\delta_2^1}$. □

Proposition 2.6.3. *Let $T \supseteq T_0$ be a Σ_2^1-singleton theory. Then $|T|_\beta^{T_0} < \delta_2^1$.*

Proof. If T is a Σ_2^1-singleton, then it is also Σ_2^1-singleton over L, and the statement 'T exists' is a Σ_2^1-statement, which is a Σ_1-statement over L. Hence $T \in L_{\delta_2^1}$. Hence by Lemma 2.6.2, T has a transitive model M in $L_{\delta_2^1}$, and $|T|_\beta^{T_0} \leq \beta\text{-rank}_{T_0}(M)$ by Lemma 2.3.2. Hence by Lemma 2.5.1, we have $\beta\text{-rank}_{T_0}(M) \leq M \cap \mathsf{Ord} < \delta_2^1$. □

To show the other direction, we construct a theory T_α whose β-rank is at least α for $\alpha < \delta_2^1$.

Definition 2.6.4. For $\alpha < \delta_2^1$, let us fix its Σ_1-definition $\phi(x)$ over L. The theory T_ϕ is the combination of the following axioms:

1. Axioms of T_0.

2. The statement $L \vDash \exists! x \phi(x)$.

3. The statement saying 'There is a $(\zeta+1)$-increasing chain of transitive models for T_0 for each $\zeta < \alpha$ such that the least model thinks $L \vDash \exists x \phi(x)$ holds.' Formally,

$$\exists x \Big[\phi^L(x) \wedge \forall \zeta \in x \exists f \colon (\zeta+1) \to V \big[\forall \xi \in x (f(\xi) \vDash T_0 \wedge f(\xi) \text{ is transitive} \\ \wedge \forall \xi \leq \zeta (f \restriction \xi \in f(\xi)) \wedge f(0) \vDash (\exists y \phi(y))^L \big] \Big].$$

Clearly, T is recursively axiomatizable or finitely axiomatizable if T_0 is. Also, we can see that it is β-satisfiable:

Lemma 2.6.5. *Suppose that $H_{\omega_1} \vDash T_0$ (which holds when $T_0 = \mathsf{PRS} + \mathsf{Beta}$ or $T_0 = \mathsf{ATR}_0^{\mathsf{set}}$.) For each Σ_1-formula ϕ defining an ordinal over L, T_ϕ has a transitive model.*

Proof. We claim that H_{ω_1} is a model of T_ϕ. By the assumption on T_0 and $\phi(x)$, we have that H_{ω_1} satisfies the first two axiom schemes of T_ϕ. To finish the proof, it suffices to see that H_{ω_1} thinks there is an ξ-increasing chain of transitive models $\langle M_\eta \mid \eta < \xi \rangle$ of T_0 for each $\xi < \alpha$ such that $M_0 \vDash \exists x \phi(x)$, where α is an ordinal defined by ϕ.

We prove the following stronger claim: There is an $\in$-increasing chain of transitive models $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ such that $M_\alpha \prec H_{\omega_1}$ and each countable section of the chain is in H_{ω_1}. For each α, let us choose a countable transitive $M_\alpha \prec H_{\omega_1}$ such that $\langle M_\beta \mid \beta < \alpha \rangle \in M_\alpha$. Then we have the desired chain of models. $\square$

The reader might think T_ϕ does not depend on the choice of ϕ as long as ϕ defines a fixed ordinal α. However, T_ϕ is sensitive to not only α but also the choice of ϕ: That is, for every $\alpha < \delta^1_2$, we can find Σ_1-formulas ϕ and ψ defining the same α over L such that T_ϕ and T_ψ are not β-equivalent. An informal reason for that is we can force ϕ to have arbitrarily large ξ, where ξ is the least ordinal satisfying $L_\xi \vDash \exists x \phi(x)$. The construction of an example is a variant of the padding argument:

Example 2.6.6. For a given $\alpha < \delta^1_2$, let us fix a Σ_1-formula $\phi(x)$ defining α over L. For a sufficiently large $\gamma < \delta^1_2$ whose size will be determined later, let us also fix a Σ_1-formula $\theta(x)$ defining γ over L. Then take

$$\psi(x) \equiv \exists y(\theta(y) \wedge \phi(x)).$$

We claim that T_ϕ and T_ψ fail to be β-equivalent: Suppose that M is a transitive model of T_ϕ. By Lemma 2.6.2, we may assume that $M \in L_{\delta^1_2}$, so $\beta := M \cap \mathrm{Ord} < \delta^1_2$. Now let us pick $\gamma > \beta$, and choose θ as before. Then consider the theory T_ψ. If $N \vDash T_\psi$, then $N \vDash \exists x \psi(x)$, so $N \vDash \exists y \theta(y)$. It means $\gamma \in N$, but $\gamma \notin M$. This shows $M \nvDash T_\phi$, so T_β and T_ψ are not β-equivalent.

To avoid the pathology illustrated in the previous example, we should restrict ϕ to be of the least *height*:

Definition 2.6.7. Let $\phi(x)$ be a Σ_1-formula defining an element over L. The *height* of $\phi(x)$ is the least α such that $L_\alpha \vDash \exists x \phi(x)$. We denote the height of $\phi(x)$ by $\mathrm{height}(\phi)$.

Note that if $\phi(x)$ defines $\alpha < \delta^1_2$, then $\mathrm{height}(\phi) \geq \alpha + 1$. Also, under the notation in Lemma 2.1.1, if ϕ defines α over L and $\alpha < \sigma(\gamma)$, then $\mathrm{height}(\phi) \leq \gamma$. The next lemma says restricting formulas of the least height yields β-equivalent theories:

Lemma 2.6.8. *Let ϕ and ψ be Σ_1-formulas defining the same $\alpha < \delta^1_2$ of the least height. Then $T_\phi \equiv_\beta T_\psi$.*

Proof. Suppose that M is a transitive model of T_ϕ. We claim that $M \vDash T_\psi$: By the assumption, $L^M = L_\xi \vDash \exists x \phi(x)$ for some ξ. Since ϕ and ψ be of least height, $\mathrm{height}(\phi) = \mathrm{height}(\psi) \leq \xi$, so $L_\xi \vDash \exists x \psi(x)$. Note that the statement

$$\forall x, y(L \vDash \phi(x) \wedge \psi(y)) \to x = y$$

is a true Π_1-statement, M sees $\phi^L(\alpha)$ and $\psi^L(\alpha)$ hold. It shows M satisfies the remaining part of T_ψ. □

Now for each $\alpha < \delta_2^1$ fix its Σ_1-definition ϕ_α of least height. Note that $\{\phi_\alpha \mid \alpha < \delta_2^1\}$ is not recursive although each T_{ϕ_α} is recursive (and finite if T_0 is.)

Lemma 2.6.9. *If* $\alpha \leq \gamma$, *then* $\text{height}(\phi_\alpha) \leq \text{height}(\phi_\gamma)$.

Proof. We claim the following to see $\text{height}(\phi_\alpha) \leq \text{height}(\phi_\gamma)$:

$$\forall \xi[\xi \geq \text{height}(\phi_\gamma) \to \xi \geq \text{height}(\phi_\alpha)].$$

If $\xi \geq \text{height}(\phi_\gamma)$, then γ is Σ_1-definable over L_ξ. This means $\gamma \in L_{\sigma(\xi)}$, and so $\alpha \in L_{\sigma(\xi)}$. Hence α is Σ_1-definable over L_ξ.

Now suppose contrary that $\text{height}(\phi_\alpha) > \xi$, and let $\phi(x)$ be a Σ_1-formula defining α *over* L_ξ. Following the argument for the proof of Lemma 2.1.3, $\phi'(x)$ becomes a formula defining α over L. Hence $\text{height}(\phi') \leq \xi < \text{height}(\phi_\alpha)$, contradicting with that ϕ_α is a Σ_1-formula defining α over L of the least height. □

Proposition 2.6.10. *If* $\alpha < \gamma$, *then* $T_{\phi_\alpha} <_\beta^{T_0} T_{\phi_\gamma}$.

Proof. Suppose that M is a transitive model of T_{ϕ_γ}, then $\gamma \in M$ and γ is the unique set defined by $\phi_\gamma^L(x)$ over M. Hence $\text{height}(\phi_\gamma) \leq M \cap \text{Ord}$, so by Lemma 2.6.9, $\phi_\alpha^L(x)$ defines α over M.

Now let $\langle M_\xi \mid \xi \leq \alpha \rangle \in M$ be a sequence of transitive models of T_0 such that

1. For each $\xi \leq \alpha$, $\langle M_\eta \mid \eta < \xi \rangle \in M_\xi$.
2. $M_0 \vDash \exists x \phi_\gamma(x)$.

We claim that $M_\alpha \vDash T_{\phi_\alpha}$: Clearly $M_\alpha \vDash T_0$. Also, we have

$$\text{height}(\phi_\alpha) \leq \text{height}(\phi_\gamma) \leq M_0 \cap \text{Ord} < M_\alpha \cap \text{Ord},$$

so $M_\alpha \vDash (\exists x \phi_\alpha(x))^L$. Lastly, we can see that $\langle M_\eta \mid \eta \leq \xi \rangle$ witnesses the last axiom of T_{ϕ_α}. □

Let us finish the proof of Theorem 2.6.1:

Proof of Theorem 2.6.1. We have $|T_{\phi_\alpha}|_\beta^{T_0} \geq \alpha$ for each $\alpha < \delta_2^1$. □

2.7 β-rank for specific theories

In this section, we calculate the β-ranks of specific systems. The ranks of some familiar theories are easy to calculate. In Simpson's book, we find the following:

Proposition 2.7.1 ([89, Lemma VII.2.9]). ACA_0 *proves: For any set X, if $\mathcal{O}^X$ exists, then there is a β-model containing X.*

Attention to the proof of this theorem shows the following:

Proposition 2.7.2. *For any set X, if $\mathcal{O}^X$ exists, then there is a β-model M containing X such that M is arithmetic in $\mathcal{O}^X$.*

These propositions easily yield the following results:

- $|\mathsf{ATR}_0|_\beta^{\mathsf{ATR}_0} = 0$
- $|\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{CA}_0^-|_\beta^{\mathsf{ATR}_0} = |\mathsf{ATR}_0 + \text{“}\mathcal{O} \text{ exists”}|_\beta^{\mathsf{ATR}_0} = 1$
- $|\mathsf{ATR}_0 + \text{“}\mathcal{O}^{(n)} \text{ exists”}|_\beta^{\mathsf{ATR}_0} = n$

From these we infer:

Proposition 2.7.3. $|\Pi^1_1\text{-}\mathsf{CA}_0|^{\mathsf{ATR}_0}_\beta = \omega$

Proof. For the lower bound, observe that $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathsf{ATR}_0 +$ "$\mathcal{O}^{(n)}$ exists" for every n. These facts are correctly computed in β-models. So

$$|\Pi^1_1\text{-}\mathsf{CA}_0|^{\mathsf{ATR}_0}_\beta \geq \sup\{|\mathsf{ATR}_0 + \text{“}\mathcal{O}^{(n)} \text{ exists ”}|^{\mathsf{ATR}_0}_\beta \mid n < \omega\} = \omega.$$

For the upper bound, suppose $T <^{\mathsf{ATR}_0}_\beta \Pi^1_1\text{-}\mathsf{CA}_0$. The important thing to note is that $\{X \mid \exists n\ X \leq_H \mathcal{O}^{(n)}\}$ is a β-model of $\Pi^1_1\text{-}\mathsf{CA}_0$. Hence some β-model of T is hyp-below $\mathcal{O}^{(n)}$ for some n. But then

$$|T|^{\mathsf{ATR}_0}_\beta \leq \left|\mathsf{ATR}_0 + \text{“}\mathcal{O}^{(n)} \text{ exists”}\right|^{\mathsf{ATR}_0}_\beta = n.$$

So $|\Pi^1_1\text{-}\mathsf{CA}_0|^{\mathsf{ATR}_0}_\beta \leq \omega$. □

To calculate the β-rank of theories extending KPi, we rely on the following general theorem:

Theorem 2.7.4. *Let $T \supseteq \mathsf{KPi} +$ Axiom of Countability be a theory such that T proves "L satisfies T plus the axiom of countability" Then*

$$|T|^{\mathsf{PRS+Beta}}_\beta = \min\{M \cap \mathrm{Ord} \mid M \textit{ is a transitive model of } T\}.$$

Proof. $\leq$ follows from Lemma 2.3.2 and $\beta\text{-rank}_{T_0}(M) \leq M \cap \mathrm{Ord}$ for a transitive model M of T_0. For $\geq$, let M be a transitive model of T. By the assumption, L^M is also a transitive model of T. Let α be the least ordinal such that $L_\alpha \vDash T$. By Lemma 2.5.7, the collection

$$A = \{\xi < \alpha \mid \xi \text{ is locally countable limit admissible and every element of } L_\xi \text{ is definable in } L_\xi\}$$

has ordertype α: Suppose not, observe that A is a Δ_0-definable class in L_α, so a class function enumerating A is Σ_1-definable in L_α. Hence by Σ_1-Replacement over L_α, A becomes a member of L_α. It is impossible since it means A is bounded below α, but T proves the axiom of countability, so L_α thinks A is unbounded by Lemma 2.5.7.

Since $\xi < \alpha < \mathrm{Ord} \cap M$ for each $\xi \in A$, M contains a transitive model of $\mathrm{Th}(L_\xi)$, so $\mathrm{Th}(L_\xi) <_\beta^{\mathsf{PRS+Beta}} T$. Then by the proof of Proposition 2.5.9, we have $\beta\text{-rank}_{\mathsf{PRS+Beta}}(\mathrm{Th}(L_\xi)) \geq \xi$ for each $\xi \in A$. Hence, the desired result follows. □

Corollary 2.7.5. *1.* $|\mathsf{KPi}|_\beta^{\mathsf{PRS+Beta}} =$ *The least recursively inaccessible ordinal.*

2. $|\Pi^1_2\text{-}\mathsf{CA}_0|_\beta^{\mathsf{PRS+Beta}} =$ *The least non-projectible ordinal.*

3. $|\mathsf{ZF}^-|_\beta^{\mathsf{PRS+Beta}} =$ *The least gap ordinal.*

Proof. We only provide the proof of the first item, as the proof for the others is similar. Suppose that ι is the least recursively inaccessible ordinal. Then we have

$$|\mathsf{KPi}|_\beta^{\mathsf{PRS+Beta}} \leq |\mathsf{KPi} + \text{Countability} + (V = L)|_\beta^{\mathsf{PRS+Beta}} = \iota.$$

For the other direction, suppose that M is a transitive model of KPi. Now, let $N \subseteq L^M$ be a set of all members $a \in L^M = L \cap M$ such that L^M thinks a is hereditarily countable. If $\mathrm{Ord}^N = \mathrm{Ord}^M$, then $N = L^M$, and N becomes a model of $\mathsf{KPi} + (V = L) + \text{Countability}$, If $\mathrm{Ord}^N < \mathrm{Ord}^M$, then N becomes a model of $\mathsf{Z}_2^{\mathsf{set}} + (V = L) + \text{Countability}$. In either case, we have a transitive model $N \subseteq M$ of $\mathsf{KPi} + (V = L) + \text{Countability}$. Hence, if $T <_\beta \mathsf{KPi} + (V = L) + \text{Countability}$, then $T <_\beta \mathsf{KPi}$. □

2.8 Final Remarks

β-rank of a theory gives a different way to gauge the strength of theories, and Theorem 2.7.4 shows that β-rank of a theory is closely tied with the least height of a transitive

model of the theory. Both Σ^1_2-proof-theoretic ordinal comparison and β-rank comparison look like natural ways to compare natural theories, so they should coincide for natural theories. However, it turns out not.

First, let us observe the following sets-as-trees interpretation results:

Proposition 2.8.1. $\mathsf{ATR}_0^{\mathsf{set}}$ *proves the following:*

1. Π^1_1-$\mathsf{CA}_0^{\mathsf{set}}$ *is equivalent to "Every set is contained in an admissible set"*
2. *For* $n \geq 1$, Π^1_{n+1}-$\mathsf{CA}_0^{\mathsf{set}}$ *is equivalent to* Σ_n*-Separation.*
3. Σ^1_{n+2}-AC_0 *is equivalent to* Σ_{n+1}*-Collection.* □

(See [21, Proposition 1.4.12] for the proof of the first item. For the remaining results, see [89, Exercise VII.3.39, Lemma VII.5.3].) Combining it with Theorem 2.7.4, we have the following:

Corollary 2.8.2. *Let* $n \geq 1$.

1. $|\Pi^1_{n+1}\text{-}\mathsf{CA}_0|_\beta^{\mathsf{ATR}_0}$ *is the least ordinal* ρ *such that* L_ρ *satisfies* Σ_n*-Separation.*
2. $|\Sigma^1_{n+2}\text{-}\mathsf{AC}_0|_\beta^{\mathsf{ATR}_0}$ *is the least ordinal* ρ *such that* L_ρ *satisfies* Σ_{n+1}*-Collection.*

In particular, we have $|\Pi^1_{n+1}\text{-}\mathsf{CA}_0|_\beta^{\mathsf{ATR}_0} < |\Sigma^1_{n+2}\text{-}\mathsf{AC}_0|_\beta^{\mathsf{ATR}_0}$. □

Let me also remark that the sets-as-trees interpretation gives a bi-interpretation from Σ^1_2-AC_0 to KPi plus the axiom of countability, if we formulate KPi with Δ_0-Set Induction in place of Full Set Induction. This also shows $|\Pi^1_1\text{-}\mathsf{CA}_0|_\beta^{\mathsf{ATR}_0} < |\Sigma^1_2\text{-}\mathsf{AC}_0|_\beta^{\mathsf{ATR}_0}$. In summary, β-rank gauges Σ^1_{n+2}-AC_0 is stronger than Π^1_{n+1}-CA_0.

However, the following result shows that Π^1_{n+1}-CA_0 and Σ^1_{n+2}-AC_0 prove the same Π^1_3-consequences:

Theorem 2.8.3 ([89, Theoerem IX.4.9, Corollary IX.4.12]). *Σ^1_2-CA_0 is conservative over Π^1_1-CA_0 for Π^1_3-sentences. For $n \geq 1$, Σ^1_{n+2}-CA_0 is conservative over Π^1_{n+1}-CA_0 for Π^1_4-sentences.* □

One way to understand this phenomenon is that comparing Γ-consequences of theories involves non-β or non-ω-models. Indeed, the proof of Theorem 2.8.3 uses recursively saturated models, which are not ω-models. However, I think this explanation may not be sufficient since we can equate the two comparisons by adding a β-valid statement, namely, the transfinite induction scheme TI.

Theorem 2.8.4. *For $n \in \mathbb{N}$, Σ^1_{n+2}-$\mathsf{AC}_0 + \mathsf{TI}$ proves Σ^1_2-$\mathsf{RFN}(\Pi^1_{n+1}$-$\mathsf{CA}_0 + \mathsf{TI})$. In fact, Σ^1_{n+2}-$\mathsf{AC}_0 + \mathsf{TI}$ proves there is a transitive model of Π^1_{n+1}-$\mathsf{CA}_0 + \mathsf{TI}$.*

Proof. We first consider the case $n = 0$. The sets-as-trees interpretation interprets KPi with Full Set Induction over Σ^1_2-$\mathsf{AC}_0 + \mathsf{TI}$. For this reason, let us work over $T_0 = \mathsf{KPi} +$ Full Set Induction instead. We will show in T_0 that there is a transitive model of Π^1_1-$\mathsf{CA}^{\mathsf{set}}_0$ by constructing $L_{\omega^{\mathsf{CK}}_\omega}$.

Working in T_0, we can see that $L \vDash T_0$, so we can work in $T_0 + (V = L)$ instead. Observe that the function $\alpha \mapsto$ The least admissible ordinal greater than α is Σ_1-definable. We can prove Σ_1-recursion with the help of Σ_1-Set Induction,[3] and we can define $\omega^{\mathsf{CK}}_\omega$ by recursion. It is easy to see that $L_{\omega^{\mathsf{CK}}_\omega}$ is a model of Π^1_1-$\mathsf{CA}_0 + \mathsf{TI}$.

Now let us consider the case $n > 1$. We may also work over KP with Σ_{n+1}-Collection and Full Set Induction instead. We may further assume $V = L$, and observe that under $\mathsf{KP} + (V = L)$, Σ_{n+1}-Collection implies Σ_n-Separation, which is equivalent to the assertion "Every ordinal is less than some Σ_n-stable ordinal; i.e., an ordinal ξ such that

[3] Here is where we need TI; The proof does not work otherwise. Also, Σ_1-Induction over ω suffices to prove $\omega^{\mathsf{CK}}_\omega$ exists.

$L_\xi \prec_{\Sigma_n} L$. The function $\xi \mapsto$ The least Σ_n-stable ordinal greater than ξ is Σ_{n+1}, and Σ_{n+1}-Collection with the Full Set Induction implies Σ_{n+1}-transfinite recursion. Hence, we can define an increasing ω-chain of Σ_n-stable ordinals $\langle \sigma_m \mid m < \omega \rangle$, and $L_{\sup_{m<\omega} \sigma_m}$ is a transitive model of KP plus Σ_n-Separation.

In sum, we proved for every $n \geq 1$, Σ^1_{n+2}-$\mathsf{AC}_0 + \mathsf{TI}$ proves the existence of a β-model of Π^1_{n+1}-$\mathsf{CA}_0 + \mathsf{TI}$. Hence the Σ^1_2-reflection of Π^1_{n+1}-$\mathsf{CA}_0 + \mathsf{TI}$ follows. □

This specific example raises the following question: Can we equate the Σ^1_2-consequence comparison with the β-rank comparison by adding transfinite induction?

Question 2.8.5. Let S, T be recursive theories extending Π^1_1-$\mathsf{CA}^{\mathsf{set}}_0$ having a transitive model.[4] Are these two equivalent?

1. $|S|^{\mathsf{ATR}^{\mathsf{set}}_0}_\beta \leq |T|^{\mathsf{ATR}^{\mathsf{set}}_0}_\beta$.
2. $s^1_2(S + \text{Full Set Induction}) \leq s^1_2(T + \text{Full Set Induction})$.

Let me also point out that a similar discrepancy phenomenon also happens at a very high level; A folklore result most likely due to Woodin (unpublished, but see [53]) shows that "ZFC + There are infinitely many Woodin cardinals" cannot force $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})}$, even though the two theories are equiconsistent. Compared it to the previous result, Π^1_n-CA_0 and Σ^1_{n+1}-AC_0 are equiconsistent but Π^1_n-CA_0 cannot force Σ^1_{n+1}-AC_0. It might be possible that the two extensions of ZFC have two different 'rank' under a higher version of β-models, like EM-models or their variants introduced in Section 0.9, and the 'rank' is invariant under forcing extensions.

I also expect that considering EM-models and their variants would be unavoidable when we gauge the strength of theories beyond "$0^\sharp$ exists": For example, one can show

[4] Every transitive model of Π^1_1-$\mathsf{CA}^{\mathsf{set}}_0$ is Σ^1_2-correct by Shoenfield absoluteness theorem, so we automatically have Σ^1_2-soundness of S and T.

that if $0^\sharp$ exists and an inaccessible cardinal exists, then the β-rank of "ZFC + $0^\sharp$ exists" is less than that of $\mathrm{Th}(L)$[5] since the statement "The theory 'ZFC + $0^\sharp$ exists' has a well-founded model" is Σ^1_2, so it is also true on L. We should be able to avoid it by considering EM-models since $0^\sharp$ is an EM-model satisfying $\mathrm{Th}(L)$, but it does not contain a non-trivial EM-model of ZFC plus '$0^\sharp$ exists.'

[5] $\mathrm{Th}(L)$ is well-defined if we have $0^\sharp$.

CHAPTER 3

PROOF-THEORETIC DILATOR AND INTERMEDIATE POINTCLASSES

> The pupil dilates in the dark, and the soul dilates in misfortune and ends by finding God there.
>
> Victor Hugo, *Les Misérables*, Vol. V, Book 3, Ch. 1.

In this chapter, we analyze the connection between Pohlers' characteristic ordinals [75, 76] and Girard's Π^1_2-proof theory. Every result in this chapter appears [41] in verbatim.

3.1 Preliminaries

In this section, we review preliminary knowledge necessary for this chapter, except for quasidendroids and β-logic, which will appear in later sections separately. We also include a summary of Pohlers' work [76] to support arguments in the Introduction, although its technical details are unnecessary for this chapter.

3.1.1 Σ^1_2-singleton reals

In recursion-theoretic context, Σ^1_2-singleton real is 'reducible' to a Π^1_1-singleton real in the following manner:

Proposition 3.1.1 (Π^1_1-CA_0)**.** *If A is a Σ^1_2-singleton real, then there is a Π^1_1-singleton real B such that A is primitive recursive in B.*

Proof. Suppose that $\exists^1 Z \phi(Z, X)$ is a Σ^1_2-formula witnessing A being a Σ^1_2-singleton real for a Π^1_1-formula ϕ. By Π^1_1-uniformization theorem, we can find a Π^1_1-formula $\hat{\phi}$ uniformizing ϕ. Then consider the Π^1_1-formula

$$\psi(Y) \equiv \phi((Y)_0, (Y)_1),$$

where $(Y)_i = \{n \in \mathbb{N} \mid \langle i, n \rangle \in Y\}$. If $\hat{\phi}(B, A)$ holds for some real B, then $B \oplus A$ is a singleton real witnessed by ψ. Clearly, A is primitive recursive in $B \oplus A$. □

The previous theorem says there is no significant recursion-theoretic difference between Σ^1_2-singleton reals and Π^1_1-singleton reals. Thus, examining the recursion-theoretic behavior for Π^1_1-singleton reals is sufficient for examining that for Σ^1_2-singleton reals. However, we stick to Σ^1_2-definable reals since Σ^1_2-definable reals are more proof-theoretically natural objects.

Suzuki proved that every Δ^1_2-real is hyperarithmetical in a Π^1_1-singleton real, hinting at the connection between Σ^1_2-statements and the intermediate pointclasses. The proof is not hard, so let us include its proof for completeness.

Theorem 3.1.2 (Suzuki [94], Σ^1_2-AC_0). *For $X \subseteq \mathbb{N}^k$, X is Δ^1_2 iff there is a Π^1_1-singleton real R such that $X \in \Delta^1_1[R]$.*

Proof. Suppose that ϕ is a Π^1_1-formula witnessing R is a Π^1_1-singleton , so R is the unique real satisfying $\phi(R)$. If ψ is a Σ^1_1-formula satisfying

$$\vec{n} \in X \iff \psi(\vec{n}, R),$$

for every $\vec{n} \in \mathbb{N}^k$, then we have

$$\vec{n} \in X \iff \exists^1 Y[\phi(Y) \wedge \psi(\vec{n}, Y)],$$

so X is Σ^1_2. By applying the same argument to $\neg X$, we can derive X is Δ^1_2. Conversely, suppose that $X \in \Delta^1_2$, and let $\chi(\vec{n}, i)$ be a Σ^1_2-formula defining the characteristic function for X:

$$[\vec{n} \in X \leftrightarrow \chi(\vec{n}, 1)] \wedge [\vec{n} \notin X \leftrightarrow \chi(\vec{n}, 0)].$$

Then we have

$$Z = X \iff \forall^0 \vec{n}[(\vec{n} \in Z \to \chi(\vec{n}, 1)) \wedge (\vec{n} \notin Z \to \chi(\vec{n}, 0))].$$

Now let $\chi(\vec{n}, i) \equiv \exists^1 Y \psi(\vec{n}, i, Y)$ for some Π^1_1-formula ψ. Then by Σ^1_2-AC_0, we have

$$Z = X \iff \exists^1 Y \forall^0 \vec{n}\big[(\vec{n} \in Z \to \psi(\vec{n}, 1, (Y)_{\vec{n}})) \wedge (\vec{n} \notin Z \to \psi(\vec{n}, 0, (Y)_{\vec{n}}))\big].$$

The right-hand side has the form $\exists^1 Y \phi(Y)$ for a Π^1_1-formula ϕ. By Π^1_1-uniformization that is provable over Π^1_1-CA_0, we can find a Π^1_1-uniformization $\hat{\phi}$ of ϕ. Now let R be the unique real satisfying $\hat{\phi}(R)$. Then we have

$$Z = X \iff \forall^0 \vec{n}\big[(\vec{n} \in Z \to \psi(\vec{n}, 1, (R)_{\vec{n}})) \wedge (\vec{n} \notin Z \to \psi(\vec{n}, 0, (R)_{\vec{n}}))\big].$$

Hence

$$\vec{n} \in X \iff \psi(\vec{n}, 1, (R)_{\vec{n}}) \iff \neg\psi(\vec{n}, 0, (R)_{\vec{n}}),$$

so X is $\Delta^1_1[R]$. □

But can we get a result showing a more proof-theoretic connection between Σ^1_2-sentences and the intermediate pointclasses? The following lemma says every Σ^1_2-*sentence* is $\Sigma^0_2[R]$-sentence for some implicitly Π^1_1-definable real R:

Lemma 3.1.3 (Π^1_1-CA_0). *If σ is a Σ^1_2-sentence, then we can find a Π^1_1-singleton real R such that σ is equivalent to a $\Sigma^0_2[R]$-sentence.*

Proof. Suppose that $\sigma \equiv \exists X \phi(X)$ for a Π^1_1-formula ϕ. Then we have

$$\exists^1 X \phi(X) \iff \exists^1 M[M \text{ is a } \beta\text{-model} \wedge M \vDash \exists^1 X \phi(X)].$$

The right-hand-side of the above formula takes the form $\exists^1 M \psi(M)$ for a Π^1_1-formula ψ. Let $\hat{\psi}$ be the uniformization of ψ, and let R be a real satisfying $\hat{\psi}(R)$. Then we have

$$\exists^1 X \phi(X) \iff R \vDash \exists^1 X \phi(X), \tag{3.1}$$

and the right-hand-side is $\Sigma^0_2[R]$. □

Lemma 3.1.3 itself has no practical information since R exists only when σ holds, so the equivalence (3.1) does not make sense if σ fails. However, Lemma 3.1.3 illustrates the possibility of decomposing Π^1_2 into $\Pi^0_2[R]$ or $\Pi^1_1[R]$ for a Π^1_1-singleton real R, which motivates the connection between proof-theoretic dilators and R-recursive well-orders.

Definition 3.1.4. Suppose that T is a theory and $\phi(X)$ is a Σ^1_2-formula witnessing R being a Σ^1_2-singleton. Furthermore, suppose that T proves $\phi(X)$ holds for at most one real; That is,

$$T \vdash \forall^1 X, Y[\phi(X) \wedge \phi(Y) \to X = Y].$$

Then let us define $T[R] := T + (\exists^1 X \phi(X))$.

The reason we assume the uniqueness assertion

$$\forall^1 X, Y[\phi(X) \wedge \phi(Y) \to X = Y]$$

is a T-theorem that is usually provable over a weak theory like ACA_0. Also relatedly, in the context of our proposed main theorem, we can add a 'true Σ^1_2-sentence' to the theory T by looking at $|T|_{\Pi^1_2}(\alpha)$ for a sufficiently large α, but adding a true Π^1_2-sentence is impossible in this manner.

3.1.2 $\Pi^1_1[R]$-proof theory

In this subsection, we define the proof-theoretic ordinal augmented with a Σ^1_2-singleton R. We want to make this notion syntactic, so instead of using a semantic notion of a

real R, we use its Σ^1_2-definition to formulate $\Gamma[R]$-formulas and $\Pi^1_1[R]$ proof-theoretic ordinal.

Definition 3.1.5. Let $\phi(X)$ be a Σ^1_2-formula witnessing R being a Σ^1_2-singleton. For a lightface class Γ of formulas, the $\Gamma[R]$-*formula* is a formula of the form $\forall^1 X[\phi(X) \to \psi(X)]$ for a Γ-formula ψ.

Note that if T proves $\exists!^1 X\phi(X)$, then T proves $\forall^1 X[\phi(X) \to \psi(X)]$ is equivalent to $\exists^1 X[\phi(X) \wedge \psi(X)]$.

Definition 3.1.6. Let T be a theory proving $\exists!^1 X\phi(X)$ for a Σ^1_2-formula $\phi(X)$ witnessing R being a Σ^1_2-singleton. We say T is $\Gamma[R]$-*sound* if every T-provable $\Gamma[R]$-sentence is true; That is, if $T \vdash \forall^1 X[\phi(X) \to \psi(X)]$ for a Γ-formula $\psi(X)$, then $\psi(R)$ holds.

For a $\Pi^1_1[R]$-sound theory T such that $T \vdash \exists!^1 X\phi(X)$, let us define

$$|T|_{\Pi^1_1[R]} := \sup\{\mathrm{otp}(\alpha) \mid \alpha \text{ is } R\text{-recursive and } T \vdash \mathsf{WO}(\alpha)\}.$$

However, the expression $\mathsf{WO}(\alpha)$ is not expressible in the language of second-order arithmetic. Hence we rephrase the above definition as follows: Every R-recursive linear order takes the form $\alpha'(R)$ for some recursive function α'. Thus we can take

$$|T|_{\Pi^1_1[R]} := \sup\big\{\mathrm{otp}(\alpha'(R)) \;\big|\; \alpha' \text{ is a recursive function and } T \vdash \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha'(X))]\big\}.$$

The above definition works when T is $\Pi^1_1[R]$-sound, otherwise $\alpha'(R)$ may not even be a linear order for some α'.

From the remaining part of this subsection, let us check the properties of $\Pi^1_1[R]$-proof-theoretic ordinal. *Throughout this subsection, we always assume that $\phi(X)$ is a Σ^1_2-formula witnessing R being a Σ^1_2-singleton real and T proves $\exists!^1 X\phi(X)$.*

Let us recall the precise statement of the Kleene normal form theorem:

Theorem 3.1.7 (Kleene, ACA_0). *Let $\chi(X, n)$ be a Π^1_1-formula. Then we can effectively find a primitive recursive linear order $\alpha(X, n)$ such that $\chi(X, n) \leftrightarrow \mathsf{WO}(\alpha(X, n))$ for every real X and a natural number n.* □

For later use, let us prove the following version of the fixed point theorem:

Lemma 3.1.8. *For a natural number k, we can find another natural number e such that*

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 n[\{\overline{e}\}^X(n) \simeq \{\{\overline{k}\}^X(\overline{e})\}^X(n)].$$

Proof. By s-m-n-theorem for oracles, we can find a primitive recursive function h (which does not depend on the choice of X) such that

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 m, n[\{h(m)\}^X(n) \simeq \{\{m\}^X(m)\}^X(n)].$$

Let us find m such that $\{k\}^X \circ h$ is $\{m\}^X$, uniformly on X. Furthermore, we have

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 n[\{\overline{k}\}^X(h(n)) \simeq \{\overline{m}\}^X(n)].$$

Then we have

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 n[\{h(\overline{m})\}^X(n) \simeq \{\{\overline{m}\}^X(\overline{m})\}^X(n) \simeq \{\{\overline{k}\}^X(h(\overline{m}))\}^X(n)].$$

Thus $m = h(e)$ is the desired natural number. □

The following effective version of $\Sigma^1_1[R]$-boundedness theorem has an important role in establishing properties of $\Pi^1_1[R]$-proof-theoretic ordinal. Its proof follows from a variation of that by Rathjen [77, Lemma 1.1], and let us define a preliminary notion for the proof:

Definition 3.1.9. Consider the $\Pi^0_2[X]$-formula

$$\mathsf{RecLO}(X, e) \equiv \text{The relation } \prec^X_e := \{\langle i, j\rangle \mid \{e\}^X(\langle i, j\rangle) = 0\} \text{ is a linear order.}$$

Let us recall that the linear order $\alpha(X, n)$ in the statement of the Kleene normal form theorem is primitive recursive, meaning that there is a primitive recursive function f such that $f(X, n, \cdot)$ computes the characteristic function for $\alpha(X, n)$. Hence we can find a natural number e, which only depends on α and not on X and n, such that

$$\forall^0 i, j[\{e\}^X(n)(\langle i, j\rangle) = 0 \iff i <_{\alpha(X,n)} j]. \tag{3.2}$$

This fact is also provable in ACA_0.

Lemma 3.1.10 (Effective $\Sigma^1_1[R]$-boundedness, ACA_0). *Let T be a theory extending ACA_0. Suppose that $\psi(X, n)$ is a $\Sigma^1_1[X]$-formula such that*

$$T \vdash \forall^1 R[\phi(R) \to [\forall^0 n[\psi(R, n) \to \mathsf{RecLO}(R, n) \wedge \mathsf{WO}(\prec^R_n)]]]. \tag{3.3}$$

(Informally, it says T proves $\{n \in \mathbb{N} \mid \psi(R, n)\}$ is a set of indices for R-recursive well-orders of the form $\prec^R_n$.) Then we can find a natural number e such that

$$T \vdash \forall^1 R\phi(R) \to [\mathsf{RecLO}(R, \overline{e}) \wedge \mathsf{WO}(\prec^R_{\overline{e}}) \wedge \neg\psi(R, \overline{e})].$$

Proof. By Kleene normal form theorem and the previous discussion about (3.2), we can find a natural number k that only depends on ψ such that

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 n\ \mathsf{RecLO}(X, \{\overline{k}\}^X(n)) \wedge [\neg\psi(X, n) \leftrightarrow \mathsf{WO}(\prec^{\mathcal{X}}_{\{\overline{k}\}^X(n)})]. \tag{3.4}$$

By Lemma 3.1.8, we can find a natural number e such that

$$\mathsf{ACA}_0 \vdash \forall^1 X \forall^0 n[\{\overline{e}\}^X(n) \simeq \{\{\overline{k}\}^X(\overline{e})\}^X(n)].$$

We claim that e is the desired natural number: *Reasoning over T*, observe that $\{\overline{e}\}^X$ and $\{\{\overline{k}\}^X(\overline{e})\}$ are extensionally the same, so $\mathsf{RecLO}(X, \overline{e})$ iff $\mathsf{RecLO}(X, \{\overline{k}\}^X(\overline{e}))$, and the latter holds. Hence $\mathsf{RecLO}(X, \overline{e})$ holds for every choice of X.

Now we claim that $\mathsf{WO}(\prec^R_{\overline{e}})$ holds. Suppose the contrary assume that $\neg\mathsf{WO}(\prec^R_{\overline{e}})$ holds, which is equivalent to $\neg\mathsf{WO}(\prec^R_{\{\overline{k}\}^X(\overline{e})})$. Then by (3.4), we have $\psi(R, \overline{e})$. However,

we assumed that (3.3) holds, so $\mathsf{WO}(\prec_{\overline{e}}^R)$, a contradiction. $\neg\psi(R,\overline{e})$ follows from a part of the previous argument. □

Then we can prove the following, which has essentially the same proof in [78]:

Proposition 3.1.11. *Let T be a $\Pi^1_1[R]$-sound extension of $\mathsf{ACA}_0 + \exists!^1 X\phi(X)$. Then*

$$|T|_{\Pi^1_1[R]} = \sup\{\mathrm{otp}(\alpha) \mid \alpha \textit{ is arithmetical-in-}R \textit{ well-order such that } T \vdash \mathsf{WO}(\alpha)\}.$$

Proof. Let us clarify the meaning of arithmetical-in-R linear order in our context: α is arithmetical-in-R means there is a first-order formula $\psi(n,X)$ such that $\alpha = \{n \mid \psi(n,R)\}$. Then the statement '$\alpha$ is a linear order' is also arithmetical-in-R, whose formal statement is

$$\forall^1 R[\phi(R) \to \forall^0 i,j,k\psi(\langle i,i\rangle,R) \wedge [\psi(\langle i,j\rangle),R) \vee \psi(\langle j,i\rangle,R)] \\ \wedge [\psi(\langle i,j\rangle,R) \wedge \psi(\langle j,k\rangle,R) \to \psi(\langle i,k\rangle,R)]].$$

Similarly, the statement 'α is a well-order' is $\Pi^1_1[R]$.

Let $\mathsf{Emb}(\alpha,\beta)$ be the Σ^1_1-statement saying 'There is an embedding from a linear order α to another linear order β.' Now let $\alpha(R)$ be an arithmetical-in-R linear order over $\mathbb{N}$ such that $T \vdash \mathsf{WO}(\alpha(R))$. Then observe that

$$T \vdash \forall^1 R\phi(R) \to [\forall^0 n[\mathsf{RecLO}(R,n) \wedge \mathsf{Emb}(\prec_n^R, \alpha(R)) \to \mathsf{WO}(\prec_n^R)]],$$

so Lemma 3.1.10 applied to the formula $\psi(R,n) \equiv \mathsf{RecLO}(R,n) \wedge \mathsf{Emb}(\prec_n^R,\alpha(R))$, we can find a natural number e such that

$$T \vdash \forall^1 R\phi(R) \to [\mathsf{RecLO}(R,\overline{e}) \wedge \mathsf{WO}(\prec_{\overline{e}}^R) \wedge \neg\mathsf{Emb}(\prec_{\overline{e}}^R, \alpha(R))].$$

The formula in the right-hand-side of the turnstile is $\Pi^1_1[R]$. Hence by $\Pi^1_1[R]$-soundness of T, $\forall^1 R\phi(R) \to \neg\mathsf{Emb}(\prec_{\overline{e}}^R,\alpha(R))$ is a true $\Pi^1_1[R]$-statement. Hence $\mathrm{otp}(\alpha(R)) \leq \mathrm{otp}(\prec_e^R) \leq |T|_{\Pi^1_1[R]}$.[1] □

[1] The second inequality is strict since we can prove $\alpha < |T|_{\Pi^1_1[R]}$ implies $\alpha + 1 < |T|_{\Pi^1_1[R]}$.

Proposition 3.1.12. *Let T be a $\Pi^1_1[R]$-sound $\Sigma^1_1[R]$-definable theory. Then $|T|_{\Pi^1_1[R]} < \delta(R)$, where $\delta(R)$ is the supremum of the ordertype of R-recursive well-orders.*

Proof. Let

$$X = \{e \in \mathbb{N} \mid \mathsf{RecLO}(R, e) \wedge T \vdash \ulcorner \forall^1 Y[\phi(Y) \to \mathsf{WO}(\prec_e^Y)]\urcorner\}.$$

Then X is $\Sigma^1_1[R]$, so the usual $\Sigma^1_1[R]$-boundedness theorem implies $\sup\{\mathrm{otp}(\prec_e^R) \mid e \in X\} < \delta(R)$. □

We prove Kriesel's theorem for $\Pi^1_1[R]$-proof-theoretic ordinal, which states adding the true $\Sigma^1_1[R]$-sentence to a theory does not change the $\Pi^1_1[R]$-proof-theoretic ordinal of the theory. [78, Proposition 2.24] uses Löb's theorem to prove the given proposition for $R = \emptyset$. We use a different way akin to the first equivalence in [102], and the following variation of [71, Lemma 5.2] has a focal role in the proof:

Lemma 3.1.13 (ACA_0)**.** *Let $\alpha(X)$ and $\beta(X)$ be recursive functions such that both of them are linear orders for every real X. Then we can define a new recursive function $(\alpha \vee \beta)(X)$ such that for every real X,*

1. *$(\alpha \vee \beta)(X)$ is a linear order.*
2. *$\mathsf{WO}((\alpha \vee \beta)(X)) \leftrightarrow \mathsf{WO}(\alpha(X)) \vee \mathsf{WO}(\beta(X))$.*
3. *If $\beta(X)$ is ill-founded, then there is an embedding $\alpha(X) \to (\alpha \vee \beta)(X)$.*

Proof. Let $T(X)$ be the set of sequences $\langle (a_i, b_i) \mid i < n\rangle$ such that each of $\langle a_i \mid i < n\rangle$ and $\langle b_i \mid i < n\rangle$ are decreasing sequences over $\alpha(X)$ and $\beta(X)$ respectively. Since both α and β are recursive functions, T is also a recursive function, and $T(X)$ is an X-recursive tree over $\mathbb{N} \times \mathbb{N}$. (We understand $\alpha(X)$ and $\beta(X)$ have the field a subset of

$\mathbb{N}$, which is possible since both of them are X-recursive.) Then consider the linear order $(\alpha \vee \beta)(X)$ defined by $(T(X), \leq_{\mathsf{KB}})$; that is,

$$\langle (a_i, b_i) \mid i < n \rangle <_{\mathsf{KB}} \langle (a'_i, b'_i) \mid i < n' \rangle$$

if and only if either

1. $\langle (a_i, b_i) \mid i < n \rangle$ is a proper initial segment of $\langle (a'_i, b'_i) \mid i < n' \rangle$, or
2. There is $i < n$ such that $(a_i, b_i) < (a'_i, b'_i)$, where the comparison is done under the lexicographic order over $\mathbb{N} \times \mathbb{N}$.

$\leq_{\mathsf{KB}}$ is Δ^0_1-definable, so $(\alpha \vee \beta)$ is a recursive function. The remaining part of the proof is identical to that of [71, Lemma 5.2], so we omit the proof. □

Proposition 3.1.14. *Let T be a $\Pi^1_1[R]$-sound extension of $\mathsf{ACA}_0 + \exists!^1 X \phi(X)$. Then $|T|_{\Pi^1_1[R]} = |T + \sigma|_{\Pi^1_1[R]}$ for a true $\Sigma^1_1[R]$-formula σ.*

Proof. Let σ be $\exists^1 X[\phi(X) \to \psi(X)]$ for some Σ^1_1-formula $\psi(X)$ with the only free variable X. By the Kleene normal form theorem, we have a primitive recursive $\alpha(X)$ such that

$$\mathsf{ACA}_0 \vdash \forall^1 X[\neg\psi(X) \leftrightarrow \mathsf{WO}(\alpha(X))].$$

Suppose that we have a primitive recursive well-order $\beta(X)$ such that

$$T + \sigma \vdash \forall^1 X[\phi(X) \to \mathsf{WO}(\beta(X))].$$

Recall that every $\Pi^1_1[R]$-sentence has the form $\forall^1 X[\phi(X) \to \mathsf{WO}(\beta(X))]$ for some primitive recursive linear order β by the Kleene normal form theorem. Then we have

$$T \vdash \neg\sigma \vee \forall^1 X[\phi(X) \to \mathsf{WO}(\beta(X))],$$

which is equivalent to

$$T \vdash \forall^1 X[\phi(X) \to [\mathsf{WO}(\alpha(X)) \vee \mathsf{WO}(\beta(X))]].$$

By Lemma 3.1.13, we have

$$T \vdash \forall^1 X[\phi(X) \to \mathsf{WO}((\alpha \vee \beta)(X))].$$

By $\Pi^1_1[R]$-soundness of T, the sentence $\forall^1 X[\phi(X) \to \mathsf{WO}((\alpha \vee \beta)(X))]$ is true. Hence we have $\mathsf{WO}((\alpha \vee \beta)(R))$. Also, the definition of $|T|_{\Pi^1_1[R]}$ implies $\mathrm{otp}((\alpha \vee \beta)(R)) < |T|_{\Pi^1_1[R]}$. Since σ is true, $\psi(R)$ is also true, which is equivalent to $\neg\mathsf{WO}(\alpha(R))$. This implies $\beta(R)$ embeds to $(\alpha \vee \beta)(R)$, so $\mathrm{otp}(\beta(R)) \leq \mathrm{otp}((\alpha \vee \beta)(R)) < |T|_{\Pi^1_1[R]}$. $\square$

3.1.3 Characterizing $\Pi^1_1[R]$-proof theory

Walsh [103] proved that for an arithmetically definable Π^1_1-sound theories, the following are equivalent:

1. Comparing their proof-theoretic ordinals.
2. Comparing their Π^1_1-consequences modulo true Σ^1_1-statements.
3. Comparing their Π^1_1-reflection, which is a Π^1_1-analogue of their consistency.

We can establish a similar characterization to $\Pi^1_1[R]$-proof-theoretic ordinal for a Σ^1_2-singleton real R. Throughout this section, let us fix a Σ^1_2-formula $\phi(X)$ witnessing a real R being a Σ^1_2-singleton.

Definition 3.1.15. Let S, T be two theories extending $\mathsf{ACA}_0 + \exists^1! X \phi(X)$. Furthermore, assume that they are $\Pi^1_1[R]$-sound. For a sentence σ, define $T \vdash^{\Sigma^1_1[R]} \sigma$ iff there is a Σ^1_1-formula $\psi(X)$ with only free variable X such that

$$T + (\exists^1 X \phi(X) \wedge \psi(X)) \vdash \sigma.$$

Also, define $S \subseteq^{\Sigma^1_1[R]}_{\Pi^1_1[R]} T$ if and only if for every Π^1_1-formula $\psi(X)$ with only free variable X,

$$S \vdash^{\Sigma^1_1[R]} \forall^1 X(\phi(X) \to \psi(X)) \implies T \vdash^{\Sigma^1_1[R]} \forall^1 X(\phi(X) \to \psi(X)).$$

Then we can prove the following:

Theorem 3.1.16. *Let S, T be $\Pi^1_1[R]$-sound extensions of $\mathsf{ACA}_0 + \exists^1! X\phi(X)$. Then*

$$S \subseteq^{\Sigma^1_1[R]}_{\Pi^1_1[R]} T \iff |S|_{\Pi^1_1[R]} \leq |T|_{\Pi^1_1[R]}.$$

Theorem 3.1.16 follows from the following lemma:

Lemma 3.1.17. *Let T be $\Pi^1_1[R]$-sound extensions of $\mathsf{ACA}_0 + \exists^1! X\phi(X)$. If $\alpha(X)$ is a recursive function, then*

$$T \vdash^{\Sigma^1_1[R]} \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))] \iff \alpha(R) < |T|_{\Pi^1_1[R]}.$$

Proof. In one direction, suppose that $T \vdash^{\Sigma^1_1[R]} \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(R))]$ holds. Let $\psi(X)$ be a Σ^1_1-formula such that

$$T + \exists^1 X[\phi(X) \wedge \psi(X)] \vdash \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(R))] \tag{3.5}$$

By Kleene normal form theorem, we can find a recursive $\beta(X)$ such that

$$\mathsf{ACA}_0 \vdash \forall^1 X[\psi(X) \leftrightarrow \neg\mathsf{WO}(\beta(X))].$$

Hence from (3.5), we have

$$T \vdash \forall^1 X[\phi(X) \to [\mathsf{WO}(\beta(X)) \vee \mathsf{WO}(\alpha(X))]].$$

By Lemma 3.1.13, we can find a new recursive function $(\alpha \vee \beta)(X)$ such that

$$\mathsf{ACA}_0 \vdash \forall^1 X[\mathsf{WO}((\alpha \vee \beta)(X)) \leftrightarrow \mathsf{WO}(\alpha(X) \vee \beta(X))].$$

Hence $T \vdash \forall^1 X[\phi(X) \to \mathsf{WO}((\alpha \vee \beta)(X))]$, so $(\alpha \vee \beta)(R) < |T|_{\Pi^1_1[R]}$.[2] However, $\beta(R)$ is ill-founded, so $\alpha(R) \leq (\alpha \vee \beta)(R)$, which proves the desired conclusion.

Conversely, assume that $\alpha(R) < |T|_{\Pi^1_1[R]}$. Then we can find a recursive $\beta(X)$ such that $\alpha(R) \leq \beta(R)$ and $T \vdash \forall^1 X[\phi(X) \to \mathsf{WO}(\beta(X))]$. $\alpha(R) \leq \beta(R)$ implies the $\Sigma^1_1[R]$-statement

$$\sigma \equiv \exists^1 X\big[\phi(X) \wedge \exists^1 F\big(F \text{ is an embedding from } \alpha(X) \text{ to } \beta(X)\big)\big]$$

is true. Then we can see that $T + \sigma$ proves $\forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$, as desired. □

Proof of Theorem 3.1.16. Suppose that $S \subseteq^{\Sigma^1_1[R]}_{\Pi^1_1[R]} T$ holds. If $\xi < |S|_{\Pi^1_1[R]}$, then we can find a recursive $\alpha(X)$ such that $\xi < \alpha(R)$ and $S \vdash \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$. By the assumption, we have $T \vdash^{\Sigma^1_1[R]} \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$, so $\alpha(R) < |T|_{\Pi^1_1[R]}$ by Lemma 3.1.17. Hence $\xi \leq \alpha(R) < |T|_{\Pi^1_1[R]}$. Since ξ is arbitrary, we have $|S|_{\Pi^1_1[R]} \leq |T|_{\Pi^1_1[R]}$.

Conversely, if $|S|_{\Pi^1_1[R]} \leq |T|_{\Pi^1_1[R]}$ and $S \vdash^{\Sigma^1_1[R]} \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$ for some recursive $\alpha(X)$, then $\alpha(R) < |T|_{\Pi^1_1[R]}$. This implies $T \vdash^{\Sigma^1_1[R]} \forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$. Kleene normal form implies every $\Pi^1_1[R]$-sentence takes the form $\forall^1 X[\phi(X) \to \mathsf{WO}(\alpha(X))]$ for some recursive $\alpha(X)$, so we have $S \subseteq^{\Sigma^1_1[R]}_{\Pi^1_1[R]} T$. □

It is possible to prove the second equivalence for $\Pi^1_1[R]$-proof theory, whose proof follows from a modification of [103, §3.2] or [39, §6.2]. Hence let me leave its proof to the reader:

Theorem 3.1.18. *Let S, T be arithmetically definable $\Pi^1_1[R]$-sound theories extending $\mathsf{ACA}_0 + \exists^1 X \phi(X)$. Then*

$$|S|_{\Pi^1_1[R]} \leq T_{\Pi^1_1[R]} \iff \mathsf{ACA}_0 + \exists^1! X \phi(X) \vdash^{\Sigma^1_1[R]} \Pi^1_1[R]\text{-}\mathsf{RFN}(T) \to \Pi^1_1[R]\text{-}\mathsf{RFN}(S),$$

[2] We can see that $|T|_{\Pi^1_1[R]}$ is always a limit ordinal, so the desired inequality follows.

where

$$\Pi^1_1[R]\text{-}\mathsf{RFN}(T) \equiv \forall \psi(X) \in \Pi^1_1[T \vdash \phi(X) \to \psi(X) \implies \vDash_{\Pi^1_1} \phi(R) \to \psi(R)].$$

3.2 Pohlers' characteristic ordinals

In this section, we review Pohlers' framework presented in [76]. We may view that proof-theoretic ordinal gauges the strength of a theory over the structure $\mathbb{N}$ of natural numbers; By Proposition 3.1.11, $|T|_{\Pi^1_1}$ is equal to the supremum of ordertype of well-orders definable in $\mathbb{N}$ whose well-foundeness is a theorem of T. Pohlers generalized proof-theoretic ordinal by replacing $\mathbb{N}$ with an *acceptable* countable structure:

Definition 3.2.1. Let $\mathfrak{M}$ be a countable structure in the language $\mathcal{L}$. A structure is *acceptable* if the following *coding scheme* are first-order definable in $\mathfrak{M}$:

1. A copy of $(\mathbb{N}, \leq)$.
2. A map $x_0, \cdots, x_{m-1} \mapsto \langle x_0, \cdots, x_{m-1} \rangle$.
3. The unary predicate Seq for the set of finite $\mathfrak{M}$-sequences.
4. A function lh returning a length of given finite sequence. (If $\neg\mathrm{Seq}(x)$, $\mathrm{lh}(x) = 0$.)
5. A projection function $q(x, i)$ taking a finite sequence x and $i \in \mathbb{N}$ (from the copy in $\mathfrak{M}$) and returning the ith component of x. (Otherwise, return 0.)

Then let us consider the new language $\mathcal{L}(\mathfrak{M})$ augmented with the elements of $\mathfrak{M}$ as constant symbols. An $\mathcal{L}(\mathfrak{M})$-theory T is an *acceptable axiomatization of* $\mathfrak{M}$ if T defines $\mathfrak{M}$ and its coding scheme, $\mathfrak{M}$-sound, and proves the following:[3]

[3] Pohlers did not state the definition of acceptable axiomatization as a single definition. The following definition is a combination of his description in [76, p38, Footnote 14, p50].

1. Every true atomic formula in the language $(\mathbb{N}, \leq, \langle\cdot\rangle, \mathrm{Seq}, \mathrm{lh}, q)$.
2. Induction schema on the copy of $(\mathbb{N}, \leq)$ for interpreted $\mathfrak{M}$-formulas.

Our choice of T is first-order. However, we want to gauge Π^1_1-consequences of T to gauge its strength even when T is a first-order arithmetic. To cope with this situation, we assume T has a unary predicate X corresponding to a free set variable, and allow it to occur in the induction scheme. We call a sentence of the form $\phi(X)$ a *pseudo* Π^1_1-*sentence.*

Definition 3.2.2 ([76, p48])**.** A countable structure $\mathfrak{M}$ over $\mathcal{L}$ is *strictly acceptable* if it is acceptable and

1. There is an $\mathfrak{M}$-definable well-order over $\mathfrak{M}$.
2. There is a first-order definable coding $\ulcorner F \urcorner$ for $\mathcal{L}$-formulas with parameters in $\mathfrak{M}$ such that $\ulcorner G \urcorner \in \mathrm{CS}(\ulcorner F \urcorner)$ is first-order expressible in $\mathfrak{M}$. (See [74, Definition 5.3.3] or [76, p40] for the definition of CS.)

Definition 3.2.3 ([76, Definition 3.15])**.** For an acceptable structure $\mathfrak{M}$ in the language $\mathcal{L}$ and an acceptable axiomatization T of $\mathfrak{M}$, define

- $\delta^{\mathfrak{M}} = \sup\{\mathrm{otp}(\prec) \mid \prec \text{ is definable in } \mathfrak{M} \text{ with parameters and } \mathfrak{M} \vDash \mathsf{TI}(\prec, X)\}$,
- $\delta^{\mathfrak{M}}(T) = \sup\{\mathrm{otp}(\prec) \mid \prec \text{ is definable in } \mathfrak{M} \text{ with parameters and } T \vdash \mathsf{TI}(\prec, X)\}$.

Here X is a fresh predicate variable, and $\mathsf{TI}(\prec, X)$ states the transfinite induction along $\prec$ over X:

$$\mathsf{TI}(\prec, X) \equiv \forall x[\forall y[y \prec x \to y \in X] \to x \in X] \to (\mathrm{field}(\prec) \subseteq X).$$

Pohlers proved for a strictly acceptable $\mathfrak{M}$ and its acceptable axiomatization T of $\mathfrak{M}$, adding the first-order truth about $\mathfrak{M}$ does not change the 'strength' of T in the sense that $\delta^{\mathfrak{M}}(T) = \delta^{\mathfrak{M}}(T + \mathrm{Th}(\mathfrak{M}))$. (See [76, Theorem 3.21].) Then Pohlers argued that the only way to increase the strength of T is by adding axioms for universes above $\mathfrak{M}$. The extension of $\mathfrak{M}$ what Pohlers considered is a *Spector class*:

Definition 3.2.4. A *Spector class* Γ is a collection of relations over $\mathfrak{M}$ such that

1. Every atomic predicate and function over $\mathfrak{M}$, and their complements are in Γ. (For functions, consider their graph instead.)
2. Γ contains coding scheme over $\mathfrak{M}$.
3. Γ is closed under $\cap$, $\cup$, $\exists^0$, $\forall^0$, and trivial combinatorial substitutions.[4]
4. Γ has a *universal set*; That is, for each $n \in \mathbb{N}$ there is an $(n+1)$-ary relation $U \in \Gamma$ such that every n-ary $R \in \Gamma$ is a section of U.
5. Γ has the *prewellordering property*; That is, for every $P \in \Gamma$ there is a *norm* $\sigma_P \colon P \to \mathrm{Ord}$ such that the relations

   (a) $\vec{m} \leq^*_P \vec{n} \iff P(\vec{m}) \wedge [P(\vec{n}) \to (\sigma(\vec{m}) \leq \sigma(\vec{n}))]$, and

   (b) $\vec{m} <^*_P \vec{n} \iff P(\vec{m}) \wedge [P(\vec{n}) \to (\sigma(\vec{m}) < \sigma(\vec{n}))]$

   are both in Γ.

An example of a Spector class is the set of Π^1_1-definable sets with a parameter real R over $\mathbb{N}$. It is well-known that every Π^1_1-set with parameter real R is the least fixedpoint of an R-recursive monotone operator; If Γ^∞ is the least fixedpoint of a monotone operator Γ, then we can 'rank' $n \in \Gamma^\infty$ by looking for the least α such that $n \in \Gamma^\alpha$. (Here we define $\Gamma^0 = \emptyset$ and $\Gamma^\alpha = \bigcup_{\xi<\alpha} \Gamma(\Gamma^\xi)$.) This rank gives the *norm* of Γ^∞.

[4] Trivial combinatorial substitution is a map that is a composition of projection maps and the tuple map; See [67, p312].

There are many Spector classes above a given structure $\mathfrak{M}$. For example, both Π^1_1-sets and Σ^1_2-sets form Spector classes over $\mathbb{N}$. Each Spector class can be viewed as a universe above $\mathfrak{M}$, and let us consider them collectively:

Definition 3.2.5 ([76, Definition 3.25])**.** For a Spector class Γ and $R \in \Gamma$, define

$$o(R) = \sup\{\sigma(\vec{x}) + 1 \mid R(\vec{x}) \wedge \sigma \text{ is a } \Gamma\text{-norm on } R.\}$$

For $\Lambda \subseteq \Gamma$, define $o(\Lambda) = \sup\{o(R) \mid R \in \Lambda\}$. We define the *Spector spectrum* of $\mathfrak{M}$ by the class of all Spector classes above $\mathfrak{M}$, an the *ordinal spectrum of* $\mathfrak{M}$ by $\mathrm{Spec}^{\mathfrak{M}} = \{o(\Gamma) \mid \Gamma \text{ is a Spector class above } \mathfrak{M}\}$. $\kappa_{\mathfrak{M}}$ is the least element of $\mathrm{Spec}^{\mathfrak{M}}$, or equivalently, $\kappa_{\mathfrak{M}} = o(\mathsf{SP}_{\mathfrak{M}})$.

Among Spector classes over $\mathfrak{M}$, the most natural one we should consider is the *least Spector class over* $\mathfrak{M}$, which is the intersection of every Spector class over $\mathfrak{M}$, which we will denote by $\mathsf{SP}_{\mathfrak{M}}$. For an acceptable $\mathfrak{M}$, we have $\delta^{\mathfrak{M}} = o(\mathsf{SP}_{\mathfrak{M}})$ (cf. [76, Theorem 3.34]), that is, the supremum of $\mathsf{SP}_{\mathfrak{M}}$-norms is precisely the supremum of the ordertype of $\mathfrak{M}$-definable well-orders. Hence we can use norms over a Spector class to describe the strength of an acceptable axiomatization of $\mathfrak{M}$:

Definition 3.2.6 ([76, Definition 3.29])**.** Let T be an acceptable axiomatization of a universe above $\mathfrak{M}$. For a Spector class Γ over $\mathfrak{M}$, define

$$o^{\Gamma}(T) = \sup\{\sigma_R(\vec{x}) + 1 \;\big|\; R = \{\vec{x} \in |\mathfrak{M}|^k \mid F(\vec{x})\} \in \Gamma \wedge T \vdash F(\vec{x})\}.$$

Then define $\mathrm{Spec}^{\mathfrak{M}}(T) = \{o^{\Gamma}(T) \mid \Gamma \text{ is a Spector class over } \mathfrak{M}\}$. We also define $\kappa^{\mathfrak{M}}(T) = o^{\mathsf{SP}_{\mathfrak{M}}}(T) = \min \mathrm{Spec}^{\mathfrak{M}}(T)$.

Again, for a strictly acceptable $\mathfrak{M}$ and accpetable axiomatization T of $\mathfrak{M}$ proving Weak Kőnig's lemma, we have $\delta^{\mathfrak{M}}(T) = \kappa^{\mathfrak{M}}(T)$ [74, Theorem 6.7.4]. We may think of a Spector class over $\mathfrak{M}$ as a 'measuring stick' gauging the strength of T with different

lengths and different scales. A larger Spector class can be thought of as a longer measuring stick with a coarser scale. Although there is no clear limit of the Spector classes over $\mathfrak{M}$, Pohlers focused only on the Spector class of the following form: For ξ less than the least recursively inaccessible ordinal,

1. $\mathsf{SP}^0_{\mathfrak{M}} = \mathfrak{M}$.
2. $\mathsf{SP}^{\xi+1}_{\mathfrak{M}}$ is the least Spector class over $(\mathfrak{M}; \mathsf{SP}^{\xi}_{\mathfrak{M}})$.
3. $\mathsf{SP}^{\xi}_{\mathfrak{M}} = \bigcup_{\eta<\xi} \mathsf{SP}^{\eta}_{\mathfrak{M}}$ if ξ is a limit ordinal.[5]

We also define $\kappa^{\mathfrak{M}}_0 = 0$, $\kappa^{\mathfrak{M}}_{\mu+1} = o(\mathsf{SP}^{\mu+1}_{\mathfrak{M}})$, and $\kappa^{\mathfrak{M}}_{\lambda} = \sup\{\kappa^{\mathfrak{M}}_{\xi} \mid \xi < \lambda\}$ for a limit λ. When we understand $\mathsf{SP}^{\xi}_{\mathfrak{M}}$ as a structure, it is the expansion of $\mathfrak{M}$ with new sets obtained from the iterated Spector operator as predicates. Note that $\mathsf{SP}^{\xi}_{\mathfrak{M}}$ is a *first-order* structure, so for example, $\mathsf{SP}^{\xi}_{\mathbb{N}}$ is a structure of the first-order arithmetic with extra predicates, and not a model of second-order arithmetic.

Theorem 3.2.7 (Pohlers [76, Remark 3.35]). *For an acceptable structure $\mathfrak{M}$ and an ordinal $\mu \geq 1$ less than the least recursively inaccessible ordinal,*

$$o^{\mathfrak{M}_{\mu+1}}(\mathrm{Th}(\mathfrak{M}_{\mu}) + \textit{Mathematical Induction}) = \varepsilon_{\kappa^{\mathfrak{M}}_{\mu}+1}.$$

Hence if $\mathfrak{M}$ is strictly acceptable, then we have

$$\delta^{\mathfrak{M}_{\mu}}(\mathrm{Th}(\mathfrak{M}_{\mu}) + \textit{Mathematical Induction}) = \varepsilon_{\kappa^{\mathfrak{M}}_{\mu}+1}.$$

In particular, if $\mathfrak{M} = \mathbb{N}$, then $\mathbb{N}$ is strictly acceptable and $\kappa^{\mathbb{N}}_{\mu} = \omega^{\mathsf{CK}}_{\mu}$ for μ less than the least recursively inaccessible ordinal. (See [76, Remark 3.33].) The following theorem shows the connection between Pohlers' characteristic ordinals and $\Pi^1_1[R]$-proof theory. Its proof involves a precise definition of iterated hyperjumps, so we postpone its proof until Section 3.A.

[5] For a limit ξ, SP^{ξ} is not a Spector class unless ξ is recursively inaccessible.

Proposition 3.2.8. *Let ξ be a successor ordinal less than the least recursively inaccessible ordinal. Suppose that T is a strictly acceptable axiomatization of $\mathsf{SP}^{\xi}_{\mathbb{N}}$ containing* $\mathrm{Th}(\mathsf{SP}^{\xi}_{\mathbb{N}})$. *Then we have*

$$\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(T) = \left|\mathsf{ACA}_0 + \text{`}\mathsf{HJ}^{\xi}(\emptyset) \textit{ exists'} + T \restriction \mathsf{HJ}^{\xi}(\emptyset)\right|_{\Pi^1_1[\mathsf{HJ}^{\xi}(\emptyset)]},$$

where $T \restriction R$ is the theory obtained from T by restricting its language to that of first-order arithmetic with an extra unary predicate for R.

We will also see later that the ξth iterated hyperjump $\mathsf{HJ}^{\xi}(\emptyset)$ of the empty set for ξ less than the least recursively inaccessible ordinal is a Π^1_1-singleton.

3.3 β-logic

In this section, we review Girard's β-logic, which is the right framework for Π^1_2-proof theory. This part is included not only because of the reader's convenience but also because arguments in this section will be reused to compute the 'rank' of a Σ^1_2-singleton real. Most results in this section are not new and appeared in Girard's [29]. But, many parts of the definitions underwent non-trivial modifications to admit their formulations over a fragment of second-order arithmetic.

3.3.1 Defining β-logic

Girard defined the notion of β-logic [29] to analyze Π^1_2-consequences of a theory. To see how β-logic works, let us recall what ω-logic is: ω-logic provides a satisfactory framework for Π^1_1-sentences. The semantics is based on ω-models, and the syntax of ω-logic

uses infinitary deduction for quantifiers over natural numbers. Likewise, β-logic uses β-models for its semantics. The β-model of arithmetic is Π^1_1-correct ω-models, and every β-model admits its set-theoretic translation (See [89, Ch. VI] for more details.) Hence we can regard β-models as models with ordinals. We want to liberate the notion of β-model to languages other than the language of arithmetic, and axiomatic systems different from ACA_0 or ATR_0. Thus we define β-models as models with well-orders, which is slightly different from, but will turn out to be equivalent to β-models in second-order arithmetic. Handling β-logic over second-order arithmetic (Especially, over systems strictly weaker than ATR_0) is inconvenient, but often unavoidable.

Definition 3.3.1. A β-*language* is a language with a distinguished type Ord with a distinguished predicate $\leq^{\mathrm{Ord}}$, and only terms of type Ord are variables. When we speak of a formula of a β-language, we always assume that it has no free variables of type Ord.

A β-*theory* is a theory over a β-language. For a β-language $\mathcal{L}$, a β-*model of* $\mathcal{L}$ is a model M in a usual sense but Ord^M is a well-order, and $(\leq^{\mathrm{Ord}})^M$ is the linear order of the well-order Ord^M. $\mathrm{Ord}^M = 0$ is possible.

Girard's original definition for a β-model requires Ord^M to be ordinal. Our definition is slightly more general, and this definition allows us to formulate β-logic in second-order arithmetic.[6] Like other logics, we can define the semantic implication:

Definition 3.3.2. A formula ϕ of a β-theory T is β-*valid in* T if every closed instantiation is valid over a β-model M of T. We write it by $T \models^{\beta} \phi$.

The following example shows the connection between Girard's β-models and β-models in second-order arithmetic:

Example 3.3.3 ([29, Example 10.1.3.])**.** Consider the language $\mathcal{L}$ extending the language

[6] Vauzeilles [98] called the modified version of β-logic an Ω-*logic*, but we will call the modified version simply by β-logic.

of second-order arithmetic by adding a new type Ord, and new binary relation symbol $\leq^{\mathrm{Ord}}$ and ternary relation symbol Emb of type $\mathbb{R} \times \mathbb{N} \times \mathrm{Ord}$. The theory RCA_0^β is an extension of RCA_0 with the following axioms:

1. $\forall^1 X \forall^0 n \forall^{\mathrm{Ord}} \xi, \eta [\mathsf{Emb}(X, n, \xi) \wedge \mathsf{Emb}(X, n, \eta) \to \xi = \eta]$.
2. $\forall^1 X \forall^0 n \exists^{\mathrm{Ord}} \xi [\mathsf{Emb}(X, n, \xi)]$.
3. $\forall^1 X \forall^0 n, m \forall^{\mathrm{Ord}} \xi, \eta [\mathsf{WO}(X) \wedge n <_X m \wedge \mathsf{Emb}(X, n, \xi) \wedge \mathsf{Emb}(X, m, \eta)] \to \eta \not\leq^{\mathrm{Ord}} \xi$.

These axioms describe 'Emb is a predicate describing a way to embed a well-order into Ord.' Then, any β-model of second-order arithmetic induces a Girard-styled β-model of RCA_0^β: Suppose that M is a Π^1_1-correct ω-model of arithmetic. Then define

1. $\mathrm{Ord}^M = \{\mathrm{otp}(\alpha) \mid \alpha \in M \wedge M \vDash \mathsf{WO}(\alpha)\}$.
2. $\mathsf{Emb}^M(X, n, \xi)$ iff either ($\neg\mathsf{WO}(X) \wedge \xi = 0$) or ($\mathsf{WO}(X)$ and ξ is isomorphic to $\{i \in \mathrm{field}(X) \mid i <_X n\}$).

It gives a Girard-styled β-model for RCA_0^β. Note that the previous construction works over a sufficiently strong set theory (more precisely, primitive recursive set theory plus Axiom Beta). If we stick to second-order arithmetic, we may take Ord^M by the ordered sum of all well-orders coded in M. Even in that case, we need ATR_0 as a metatheory to define Emb^M correctly.

Conversely, if N is a Girard-styled β-model for RCA_0^β, then it is a Π^1_1-correct ω-model: RCA_0 proves $\mathbb{N}$ with the usual order is a well-order, so $\mathbb{N}^M$ is isomorphic to a well-order. Since RCA_0 also proves $\mathbb{N}$ has no limit point, the same also holds for $\mathbb{N}^M$, which implies $\mathbb{N}^M \cong M$. It is clear that for a recursive linear order α if $M \vDash \mathsf{WO}(\alpha)$, then α is a well-order. Thus by the Kleene normal form theorem, M is Π^1_1-correct.

So far, we have defined the semantics for a β-logic. Then what is the syntax of β-logic? Following [29, Definition 10.1.6], let us define the first-order for β-logic and the notion of α-proof for a fixed well-order α:

Definition 3.3.4. Let $\mathcal{L}$ be a β-language and α be a well-order. The language $\mathcal{L}[\alpha]$ is obtained from $\mathcal{L}$ by adding constants $\overline{\xi}$ of type Ord for $\xi \in \alpha$. The new language is not technically a β-language since it has constant symbols of type Ord, and we now do not allow free variables of type Ord.

The sequent calculus LK_α is defined as follows: The sequents are made of formulas of $\mathcal{L}[\alpha]$, and we have the following axioms and inference rules, where τ is a type other than Ord, and † notifies which formula is changed by the rule, which is not part of the syntax of the β-logic:

$$\frac{}{A \vdash A}\ \text{Id}$$

$$\frac{}{\vdash \overline{\zeta} \leq^{\text{Ord}} \overline{\xi}}\ \text{When } \zeta \leq \xi \in \alpha \qquad \frac{}{\overline{\zeta} \leq^{\text{Ord}} \overline{\xi} \vdash}\ \text{When } \xi < \zeta \in \alpha$$

$$\frac{\Gamma, A^\dagger \vdash \Delta}{\Gamma, (A \wedge B)^\dagger \vdash \Delta}\ \wedge\text{L0}$$

$$\frac{\Gamma, B^\dagger \vdash \Delta}{\Gamma, (A \wedge B)^\dagger \vdash \Delta}\ \wedge\text{L1} \qquad \frac{\Gamma \vdash A^\dagger, \Delta \qquad \Lambda \vdash B^\dagger, \Pi}{\Gamma, \Lambda \vdash (A \wedge B)^\dagger, \Delta, \Pi}\ \wedge\text{R}$$

$$\frac{\Gamma A^\dagger \vdash \Delta \qquad \Lambda B^\dagger \vdash \Pi}{\Gamma, \Lambda, (A \vee B)^\dagger \vdash \Delta, \Pi}\ \vee\text{L} \qquad \frac{\Gamma \vdash A^\dagger, \Delta}{\Gamma \vdash (A \vee B)^\dagger, \Delta}\ \vee\text{R0}$$

$$\frac{\Gamma \vdash B^\dagger, \Delta}{\Gamma \vdash (A \vee B)^\dagger, \Delta}\ \vee\text{R1}$$

$$\frac{\Gamma \vdash A^\dagger, \Delta}{\Gamma, \neg A^\dagger \vdash \Delta}\ \neg\text{L} \qquad \frac{\Gamma, A^\dagger \vdash \Delta}{\Gamma \vdash \neg A^\dagger, \Delta}\ \neg\text{R}$$

$$\frac{\Gamma \vdash A^\dagger, \Delta \qquad \Lambda, B^\dagger \vdash \Pi}{\Gamma, \Lambda, (A \to B)^\dagger \vdash \Delta, \Pi} \to\mathrm{L} \qquad \frac{\Gamma, A^\dagger \vdash B^\dagger, \Delta}{\Gamma \vdash (A \to B)^\dagger, \Delta} \to\mathrm{R}$$

$$\frac{\Gamma, A(t)^\dagger \vdash \Delta}{\Gamma, \forall^\tau x A(x) \vdash \Delta} \forall^\tau\mathrm{L} \qquad \frac{\Gamma \vdash A(t)^\dagger, \Delta}{\Gamma \vdash \forall^\tau x A(x)^\dagger, \Delta} \forall^\tau\mathrm{R}^*$$

$$\frac{\Gamma, A(x)^\dagger \vdash \Delta}{\Gamma, \exists^\tau x A(x)^\dagger \vdash \Delta} \exists^\tau\mathrm{L}^* \qquad \frac{\Gamma \vdash A(t)^\dagger, \Delta}{\Gamma \vdash \exists^\tau x A(x), \Delta} \exists^\tau\mathrm{R}$$

$$\frac{\Gamma, A(\bar{\zeta})^\dagger \vdash \Delta}{\Gamma, \forall^{\mathrm{Ord}} x A(x)^\dagger \vdash \Delta} \forall^{\mathrm{Ord}}\mathrm{L} \qquad \frac{\cdots \Gamma \vdash A(\bar{\zeta})^\dagger, \Delta \cdots (\text{all } \zeta \in \alpha)}{\Gamma \vdash \forall^{\mathrm{Ord}} x A(x)^\dagger, \Delta} \forall^{\mathrm{Ord}}\mathrm{R}$$

$$\frac{\cdots \Gamma, A(\bar{\zeta})^\dagger \vdash \Delta \cdots (\text{all } \zeta \in \alpha)}{\Gamma, \exists^{\mathrm{Ord}} x A(x)^\dagger \vdash \Delta} \exists^{\mathrm{Ord}}\mathrm{L} \qquad \frac{\Gamma, \vdash A(\bar{\zeta})^\dagger, \Delta}{\Gamma, \vdash \exists^{\mathrm{Ord}} x A(x)^\dagger, \Delta} \exists^{\mathrm{Ord}}\mathrm{R}$$

$$\frac{\Gamma, \vdash \Delta}{\Gamma, A^\dagger \vdash \Delta} \mathrm{WL} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^\dagger, \Delta} \mathrm{WR}$$

$$\frac{\Gamma', A^\dagger, B^\dagger, \Gamma'' \vdash \Delta}{\Gamma', B^\dagger, A^\dagger, \Gamma'' \vdash \Delta} \mathrm{EL} \qquad \frac{\Gamma \vdash \Delta', A^\dagger, B^\dagger, \Delta''}{\Gamma \vdash \Delta', B^\dagger, A^\dagger, \Delta''} \mathrm{ER}$$

$$\frac{\Gamma, A^\dagger, A^\dagger \vdash \Delta}{\Gamma, A^\dagger \vdash \Delta} \mathrm{CL} \qquad \frac{\Gamma \vdash A^\dagger, A^\dagger, \Delta}{\Gamma \vdash A^\dagger, \Delta} \mathrm{CR}$$

$$\frac{\Gamma \vdash A^\dagger, \Delta \qquad \Lambda, A^\dagger \vdash \Pi}{\Gamma, \Lambda \vdash \Delta, \Pi} \mathrm{Cut}$$

In the asterisked rules, the *eigenvariable* x which applies the rule must not appear free in Γ, Δ. Following the standard terminology, we use double lines not to write structural rules WL, WR, EL, ER, CL, and CR. We call a proof tree generated by the previous rules an α-*preproof*. If an α-preproof has no infinite branch, then we call the α-preproof an α-*proof*.

We have only defined β-proof system LK_α for predicate logic, but we will see a modification of LK by adding additional rules (like ω-rules or rules for an inductive definition

for a fixed arithmetical positive operator) into LK_α. However, the previous notion of LK_α and α-proof have a fault in that both LK_α and an α-proof are not syntactic objects. For example, we expect that an α-proof should have a 'recursive' structure, but the current definition of an α-preproof is a mixture of arbitrary well-orders α and syntactic objects. Also, we expect LK_α be defined uniformly on α, but we have not stated what the uniformity is. The latter issue is not a big problem if we restrict ourselves to LK_α or its trivial modification (like $T + \mathsf{LK}_\alpha$, obtained from LK_α by adding statements of T as axioms), but it becomes problematic if we consider the expansion of LK_α by adding new inference rules. To resolve these issues, we provide a formal definition of a β-*proof system*, which generalizes the system $\langle \mathsf{LK}_\alpha \mid \alpha \in \mathsf{WO} \rangle$ and other formal systems we will introduce.

Definition 3.3.5. Let $\mathcal{L}$ be a β-language and α be a linear order. An *α-proof system* $\mathfrak{T}$ consists of the *set of inference rules* $\mathsf{Rule}^{\mathfrak{T}} = \mathsf{Rule}$, the *base linear order* $\mathbf{B}^{\mathfrak{T}} = \mathbf{B}$, and an index set I_{R} for each $\mathsf{R} \in \mathsf{Rule}$. Each inference rule takes the form

$$\frac{\langle \Gamma_\iota \vdash \Delta_\iota \mid \iota \in I_{\mathsf{R}} \rangle}{\Gamma \vdash \Delta}\,\mathsf{R} \tag{3.6}$$

We also require that I_{R} is either a (possibly empty) subset of $\mathbf{B}$ that only depends on the choice of R or equal to α. Typically $\mathbf{B} = \mathbb{N}$ and I_{R} is an initial segment of either $\mathbf{B}$ or equal to α. We call I_{R} the *arity of* R, and we say R *has an ordinal arity* when $I_{\mathsf{R}} = \alpha$.

A *$\mathfrak{T}$-preproof* is a pair (D, ϕ) of a tree $D \subseteq [\alpha \cup \mathbf{B}]^{<\omega}$ with a labeling ϕ satisfying the following: For each $\sigma \in D$, $\phi(\sigma) = (\mathsf{R}, \Gamma \vdash \Delta)$ for some $\mathsf{R} \in \mathsf{Rule}$ and $\Gamma \vdash \Delta$ is a sequent such that

$$I_{\mathsf{R}} = \{\iota \mid \sigma^\frown\langle\iota\rangle \in D\}$$

and if $\phi(\sigma^\frown\langle\iota\rangle) = (\mathsf{R}_\iota, \Gamma_\iota \vdash \Delta_\iota)$ then $\langle \Gamma_\iota \vdash \Delta_\iota \mid \iota \in I_{\mathsf{R}} \rangle$ and $\Gamma \vdash \Delta$ satisfy (3.6).

When we define an α-proof system, we do not build just a single α-proof system but a family of α-proof systems for each well-order α. The family of proof systems shares

the same β-language, and is somehow functorial. We will define the precise meaning of being functorial, and let us start with a preliminary notion:

Definition 3.3.6. Let $\mathcal{L}$ be a β-language. For two well-orders α, γ and an increasing map $f : \alpha \to \gamma$, let us define $\mathsf{m}_f(\psi)$ for an $\mathcal{L}[\alpha]$-formula ψ by replacing every occurance of $\overline{\xi}$ in ψ for $\xi \in \alpha$ with $\overline{f(\xi)}$.

We extend the notion m_f to finite sequences of formulas or sequents. For a sequence $\Gamma = A_0, \cdots, A_{m-1}$ of formulas, $\mathsf{m}_f(\Gamma) = \mathsf{m}_f(A_0), \cdots, \mathsf{m}_f(A_{m-1})$, and for a sequent $\Gamma \vdash \Delta$, $\mathsf{m}_f(\Gamma \vdash \Delta)$ is the sequent $\mathsf{m}_f(\Gamma) \vdash \mathsf{m}_f(\Delta)$.

Definition 3.3.7. Let $\mathfrak{T}$ be an ω-proof system (not to be confused with ω-logic.) We say $\mathfrak{T}$ is a β-*proof system* if $\mathbf{B}$ is well-founded and moreover the following holds:

1. If $I_{\mathsf{R}} = \omega$, then the premises of R must have the form $\Gamma'[\overline{\iota}/x] \vdash \Delta'[\overline{\iota}/x]$ for $\iota \in I_{\mathsf{R}} = \omega$, where Γ', Δ' are sequents over the language $\mathcal{L}[\omega]$ with an ordinal variable x.
2. If $I_{\mathsf{R}} = \omega$, then $\mathrm{Occ}(\Gamma_\iota \cup \Delta_\iota) \subseteq \mathrm{Occ}(\Gamma \cup \Delta) \cup \{\underline{\iota}\}$, where the sequents are given as (3.6) and $\mathrm{Occ}(\Gamma)$ is the set of ordinals occurring in a formula in Γ.
3. If $I_{\mathsf{R}} \subseteq \mathbf{B}$, then $\mathrm{Occ}(\Gamma_\iota \cup \Delta_\iota) \subseteq \mathrm{Occ}(\Gamma \cup \Delta)$ for each $\iota \in I_{\mathsf{R}}$, where the sequents are given as (3.6).

Remark 3.3.8. We will see later the case when $\mathbb{N} \subseteq \mathbf{B}$ and $\alpha = \omega$. In this case, denoting both natural numbers as members of $\mathbf{B}$ and as members of the ordinal parameter set ω by n will be confusing. To avoid confusion, we always underline the parameter elements when they appear in the formal definition of β-logic, like, when we label sequents by members of α. A comparable notation is the underlined parameters in quasidendroids. Overlined ordinals will still denote the corresponding constant symbols in the expanded language occurring in formulas.

The next lemma easily follows from the induction on σ, which roughly states that in a β-proof system, the parameter ordinals of a sequent in a certain position of a preproof only depend on the parameters in the initial sequent and the position itself.

Lemma 3.3.9. *Let $\mathfrak{T}$ be a β-proof system and $\pi = (D, \phi)$ be a $\mathfrak{T}$-preproof. For $\sigma \in D$ with $\phi(\sigma) = (\mathsf{R}_\sigma, \Gamma_\sigma \vdash \Delta_\sigma)$, we have*

$$\mathrm{Occ}(\Gamma_\sigma \cup \Delta_\sigma) \subseteq \mathrm{Occ}(\Gamma_{\langle\rangle} \cup \Delta_{\langle\rangle}) \cup \mathrm{Occ}(\sigma) \qquad \square$$

We can instantiate every β-system to an α-proof system uniformly:

Definition 3.3.10. Let $\mathfrak{T}$ be a β-proof system and α a well-order. Let us define its *instantiation* $\mathfrak{T}_\alpha$ as follows: $\mathfrak{T}_\alpha$ and $\mathfrak{T}$ have the same set of inference rules and base linear order, but $I_{\mathsf{R}}^{\mathfrak{T}_\alpha} = \alpha$ when $I_{\mathsf{R}}^{\mathfrak{T}} = \omega$. The rule R is well-defined in this case since the definition of a β-proof system requires the hypotheses of R to take the same form except for parameters $\iota \in I_{\mathsf{R}}$.

We also require a homogeneity to proofs of a β-proof system:

Definition 3.3.11. Let $\mathfrak{T}$ be an α-proof system. For a $\mathfrak{T}$-preproof $\pi = (D, \phi)$ and an increasing $f \colon \gamma \to \alpha$ if we define the *mutilation* ${}^f\pi = ({}^fD, {}^f\phi)$ by

1. ${}^fD = \{\sigma \mid \mathsf{m}_f(\sigma) \in D\}$,
2. If $\sigma \in {}^fD$, then $f(\sigma) \in D$ and every overlined element (i.e., ordinal) appearing in $\phi(f(\sigma))$ is in ran f. Then define

$$({}^f\phi)(\sigma) = \mathsf{m}_f^{-1}(\phi(\mathsf{m}_f^D(\sigma))).$$

In a more plain term, we construct ${}^f\pi$ as follows:

1. In a γ-preproof π, remove all premises containing overlined ordinals not in ran f.

2. Replace every $\overline{f(\xi)}$ with $\overline{\xi}$.

We say a $\mathfrak{T}$-preproof π is *homogeneous* if for every increasing $f, g\colon \gamma \to \alpha$, we have ${}^{f}\pi = {}^{g}\pi$ as $\mathfrak{T}_\gamma$-preproofs.

Note that the homogeneity of a preproof has the following finitary characterization:

Lemma 3.3.12. *Let $\mathfrak{T}$ be a β-proof system and α a well-order. For a $\mathfrak{T}_\alpha$-preproof π, π is homogeneous if and only if for every $n \in \mathbb{N}$ and increasing $f, g\colon n \to \alpha$, we have ${}^{f}\pi = {}^{g}\pi$.*

Proof. One direction is clear. For the remaining direction, suppose that $f, g\colon \gamma \to \alpha$ are two increasing functions. We claim that for $\sigma \in [\alpha \cup \mathbf{B}]^{<\omega}$,

$$\mathsf{m}_f(\sigma) \in D \iff \mathsf{m}_g(\sigma) \in D.$$

Let $h\colon n \to \mathrm{Occ}(\sigma)$ be the unique increasing bijection. Then we have $\sigma \in \operatorname{ran} \mathsf{m}_h$ and

$$\mathsf{m}_f(\sigma) = \mathsf{m}_{f\circ h}(\mathsf{m}_h^{-1}(\sigma)) \in D \iff \mathsf{m}_g(\sigma) = \mathsf{m}_{g\circ h}(\mathsf{m}_h^{-1}(\sigma)) \in D.$$

It gives ${}^{f}D = {}^{g}D$. Proving ${}^{f}\phi = {}^{g}\phi$ is similar, so we omit its proof. □

For a homogeneous $\mathfrak{T}$-preproof π for a sequent $\Gamma^* \vdash \Delta^*$ in $\mathcal{L}$ (so $\mathrm{Occ}(\Gamma^* \cup \Delta^*)$ is empty), we can instantiate it and get a $\mathfrak{T}_\alpha$-preproof:

Proposition 3.3.13. *Let $\mathfrak{T}$ be a β-proof system and α a well-order. Suppose that $\pi = (D, \phi)$ is a homogeneous $\mathfrak{T}$-preproof with the conclusion $\Gamma^* \vdash \Delta^*$ such that $\mathrm{Occ}(\Gamma^* \cup \Delta^*) = \emptyset$. Let us define the instantiation $\pi_\alpha = (D_\alpha, \phi_\alpha)$ to α as follows:*

1. *D_α is the set of all $\mathsf{m}_f(\sigma)$ for each $\sigma \in D$ and an increasing $f\colon \mathrm{Occ}(\sigma) \to \alpha$.*

2. *For the same $\sigma \in D$ and f, we define*

$$\phi_\alpha(\mathsf{m}_f(\sigma)) = \mathsf{m}_f(\phi(\sigma)).$$

Then π_α is a well-defined $\mathfrak{T}_\alpha$-preproof and is homogeneous.

Proof. First, we need to ensure that ϕ_α is well-defined. Note that for $\sigma \in D$ and $f\colon \mathrm{Occ}(\sigma) \to \alpha$, $\mathsf{m}_f(\phi(\sigma))$ is well-defined by Lemma 3.3.9: The lemma guarantees every ordinal parameter occurring in the second component of $\phi(\sigma)$ belongs to the domain of f. We claim that the definition of ϕ_α does not depend on the choice of f and σ in the following sense: If $\sigma, \tau \in D$, $f\colon \mathrm{Occ}(\sigma) \to \alpha$, and $g\colon \mathrm{Occ}(\tau) \to \alpha$ satisfy $\mathsf{m}_f(\sigma) = \mathsf{m}_g(\tau)$ then $\mathsf{m}_f(\phi(\sigma)) = \mathsf{m}_g(\phi(\tau))$.

Let $e\colon |\operatorname{ran} f \cup \operatorname{ran} g| \to \operatorname{ran} f \cup \operatorname{ran} g$ be the enumeration function and $h\colon |\operatorname{ran} f| \to |\operatorname{ran} f \cup \operatorname{ran} g|$, $k\colon |\operatorname{ran} g| \to |\operatorname{ran} f \cup \operatorname{ran} g|$ be such that $f = e \circ h$ and $g = e \circ k$. Then $\mathsf{m}_f(\sigma) = \mathsf{m}_g(\tau)$ implies $\mathsf{m}_h(\sigma) = \mathsf{m}_k(\tau)$. Also by homogeneity, we have ${}^{\mathsf{Id}}\phi = {}^{h}\phi = {}^{k}\phi$, so

$$\begin{aligned}\mathsf{m}_h(\phi(\sigma)) = \mathsf{m}_h(\mathsf{m}_h^{-1}(\phi(\mathsf{m}_h(\sigma)))) &= \phi(\mathsf{m}_h(\sigma)) \\ &= \phi(\mathsf{m}_k(\tau)) = \mathsf{m}_k(\mathsf{m}_k^{-1}(\phi(\mathsf{m}_k(\tau)))) = \mathsf{m}_k(\phi(\tau)),\end{aligned}$$

which implies $\mathsf{m}_f(\phi(\sigma)) = \mathsf{m}_g(\phi(\tau))$.

Then let us prove that π_α is a $\mathfrak{T}_\alpha$-preproof. Suppose that we are given $\sigma \in D$, $f\colon \mathrm{Occ}(\sigma) \to \alpha$, and $\phi(\sigma) = (\mathsf{R}, \Gamma \vdash \Delta)$. Then we have $\phi(\mathsf{m}_f(\sigma)) = (\mathsf{R}, \mathsf{m}_f(\Gamma \vdash \Delta))$. If $I_{\mathsf{R}} \subseteq \mathbf{B}$, then $\langle \mathsf{m}_f(\Gamma_\iota \vdash \Delta_\iota) \mid \iota \in I_{\mathsf{R}} \rangle$ and $\mathsf{m}_f(\Gamma \vdash \Delta)$ clearly satisfy (3.6). Otherwise, I_{R} over $\mathfrak{T}_\alpha$ changes so we need additional steps to verify (3.6). In this case, $\Gamma_\iota \vdash \Delta_\iota$ takes the form $\Gamma'[\bar{\iota}/x] \vdash \Delta'[\bar{\iota}/x]$, and $\sigma^\frown\langle \underline{\iota} \rangle \in D$ for each $\iota \in \omega$. We claim that for every $\xi \in \alpha$, $\mathsf{m}_f(\sigma)^\frown\langle \underline{\xi} \rangle \in D_\alpha$ and

$$\phi_\alpha(\mathsf{m}_f(\sigma)^\frown\langle \underline{\xi} \rangle) = (\mathsf{R}', \mathsf{m}_f(\Gamma' \vdash \Delta')[\bar{\xi}/x]).$$

for some R'. If $\xi \in \operatorname{ran} f$ so there is $k \in \omega$ such that $f(k) = \xi$, then $\mathsf{m}_f(\sigma^\frown\langle \underline{k}\rangle) = \mathsf{m}_f(\sigma)^\frown\langle \underline{\xi}\rangle \in D_\alpha$ and

$$\begin{aligned}\phi_\alpha(\mathsf{m}_f(\sigma^\frown\langle \underline{\xi}\rangle)) &= \mathsf{m}_f(\phi_\alpha(\sigma^\frown\langle \underline{k}\rangle)) \\ &= \mathsf{m}_f(\mathsf{R}', \Gamma'[\overline{k}/x] \vdash \Delta'[\overline{k}/x]) = (\mathsf{R}', \mathsf{m}_f(\Gamma' \vdash \Delta')[\overline{\xi}/x]).\end{aligned}$$

If $\xi \notin \operatorname{ran} f$, let $g\colon |\mathrm{Occ}(\sigma)| + 1 \to \operatorname{ran} f \cup \{\xi\}$ be the increasing enumeration. Then by homogeneity, ${}^{g^{-1}\circ f}D = {}^{\mathsf{Id}_{\mathrm{Occ}(\sigma)}}D \ni \sigma$, so $\mathsf{m}_{g^{-1}\circ f}(\sigma) \in D$. Moreover, by the homogeneity of π, we also have $\phi(\sigma) = ({}^{g^{-1}\circ f}\phi)(\sigma)$. It implies

$$\phi(m_{g^{-1}\circ f}(\sigma)) = \mathsf{m}_{g^{-1}\circ f}(\phi(\sigma)) = (\mathsf{R}, \mathsf{m}_{g^{-1}\circ f}(\Gamma \vdash \Delta)).$$

In particular, we have $\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle \in D$ for every $k \in \omega$.

Hence if k be the natural number such that $g(k) = \xi$, then

$$\mathsf{m}_f(\sigma)^\frown\langle \underline{\xi}\rangle = \mathsf{m}_g(\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle) \in D_\alpha$$

Moreover,

$$\phi_\alpha(\mathsf{m}_f(\sigma)^\frown\langle \underline{\xi}\rangle) = \phi_\alpha(\mathsf{m}_g(\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle)) = \mathsf{m}_g(\phi(\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle).$$

By homogeneity again, we have

- $\phi(\mathsf{m}_{g^{-1}\circ f}(\sigma)) = \mathsf{m}_{g^{-1}\circ f}(\phi(\sigma)) = (\mathsf{R}, \mathsf{m}_{g^{-1}\circ f}(\Gamma \vdash \Delta))$, and
- $\phi(\mathsf{m}_{g^{-1}\circ f}(\sigma^\frown\langle \underline{j}\rangle)) = \mathsf{m}_{g^{-1}\circ f}(\phi(\sigma^\frown\langle \underline{j}\rangle)) = (\mathsf{R}', \mathsf{m}_{g^{-1}\circ f}(\Gamma' \vdash \Delta')[\underline{g^{-1}(f(j))}/x])$ for $j \in \operatorname{dom} f$ and some R',

meaning that $\phi(\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle) = (\mathsf{R}'', \mathsf{m}_{g^{-1}\circ f}(\Gamma' \vdash \Delta')[\underline{k}/x])$ for some R''. Hence we have

$$\phi_\alpha(\mathsf{m}_f(\sigma)^\frown\langle \underline{\xi}\rangle) = \mathsf{m}_g(\phi(\mathsf{m}_{g^{-1}\circ f}(\sigma)^\frown\langle \underline{k}\rangle)) = (\mathsf{R}'', \mathsf{m}_f(\Gamma' \vdash \Delta')[\underline{\xi}/x])$$

finishing the proof of that π_α is a $\mathfrak{T}_\alpha$-preproof. For homogeneity, observe that for $f : n \to \alpha$, we have ${}^f(D_\alpha) = \{\sigma \in D \mid \mathrm{Occ}(\sigma) \subseteq n\}$, which does not depend on f. Moreover, $({}^f(\phi_\alpha))(\sigma) = \phi(\sigma)$, finishing the proof. □

Now let us define the provability relation for β-logic:

Definition 3.3.14. Let $\mathfrak{T}$ be a β-proof system. We say a $\mathfrak{T}$-preproof $\pi = (D, \phi)$ is a β-*preproof from* $\mathfrak{T}$ if it is homogeneous. Furthermore, π is a β-*proof* if, in addition, the instantiation π_α is an α-proof (so π_α has no infinite branch) for every infinite well-order α. We say an ω-preproof π is *(primitive) recursive* if there is a (primitive) recursive code for π.

We say a sequent $\Gamma \vdash \Delta$ over a β-language $\mathcal{L}$ is β-*provable* if there is an β-proof from LK for the sequent $\Gamma \vdash \Delta$. For an $\mathcal{L}$-formula ϕ and an $\mathcal{L}$-theory T, we say ϕ *is* β-*provable from* T if there is a β-proof for ϕ from the system $\mathsf{LK} + T$, whose definition is the same with LK but has extra sequents $\vdash A$ for each $A \in T$ as axioms. We write $T \vdash^\beta \phi$ if the sequence $\vdash \phi$ is β-provable from T.

Note that a β-preproof itself does not have a quasipredendroid structure, but we can extract a quasipredendroid from it:

Definition 3.3.15. Let $\mathfrak{T}$ be a β-proof system and $\pi = (D, \phi)$ be a β-preproof. Let us define the associated prequasidendroid $\mathrm{Den}(\pi)$ as follows: It is a prequasidendroid over the linear order $\mathbf{B} \cup \{\infty\}$, where ∞ is a new element larger than any elements of $\mathbf{B}$. We define $\mathrm{Den}(\pi)$ by

$$\mathrm{Den}(\pi) = \big\{\langle \sigma(0), \infty, \sigma(1), \infty, \cdots, \sigma(m-1), \infty\rangle \;\big|\; \sigma \in D^* \wedge |\sigma| = m\big\}.$$

A simpler definition should be

$$\mathrm{Den}(\pi) = \{\sigma^\frown\langle\infty\rangle \mid \sigma \in D^*\},$$

but it does not work since if $\phi(\sigma) = (\mathsf{R}, \Gamma \vdash \Delta)$ for some $\sigma \in D^*$ and an ordinal arity R, then every immediate successor of σ is of the form $\sigma^\frown\langle \underline{m} \rangle$, so adding $\sigma^\frown\langle \infty \rangle$ could violate the definition of a prequasidendroid.

Proposition 3.3.16. *Let $\mathfrak{T}$ be a β-proof system and $\pi = (D, \phi)$ be a β-preproof. Then* $\mathrm{Den}(\pi)$ *is a prequasidendroid. Also, for each linear order α,*

$$\mathrm{Den}(\pi)_\alpha = \{\langle \sigma(0), \infty, \sigma(1), \infty, \cdots, \sigma(m-1), \infty \rangle \mid \sigma \in D^*_\alpha \wedge |\sigma| = m\}. \tag{3.7}$$

Moreover, π is a β-proof if and only if $\mathrm{Den}(\pi)$ *is a quasidendroid.*

Proof. $\mathrm{Den}(\pi)$ being a prequasidendroid is easy to check. For (3.7), observe that we defined D_α by the set of all $\mathsf{m}_f(\sigma)$ for $\sigma \in D$ and an increasing $f\colon \mathrm{Occ}(\sigma) \to \alpha$, and

$$\mathrm{Occ}(\langle \sigma(0), \infty, \cdots, \sigma(m-1), \infty \rangle) = \mathrm{Occ}(\sigma).$$

For the moreover part, observe that π_α has an infinite branch if and only if D_α has an infinite branch, which is equivalent to that $\mathrm{Den}(\pi)_\alpha$ has an infinite branch. □

3.3.2 Completeness of β-logic

We have defined the semantics and the syntax of β-logic. To ensure they cohere, we need to prove the soundness and completeness of β-logic,

Proposition 3.3.17 (Soundness). *Let ϕ be a sentence over $\mathcal{L}$ and T be a $\mathcal{L}$-theory for a β-language $\mathcal{L}$. If there is an α-proof for $T \vdash \phi$, then every β-model M of T such that* $\mathrm{Ord}^M = \alpha$ *satisfies ϕ. As a corollary, if $T \vdash^\beta \phi$, then $T \vDash^\beta \phi$.*

Proof. We can prove that if there is an α-proof for a sequent $\Gamma \vdash \Delta$, and if M is a β-model with $\mathrm{Ord}^M = \alpha$, then $M \vDash \bigwedge \Gamma \to \bigvee \Delta$, which follows from an induction on the

proof tree. If $T \vdash^{\beta} \phi$, then we can find a finite $T_0 \subseteq T$ and an α-proof for $T_0 \vdash \phi$ for each ordinal α. Thus for every β-model $M \vDash T$, we have $M \vDash \phi$. $\square$

Completeness follows from Schütte-style argument for the completeness of ω-logic:

Theorem 3.3.18 (Completeness, Girard [29, Theorem 10.1.22]). *If $T \vDash^{\beta} \phi$, then $T \vdash^{\beta} \phi$.*

The completeness will follow from the preproof property:

Theorem 3.3.19 (Preproof property of β-logic, ACA_0). *Let $\Gamma \vdash \Delta$ be a sequent of $\mathcal{L}$-sentences such that $\Gamma \cup \Delta \neq \emptyset$. Then we can construct a primitive recursive β-preproof π from LK such that*

1. *If the β-preproof π is a β-proof, then $\bigwedge \Gamma \to \bigvee \Delta$ is β-valid.*
2. *Conversely, if π is not a β-proof and π_α is ill-founded, then we can recursively build a β-model M for $\neg(\bigwedge \Gamma \to \bigvee \Delta)$ such that $\mathrm{Ord}^M \subseteq \alpha$ from an infinite branch of π_α.*

Suppose we are given the sequent $\Gamma \vdash \Delta$, the level n (corresponds to the number of conclusions below the sequent in the preproof), and a distinguished formula $B^{\dagger}$ in the sequent $\Gamma \vdash \Delta$, the stage number $\varsigma \leq n$, and the finite set $\mathcal{X}_{\Gamma \vdash \Delta} = \{\alpha_0, \cdots, \alpha_{q-1}\}$ of all ordinal parameters of $\Gamma \vdash \Delta$. Let us remark that

- For each sequenct $\Gamma \vdash \Delta$ in an α-preproof of level n, there are n many conclusions below $\Gamma \vdash \Delta$. In particular, for each $i \leq n$ we can find the *ith conclusion* $\Gamma_i \vdash \Delta_i$ for $\Gamma \vdash \Delta$, which is placed at the ith position from the root of the α-preproof to the sequent $\Gamma \vdash \Delta$ (which itself is the nth conclusion.)

- The set $\mathcal{X}_{\Gamma\vdash\Delta}$ will be changed only when the rule $\forall^{\mathrm{Ord}}\mathrm{R}$ or $\exists^{\mathrm{Ord}}\mathrm{L}$ is used.

Then let us consider the following *generation scheme* for an α-preproof π_α from the initial sequent $\Gamma \vdash \Delta$ with the initial stage number $\varsigma = 0$:

1. If $\Gamma \vdash \Delta$ is a weakening of an axiom, then we obtain an α-proof for the sequent by applying WL first, then applying WR to reach into the corresponding axiom.
2. Suppose that we are not in the first case, and assume that $B^\dagger$ appears in Δ. Then we have the following subcases:

   (a) $B^\dagger$ is atomic. Let C be the smallest formula (in the sense of Gödel numbering) whose parameters are among the ordinals occurring in the ς*th conclusion for* $\Gamma \vdash \Delta$.[7] (That is, every ordinal parameter in C is in $\mathcal{X}_{\Gamma_\varsigma\vdash\Delta_\varsigma}$.) Then our portion of proof is

   $$\dfrac{\Gamma \vdash C, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{Cut}$$

   If there is no such C, let us duplicate $\Gamma \vdash \Delta$ with structural rules by weakening and contraction to the leftmost formula of Δ:

   $$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

   The new distinguished formula, the stage number, and the parameters will be specified later.

   (b) $B^\dagger$ is a conjunction $B_0 \wedge B_1$. Then our portion of proof is

   $$\dfrac{\Gamma \vdash B_0, \Delta \qquad \Gamma \vdash B_1, \Delta}{\Gamma \vdash \Delta}\,\wedge\mathrm{R}$$

   (c) $B^\dagger$ is a disjunction $B_0 \vee B_1$. Then our portion of proof is

[7] This part is different from Girard's original presentation in [29, Theorem 10.1.22].

$$\dfrac{\dfrac{\Gamma \vdash B_0, B_1, \Delta}{\Gamma \vdash B_0 \vee B_1, B_1, \Delta}\vee\text{R0}}{\Gamma \vdash \Delta}\vee\text{R1}$$

(d) $B^\dagger$ is a negation $\neg B_0$. Then our portion of proof is

$$\dfrac{\Gamma, B_0 \vdash \Delta}{\Gamma \vdash \Delta}\neg\text{R}$$

(e) $B^\dagger$ is an implication $B_0 \to B_1$. Then our portion of proof is

$$\dfrac{\Gamma, B_0 \vdash B_1, \Delta}{\Gamma \vdash \Delta}\to\text{R}$$

(f) $B^\dagger$ is $\forall^\tau x B_0(x)$ for $\tau \neq$ Ord. The portion is

$$\dfrac{\Gamma \vdash B_0(x_p), \Delta}{\Gamma \vdash \Delta}\forall^\tau\text{R}$$

where x_p is a variable of type τ not occurring in $\Gamma \vdash \Delta$.

(g) $B^\dagger$ is $\exists^\tau x_p B_0(x_p)$ for $\tau \neq$ Ord. The portion is

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash B_0(t_0^\tau), \cdots, B_0(t_n^\tau), \Delta}{\Gamma \vdash B_0(t_1^\tau), \cdots, B_0(t_n^\tau), \Delta}\exists^\tau\text{R}}{\vdots}\exists^\tau\text{R}}{\dfrac{\Gamma \vdash B_0(t_n^\tau), \Delta}{\Gamma \vdash \Delta}\exists^\tau\text{R}}$$

Here $t_0, \cdots, t_n$ are the first $(n+1)$ terms of type τ whose only ordinal parameters are among in the set $\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}$ of ordinals occurring in the ςth conclusion for $\Gamma \vdash \Delta$.

(h) $B^\dagger$ is $\forall^{\text{Ord}} x B_0(x)$. The portion is

$$\dfrac{\cdots \Gamma \vdash B_0(\overline{\xi}), \Delta \cdots \text{ for all } \xi \in \alpha}{\Gamma \vdash \Delta}\forall^{\text{Ord}}\text{R}$$

(i) $B^\dagger$ is $\exists^{\text{Ord}} x B_0(x)$. The portion is

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash B_0(\overline{\alpha}_0), \cdots, B_0(\overline{\alpha}_{q-1}), \Delta}{\Gamma \vdash B_0(\overline{\alpha}_1), \cdots, B_0(\overline{\alpha}_{q-1}), \Delta}\exists^{\text{Ord}}\text{R}}{\vdots}\exists^{\text{Ord}}\text{R}}{\dfrac{\Gamma \vdash B_0(\overline{\alpha}_{q-1}), \Delta}{\Gamma \vdash \Delta}\exists^{\text{Ord}}\text{R}}$$

Here $\alpha_0, \cdots, \alpha_{q-1}$ constitute every element of the set $\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}$ of ordinals occurring in ςth conclusion for $\Gamma \vdash \Delta$.

3. Now suppose that we are not in the first case and $B^\dagger$ appears in Γ. Then we have the following subcases:

   (a) $B^\dagger$ is atomic. Again, let C be the smallest formula whose parameters are among the ordinals occurring in the ςth conclusion for $\Gamma \vdash \Delta$. Then our portion of proof is

   $$\dfrac{\Gamma \vdash C, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{Cut}$$

   If there is no such C, let us duplicate $\Gamma \vdash \Delta$ with structural rules by weakening and contraction to the rightmost formula of Γ:

   $$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

   (b) $B^\dagger$ is a conjunction $B_0 \wedge B_1$. Then our portion of proof is

   $$\dfrac{\dfrac{\Gamma, B_0, B_1 \vdash \Delta}{\Gamma, B_0 \vee B_1, B_1 \vdash \Delta}\,\wedge\mathrm{L0}}{\Gamma \vdash \Delta}\,\wedge\mathrm{L1}$$

   (c) $B^\dagger$ is a disjunction $B_0 \vee B_1$. Then our portion of proof is

   $$\dfrac{\Gamma, B_0 \vdash \Delta \qquad \Gamma, B_1 \vdash \Delta}{\Gamma \vdash \Delta}\,\vee\mathrm{L}$$

   (d) $B^\dagger$ is a negation $\neg B_0$. Then our portion of proof is

   $$\dfrac{\Gamma \vdash B_0, \Delta}{\Gamma \vdash \Delta}\,\neg\mathrm{L}$$

   (e) $B^\dagger$ is an implication $B_0 \to B_1$. Then our portion of proof is

   $$\dfrac{\Gamma \vdash B_1, \Delta \qquad \Gamma, B_0 \vdash \Delta}{\Gamma, B_0 \to B_1 \vdash \Delta}\,\to\mathrm{L}$$

   (f) $B^\dagger$ is $\forall^\tau x B_0(x)$ for $\tau \neq \mathrm{Ord}$. The portion is

$$\dfrac{\dfrac{\dfrac{\Gamma, B_0(t_0^\tau), \cdots, B_0(t_n^\tau) \vdash \Delta}{\Gamma, B_0(t_0^\tau), \cdots, B_0(t_{n-1}^\tau) \vdash \Delta}\,\forall^\tau \mathrm{L}}{\vdots}\,\forall^\tau \mathrm{L}}{\dfrac{\Gamma, B_0(t_0^\tau) \vdash \Delta}{\Gamma \vdash \Delta}\,\forall^\tau \mathrm{L}}$$

Here $t_0, \cdots, t_n$ are the first $(n+1)$ terms of type τ whose only ordinal parameters are among in the set $\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}$ of ordinals occurring in ςth conclusion for $\Gamma \vdash \Delta$.

(g) $B^\dagger$ is $\exists^\tau x B_0(x)$ for $\tau \neq$ Ord. The portion is

$$\dfrac{\Gamma, B_0(x_p) \vdash \Delta}{\Gamma \vdash \Delta}\,\exists^\tau \mathrm{L}$$

where x_p is a variable of type τ not occurring in $\Gamma \vdash \Delta$.

(h) $B^\dagger$ is $\forall^{\mathrm{Ord}} x B_0(x)$. The portion is

$$\dfrac{\dfrac{\dfrac{\Gamma, B_0(\overline{\alpha}_0), \cdots, B_0(\overline{\alpha}_{q-1}) \vdash \Delta}{\Gamma, B_0(\overline{\alpha}_0), \cdots, B_0(\overline{\alpha}_{q-2}) \vdash \Delta}\,\forall^{\mathrm{Ord}} \mathrm{L}}{\vdots}\,\forall^{\mathrm{Ord}} \mathrm{L}}{\dfrac{\Gamma, B_0(\overline{\alpha}_0) \vdash \Delta}{\Gamma \vdash \Delta}\,\forall^{\mathrm{Ord}} \mathrm{L}}$$

Here $\alpha_0, \cdots, \alpha_{q-1}$ constitute every element of the set $\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}$ of ordinals occurring in ςth conclusion for $\Gamma \vdash \Delta$.

(i) $B^\dagger$ is $\exists^{\mathrm{Ord}} x B_0(x)$. The portion is

$$\dfrac{\cdots \Gamma, B_0(\overline{\xi}) \vdash \Delta \cdots (\xi \in \alpha)}{\Gamma \vdash \Delta}\,\exists^{\mathrm{Ord}} \mathrm{L}$$

We can see that we obtain new sequents in the hypothesis part, taking the form $\Gamma, \Gamma' \vdash \Delta', \Delta$. Then we determine the distinguished formulas in the hypotheses of the portions of proofs constructed above as follows:

1. If $B^\dagger$ was the ith formula in Δ, then the new distinguished formula is

   - The $(i+1)$th formula in Δ if $B^\dagger$ was not the last formula in Δ.

- The last formula of Γ, Γ' if $B^\dagger$ was the last formula in Δ and $\Gamma, \Gamma' \neq \emptyset$. *(Interim stage phase 1)*
- The zeroth formula in Δ', Δ if $B^\dagger$ was the last formula in Δ and $\Gamma, \Gamma' = \emptyset$. *(Interim stage phase 2)*

2. If $B^\dagger$ was the ith formula in Γ, then the new distinguished formula is
   - The $(i-1)$th formula in Γ if $B^\dagger$ was not the zeroth formula in Γ.
   - The zeroth formula of Δ', Δ if $B^\dagger$ was the zeroth formula in Γ and $\Delta', \Delta \neq \emptyset$. *(Last stage phase 1)*
   - The last formula of Γ, Γ' if $B^\dagger$ was the zeroth formula in Γ and $\Delta', \Delta = \emptyset$. *(Last stage phase 2)*

That is, the dagger mark for distinguishing a formula starts from the leftmost formula in the succedent, and moves to the right until it reaches the rightmost formula of the succedent. The generation scheme adds formulas, but does not add a formula on the right side of the dagger. Hence the dagger only travels formulas in the succedent of the *initial stage phase* (i.e., the closest conclusion sequent in which the dagger was at the leftmost side of the succedent.)

If the dagger hits the rightmost part of the succedent, we declare the interim stage phase, and the dagger moves to the rightmost formula of the antecedent. The dagger resumes travel, from the right to the left of the antecedent. The generation procedure adds formulas, but does not add a formula on the left side of the dagger. Hence the dagger only travels formulas in the antecedent of the interim stage.

If the dagger hits the leftmost side of the antecedent, then we declare the last stage phase. In the next stage, the dagger moves to the leftmost side of the succedent, we declare the initial stage phase and change the stage number:

1. If the previous distinguished formula $B^\dagger$ was not occurred in any of the *last stage phase sequents*, then do not change ς.
2. If the previous distinguished formula $B^\dagger$ has occurred in one of the last stage phase sequents and $\varsigma > 0$, then the new stage number is $\varsigma - 1$.
3. If the previous distinguished formula $B^\dagger$ has occurred in one of the last stage phase sequents and $\varsigma = 0$, then the new stage number is the level of the hypothesis sequent.

Of course, we start with the initial stage phase at level 0. Then we determine the set of distinguished parameters as follows:

1. If the rule was not $\forall^{\mathrm{Ord}}\mathrm{R}$ or $\exists^{\mathrm{Ord}}\mathrm{L}$, do not change the set.
2. If the rule was $\forall^{\mathrm{Ord}}\mathrm{R}$ or $\exists^{\mathrm{Ord}}\mathrm{L}$, we add ξ to the the set of distinguished parameters at the ξth sequent.

The following lemma shows the homogeneity of π_α and the functoriality of $\langle \pi_\alpha \mid \alpha \in \mathsf{WO} \rangle$:

Lemma 3.3.20. *For every $f \colon \alpha \to \beta$, we have ${}^f(\pi_\beta) = \pi_\alpha$.*

Proof. We can prove the following by induction on level n: Suppose that $\Gamma \vdash \Delta$ is a sequent occurring in π_β of level n such that every ordinal parameter occurring in the sequent is in ran f, then

- $\mathsf{m}_f^{-1}(\Gamma \vdash \Delta)$ occurs in π_α at level n with the same derivation rule.
- The position of the distinguished formula, the stage number, and the stage phase of $\Gamma \vdash \Delta$ in π_β are the same as that of $\mathsf{m}_f^{-1}(\Gamma \vdash \Delta)$ in π_α.

- $\mathcal{X}_{\Gamma\vdash\Delta}$ in π_β is equal to $f[\mathcal{X}_{\mathsf{m}_f^{-1}(\Gamma\vdash\Delta)}]$, where $\mathcal{X}_{\mathsf{m}_f^{-1}(\Gamma\vdash\Delta)}$ is the designated set of ordinal parameters in $\mathsf{m}_f^{-1}(\Gamma \vdash \Delta)$ in the proof π_α.

The proof follows from the long case division by the generation scheme, and we omit its details. But as Girard pointed out in [29, p231], the functoriality of the Gödel numbering is important to ensure the proof works for the quantifier cases, introducing many new terms into the new hypothesis sequents. □

By soundness, if the resulting β-preproof is a β-proof, then $T \vDash^\beta (\bigwedge \Gamma \to \bigvee \Delta)$.

Conversely, assume that π_α is ill-founded. Then for each n we can find a sequent $\Gamma_n \vdash \Delta_n$ such that $\Gamma_n \vdash \Delta_n$ is a hypothesis at level n and $\Gamma_{n+1} \vdash \Delta_{n+1}$ is a hypothesis of the portion of proof above $\Gamma_n \vdash \Delta_n$. Again, $\mathcal{X}_{\Gamma_n\vdash\Delta_n} = \{\alpha_0^n, \cdots, \alpha_{q_n-1}^n\}$ is the distinguished set of parameters associated with $\Gamma_n \vdash \Delta_n$, and define $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{X}_{\Gamma_n\vdash\Delta_n} \subseteq \alpha$. *Throughout the remaining part of the proof, we will fix an infinite branch* $\langle \Gamma_n \vdash \Delta_n \mid n \in \mathbb{N}\rangle$.

First, let us check the following fact about the stage number, saying that a stage number repeats infinitely many times along the infinite branch:

Lemma 3.3.21. *For each natural number m, there are infinitely many levels n such that the stage number for $\Gamma_n \vdash \Delta_n$ is m.*

Proof. Let us recall how we defined the new stage number: For each n, let ς_n be the stage number for $\Gamma_n \vdash \Delta_n$. Starting from the zeroth sequent $\Gamma_0 \vdash \Delta_0$, the distinguished formula starts from the rightmost formula of the succedent. It travels the formulas in the succedent of the initial stage (i.e., formulas occurring in Δ_0 in this case) rightwards. After that, the dagger moves to the rightmost formula of the antecedent and travels the

formulas of the antecedent of the interim stage leftwards. We change the stage number in the last stage and replay the same travel with the new sequent.

Every travel ends in a finite level since each sequent has finitely many formulas. Hence if we start from the non-zero stage number, then it eventually drops to zero as the level goes large. If we end the travel at stage number zero, then the stage number becomes the level of the sequent, which can be arbitrarily large. Hence, the stage number repeats infinitely many times. □

Also, the sequent in the infinite branch does not shrink:

Remark 3.3.22. Note that the only manipulation removing a formula in our construction scheme is weakening. However, if weakening occurs in a branch, then the branch ends in a finite length. In this case, there is no chance to apply the weakening to the infinite branch. Hence, the set of formulas occurring in $\Gamma_n \vdash \Delta_n$ is monotonically increasing as n goes large.

Now let us define the 'complexity' of a formula, usually known as the *degree* of a formula:

Definition 3.3.23. Let A be an $\mathcal{L}[\alpha]$-formula. Then we define $\deg(A)$ inductively as follows:

1. $\deg(A) = 0$ if A is atomic.
2. $\deg(A \circ B) = \max(\deg(A), \deg(B)) + 1$ if $\circ$ is a binary connective.
3. $\deg(\neg A) = \deg(A) + 1$.
4. $\deg(\mathsf{Q}^\tau x A(x)) = \deg(A) + 1$ if Q is a quantifier and $\tau \neq \mathrm{Ord}$ is a type.
5. $\deg(\mathsf{Q}^{\mathrm{Ord}} x A(x)) = \sup\{\deg(A(\bar{\xi})) + 1 \mid \xi < \alpha\}$.

Note that $\deg(A) < \omega$ for every $\mathcal{L}[\alpha]$-formula A.

The rank of a formula is necessary to prove the following:

Lemma 3.3.24. *Suppose that the succedent Δ_{n_0} of the sequent $\Gamma_{n_0} \vdash \Delta_{n_0}$ is non-empty for some n_0. Then there is $n \geq n_0$ such that Δ_n has an atomic formula. (Hence for every $m \leq n$, the succedent Δ_m of $\Gamma_m \vdash \Delta_m$ contains an atomic formula.)*

Proof. Start from the sequent $\Gamma_{n_0} \vdash \Delta_{n_0}$ with nonempty Δ_{n_0}. We are done if Δ_{n_0} has an atomic formula. If not, then every formula in Δ_{n_0} has a degree greater than 0. We can find $m \geq n_0$ such that the distinguished formula $B^\dagger$ of $\Gamma_m \vdash \Delta_m$ is in Δ_m, and is a formula in Δ_{n_0} of the least degree. Then the generation scheme ensures Δ_{m+1} contains a formula whose degree is less than $B^\dagger$; That is, for a given n_0 we can find $n_1 = m + 1 > n_0$ such that

$$\min\{\deg A \mid A \in \Delta_{n_1}\} < \min\{\deg A \mid A \in \Delta_{n_0}\}.$$

If the left-hand-side is 0, then Δ_{n_1} has an atomic formula. If not, repeat the same argument to get $n_2 > n_1$ such that Δ_{n_2} has a formula of lower degree, and repeat it finitely many times until we get an atomic formula. □

By the symmetric argument, we have

Lemma 3.3.25. *Suppose that the antecedent Γ_{n_0} of the sequent $\Gamma_{n_0} \vdash \Delta_{n_0}$ is non-empty for some n_0. Then there is $n \geq n_0$ such that Γ_n has an atomic formula. (Hence for every $m \leq n$, the antecedent Γ_m of $\Gamma_m \vdash \Delta_m$ contains an atomic formula.)* □

Then we can show the following:

Lemma 3.3.26. *If B is a formula in $\mathcal{L}[\alpha]$ whose ordinal parameters belong to $\mathcal{X}$, then there is n such that B appears in $\Gamma_n \vdash \Delta_n$ of an infinite branch, but not in both sides of the sequent.*

Proof. First, B does not occur in both Γ_n and Δ_n: If it is, then the hypothesis for $\Gamma_n \vdash \Delta_n$ over the infinite branch must be weakening, and the branch ends in finite stages. It contradicts that the branch is infinite.

Now let us prove that every formula occurs in some $\Gamma_n \vdash \Delta_n$. Let us fix m, and prove it by induction on the Gödel number of the formula C in the language $\mathcal{L}[\alpha]$ whose ordinal parameters belong to $\mathcal{X}_{\Gamma_m \vdash \Delta_m}$. Note that $\mathcal{X}_{\Gamma_m \vdash \Delta_m}$ is finite, so the enumeration of all Gödel numbers for formulas in $\mathcal{L}[\mathcal{X}_{\Gamma_m \vdash \Delta_m}]$ has ordertype $\mathbb{N}$. Hence, the induction argument does not require any form of transfinite induction.

Let $C = C_k$ be the kth formula in $\mathcal{L}[\mathcal{X}_{\Gamma_m \vdash \Delta_m}]$, and suppose that the induction hypothesis holds for the first k formulas $C_0, \cdots, C_{k-1}$ in $\mathcal{L}[\mathcal{X}_{\Gamma_m \vdash \Delta_m}]$. By the inductive hypothesis, there are levels $l_0, \cdots, l_{k-1}$ such that C_i occurs in the l_ith sequent in the infinite branch. By Remark 3.3.22, if we let $l = \max(l_0, \cdots, l_{k-1}) + 1$, then $C_0, \cdots, C_{k-1}$ occurs in the lth sequent $\Gamma_l \vdash \Delta_l$.

By combining Lemmas 3.3.21, 3.3.24 and 3.3.25, we have the following: There is a level $l' \geq l$ of stage number $(m+1)$ such that the l'th sequent has an atomic formula A. By proceeding with the generation scheme further, we can find $l'' > l'$ of stage number m such that A is the designated formula. Again, $C_0, \cdots, C_{k-1}$ occurs in the l''th sequent, and the stage number is m. Thus, the generation scheme introduces the cut rule applied to C_k, and the $(l''+1)$th sequent must have C_k. □

Note that Girard [29, Lemma 10.1.24] stated that its proof is similar to [28, Lemma 6.1.14], but it is not quite true. One of the main reasons is that $\mathcal{X}_{\Gamma_n \vdash \Delta_n}$ may change as the level n becomes larger, so choosing a Gödel numbering least formula in the generation scheme when $B^\dagger$ is atomic does not immediately guarantee we will enumerate every formula in $\mathcal{L}[\bigcup_n \mathcal{X}_{\Gamma_n \vdash \Delta_n}]$.

Hence we can define a model M from the infinite branch $\langle \Gamma_n \vdash \Delta_n \mid n \in \mathbb{N} \rangle$ as follows: For a function symbol f and a relation symbol P of $\mathcal{L}$,

1. The domain M^τ for type τ is the set of all terms of type τ appearing in some $\Gamma_n \vdash \Delta_n$; In particular, $\mathrm{Ord}^M = \mathcal{X}$.
2. $f^M(t_0, \cdots, t_{k-1}) = f(t_0, \cdots, t_{k-1})$.
3. $P^M(t_0, \cdots, t_{k-1})$ holds iff $P(t_0, \cdots, t_{k-1})$ appears in Γ_n for some n.

Then we have

Lemma 3.3.27. *If B is a $\mathcal{L}[\alpha]$-formula whose ordinal parameters belong to $\mathcal{X}$, then*

1. *$M \vDash B$ iff there is $n \in \mathbb{N}$ such that B occurs in Γ_n.*
2. *$M \vDash \neg B$ iff there is $n \in \mathbb{N}$ such that B occurs in Δ_n.*

Proof. We prove it by the complexity of B.

1. The case when $\deg B = 0$ is trivial since it implies B is atomic.
2. Now consider the case when B is $B_0 \wedge B_1$. If B occurs in Γ_n, then there is $m \geq n$ such that one of B_0 or B_1 occurs in Γ_m. By the inductive hypothesis, we have either $M \nvDash B_0$ or $M \nvDash B_1$. Both of them imply $M \nvDash B$. If B occurs in Δ_n, then there is $m \geq n$ such that both B_0 and B_1 occurs in Δ_m. By the inductive hypothesis, we have $M \vDash B_0$ and $M \vDash B_1$, so we have $M \vDash B_0 \wedge B_1$. It implies the equivalence since 'B occurs in Γ_m for some m' and 'B occurs in Δ_m for some m' are exclusive.
3. The case when B is $B_0 \vee B_1$ follows from the symmetric argument for $B_0 \wedge B_1$.

4. Consider when B is $B_0 \to B_1$. If B occurs in Γ_n, then there is $m > n$ such that Γ_m contains B_0 and Δ_m contains B_1. By the inductive hypothesis, we have $M \vDash B_0$ and $M \vDash \neg B_1$, so $M \vDash \neg(B_0 \to B_1)$. If B occurs in Δ_n, then there is $m \geq n$ such that either Γ_m contains B_1, or Δ_m contains B_0. Thus either $M \vDash B_1$ or $M \vDash \neg B_0$, so $B \vDash B_0 \to B_1$.

5. The case when B is $\neg B_0$ follows from simplifying the case when B is $B_0 \to B_1$.

6. Consider when B is $\forall^\tau x B_0(x)$ for $\tau \neq$ Ord. Suppose that B occurs in Γ_n, and t is a fixed term in the domain of M whose ordinal parameters are in $\mathcal{X}_{\Gamma_k \vdash \Delta_k}$, and t is the lth formula of $\mathcal{L}[\mathcal{X}_{\Gamma_k \vdash \Delta_k}]$. Then we can find $m > n, l$ such that the stage number at m is k and B is the distinguished formula at level m. By the generation scheme, there is $m' > m$ such that $B_0(t)$ occurs in $\Gamma_{m'}$, so the inductive hypothesis implies $M \vDash B_0(t)$. Since t is arbitary, we have $M \vDash \forall^\tau x B_0(x)$. If B occurs in Δ_n for some n, then there is $m > n$ and a fresh variable x_p of type τ such that $B_0(x_p)$ occurs in Δ_m. By the inductive hypothesis, we have $M \nvDash B_0(x_p)$, so $M \nvDash \forall^\tau x B_0(x_p)$

7. The case when B is $\exists^\tau x B_0(x)$ for $\tau \neq$ Ord follows from the symmetric argument for when B is $\forall^\tau x B_0(x)$.

8. The cases for ordinal quantifiers are more or less similar to the usual quantifiers.

□

Hence M satisfies every formula in Γ_n, and the negation of every formula in Δ_n. It finishes the proof of the preproof property of β-logic.

3.4 Genedendrons

In this section, we introduce the notion of *genedendrons*.

3.4.1 What is a genedendron?

Let us start with the following question: *What is the relationship between* $\mathsf{HJ}^{\xi}(\emptyset)$ *and* $\omega_{\xi}^{\mathsf{CK}}$*?* One possible answer is the following: $\omega_{\xi}^{\mathsf{CK}}$ describes the *complexity* of a real $\mathsf{HJ}^{\xi}(\emptyset)$. Then what is the precise meaning of complexity? If we view $\mathsf{HJ}^{\xi}(\emptyset)$ as a β-model of arithmetic, then $\mathsf{HJ}^{\xi}(\emptyset)$ has height $\omega_{\xi}^{\mathsf{CK}}$. This viewpoint works for the iterated hyperjumps of the empty set but does not work in general since not every real takes the form of a β-model of arithmetic. Despite that, the keyword β-model will shed light on the correct way to define complexity for Σ^1_2-singleton reals since we will use β-logic to define the correct notion of complexity.

Turning our viewpoint to the recursion-theoretic side, Suzuki [94] (see also [37, §V.6]) defined a notion of rank for Σ^1_2-singleton reals. Although Suzuki's notion is sufficient to establish properties of Δ^1_2 sets (e.g., [37, Theorem 6.8, 6.9]), this rank is difficult to calculate by hand. Despite that, the way Suzuki defined the rank is worth examining: Following terminologies in [37, §V.6], for each Σ^1_2-formula $\exists^1 Y \forall^1 Z \phi(X, Y, Z)$ witnessing R being a Σ^1_2-singleton, we associate a *matrix* F_ϕ for the real R. The matrix F_ϕ is a well-order, so we can define the rank of R by the least ordertype of F_ϕ for a Σ^1_2-formula ϕ witnessing R being a Σ^1_2-singleton.

Suzuki's rank is hard to compute for a specific real since a matrix is hard to calculate. However, this definition still illustrates the idea that defining the rank follows from extracting an ordinal from an implicit Σ^1_2-definition of a real. Suzuki's definition shows a similarity with a framework of Σ^1_2-ordinal analysis presented in [39]: Σ^1_2-sentences are represented by a recursive *pseudodilator*, and a pseudodilator is associated with its *climax*, an ordinal less than δ^1_2. However, we need more from what [39] presented for the following reasons: First, we have not provided a way to calculate a pseudodilator from a given Σ^1_2-formula. Second, we do not have a way to extract a real from a pseudodilator

whose corresponding Σ^1_2-formula implicitly defines the real.

We overcome these two issues via proof-theoretic means. To illustrate how it works, let us examine a proof-theoretic proof for the Kleene normal form theorem, stating that for every Π^1_1-formula ϕ, we can compute a recursive linear order α such that $\phi \leftrightarrow \mathsf{WO}(\alpha)$. Let $\phi \equiv \forall^1 X \psi(X)$ for an arithmetical formula $\psi(X)$. Then consider the ω-logic presented in [28, §6] with an extra unary symbol $\overline{X}$. The ω-logic enjoys the following property we will call the *preproof property*:

Theorem 3.4.1 ([28, Theorem 6.1.13]). *Let $\mathcal{L}$ be a language of a recursive ω-theory T and let $\Gamma \vdash \Delta$ be a sequent in $\mathcal{L}$. Then we can construct a recursive ω-preproof π for $\Gamma \vdash \Delta$ such that $\bigwedge \Gamma \to \bigvee \Delta$ holds over every ω-model of T iff π is well-founded. Furthermore, if π is ill-founded with an infinite branch B, then there is a B-recursive ω-model of $T + \neg(\bigwedge \Gamma \to \bigvee \Delta)$.*

In particular, if ϕ is a formula in the language of first-order arithmetic, then ϕ is true iff the ω-preproof π for $\vdash \phi$ is well-founded. □

Let π be an ω-preproof for the sequent $\vdash \psi(\overline{X})$ given by the preproof property. Then $\forall^1 X \phi(X)$ holds iff π is well-founded. Furthermore, if π has an infinite branch, then we can construct an ω-model $M = (\mathbb{N}, \overline{X}^M)$ from the infinite branch such that $M \vDash \neg\phi(\overline{X})$. In particular, if $\forall^1 X \phi(X)$ fails so π has an infinite branch, then we can find a real R satisfying $\neg\phi(R)$ by examining sequents of the form $\vdash \overline{X}(\overline{n})$ and $\overline{X}(\overline{n}) \vdash$.

Observe from the previous explanation that the construction of the ω-preproof π is an important step for the proof of the ω-completeness theorem (cf. [28, Theorem 6.1.12].) Similarly, we may use a proof of β-completeness theorem to construct a predilator D from a Π^1_2-sentence ϕ such that $\phi \leftrightarrow \mathsf{Dil}(D)$. Furthermore, the resulting predilator D comes from a functorial β-preproof $\langle \pi_\alpha \mid \alpha \in \mathrm{Ord} \rangle$. If ϕ fails, then we have the least α

such that π_α has an infinite branch, and we should be able to extract a real R such that $\neg\psi(R)$, where $\psi(X)$ is a Σ^1_1-formula satisfying $\phi \equiv \forall^1 X\psi(X)$. The ordinal α should be read as a complexity of the real R.[8]

In particular, if $\langle \pi_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is a β-preproof for a Π^1_2-formula $\forall^1 X\psi(X)$ such that $\neg\psi(X)$ implicitly defines a real, then there is the least α making π_α ill-founded, and we can understand α as a complexity of a real implicitly defined by $\neg\psi(X)$. It captures the idea of the definition of a *genedendron* we will introduce in Section 3.4. Since β-preproofs are *prequasidendroids* that we introduced in Section D.7, we will define genedendrons in terms of prequasidendroids. Genedendrons give the right notion of a complexity for Σ^1_2-singleton reals called the Σ^1_2-*altitude* (See Definition 3.4.6), and we will see in subsection 3.4.2 that the Σ^1_2-altitude of $\mathsf{HJ}(\emptyset)$ is indeed ω_1^{CK} by proving a form of β-completeness theorem for a certain β-system for Peano arithmetic with additional predicates for Kleene's $\mathcal{O}$. Just as we use cut-elimination of infinitary systems to obtain Π^1_1- or Π^1_2-consequences of a theory, we use the preproof property for a β-system to extract information about a Σ^1_2-singleton real. In this sense, we may view the computation in subsection 3.4.2 as a form of Σ^1_2-proof theoretic analysis.

Definition 3.4.2. A *semi-genedendron* is a pair (D, ϱ) satisfying the following:

1. D is a prequasidendroid.
2. $\varrho = \langle \varrho_\alpha \colon D^*_\alpha \to \mathbb{N}^{<\omega} \mid \alpha \in \mathsf{WO}\rangle$ is a functorial continuous map taking a finite branch of D^*_α and returning a finite sequence in the following sense:
   (a) If $\sigma, \tau \in D^*_\alpha$, $\sigma \subseteq \tau$, then $\varrho_\alpha(\sigma) \subseteq \varrho_\alpha(\tau)$.

[8] One may wonder that the previous argument only gives a way to define an ordinal complexity for Π^1_1-singleton reals. However, a slight modification gives a way to compute an ordinal complexity for Σ^1_2-singleton reals. In computability-theoretic view, every Σ^1_2-singleton real is Turing reducible to a Π^1_1-singleton real (See Proposition 3.1.1), so their computability-theoretic properties are not too different.

(b) For an increasing map $f\colon \gamma \to \alpha$, if $\sigma \in D(\gamma) = {}^{f}D(\alpha)$, then $\varrho_\alpha(f(\sigma)) = \varrho_\gamma(\sigma)$.

We say ϱ the *extraction scheme*.

We say (D, ϱ) is *locally well-founded* if D is locally well-founded. A semi-genedendron (D, ϱ) is *genedendron*[9] if for every ordinal α and an infinite branch b of D^*_α, $\varrho_\alpha(b) := \bigcup_{n\in\mathbb{N}} \varrho_\alpha(b \restriction n)$ is constant. If $\varrho_\alpha(b)$ has a constant value R, we say (D, ϱ) *generates* R.[10]

Note that the definition of a (semi-)genedendron does not include that D_α has an infinite branch for some α. The following lemma says, like a prequasidendroid, a semi-genedendron is determined by its small fragment:

Lemma 3.4.3. *Every semi-genedendron (D, ϱ) is determined by $(D_\omega, \varrho_\omega)$.*

Proof. D being a prequasidendroid implies D is determined by D_ω by extension. For ϱ, observe that for every $\sigma \in D^*_\alpha$ there is an increasing map $f\colon n \to \alpha$ and $\tau \in D^*_n$ such that $f[\tau] = \sigma$. By the definition, we have $\varrho_n(\tau) = \varrho_\alpha(\sigma)$. □

The previous lemma says the definition of semi-genedendron and genedendron can be formulated over a fragment of second-order arithmetic. Of course, we should replace ordinals with well-orders in the definition of a genedendron to fit the definition into the second-order arithmetic. Also, we can talk about a (semi-)genedendron being recursive: A (semi-)genedendron (D, ϱ) is recursive if $(D_\omega, \varrho_\omega)$ is recursive.

The next proposition says genedendron is a way to formulate Σ^1_2-definable reals:

[9] Ancient Greek 'Birth tree' or 'Generating tree'

[10] There is a small prickle in the definition of a genedendron: The only examples of genedendrons appearing in this chapter come from functorial trees (e.g. [30, Definition 4.1]) and the author does not think there are other important examples. It might be better to define genedendrons in terms of functorial trees, but we will stick to prequasidendroids to avoid developing the theory of functorial trees.

Proposition 3.4.4 (ACA_0)**.** *Let $\phi(X)$ be a Σ^1_2-formula witnessing a real R being a Σ^1_2-singleton. Then we can find a recursive locally well-founded genedendron (D, ϱ) such that there is a well-order α and an infinite branch B over $D(\alpha)$ such that $R = \varrho_\alpha(B)$.*

Proof. Its proof follows from a usual Shoenfield tree construction: Let $\psi(Y, X)$ be a Π^1_1-formula such that $\phi(X) \equiv \exists^1 Y \psi(Y, X)$. Then we can effectively associate a recursive tree T over $\omega \times \omega \times \omega$ such that

$$\psi(Y, X) \iff \forall^1 Z(\langle Z, Y, X\rangle \notin [T]).$$

Here $[T]$ is the set of infinite branches of T. Fix a standard enumeration $\langle \mathbf{s}_i \mid i \in \mathbb{N}\rangle$ of $\mathbb{N}^{<\omega}$, and for a fixed well-order α define

$$\begin{aligned}\widehat{T}(\alpha) = \{\langle \vec{t}, \vec{u}, \vec{\zeta}\rangle \in \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega} \times \alpha^{<\omega} \mid |\vec{t}| = |\vec{u}| = |\vec{\zeta}| \\ \wedge \forall i, j < |\vec{t}|[\langle \mathbf{s}_i, \vec{t}\restriction |\mathbf{s}_i|, \vec{u}\restriction |\mathbf{s}_i|\rangle \in T \wedge \mathbf{s}_i \supsetneq \mathbf{s}_j \leftrightarrow \zeta_i < \zeta_j]\}.\end{aligned}$$

Then let us turn $\widehat{T}(\alpha)$ to an α-prequasidendroid D_α over $(\mathbb{N}^2, \leq_{\mathsf{lex}})$ as follows:

$$D_\alpha = \{\langle (t_0, u_0), \zeta_0, \cdots, (t_{m-1}, u_{m-1}), \zeta_{m-1}, (0,0)\rangle \mid \langle \vec{t}, \vec{u}, \vec{\zeta}\rangle \in \widehat{T}(\alpha),\ |\vec{t}| = m\}.$$

We claim that for every increasing map $f \colon \alpha \to \beta$, we have ${}^f D_\beta = D_\alpha$, which proves the homogeneity of D_β: It follows from proving the equivalence

$$\langle \vec{t}, \vec{u}, \vec{\zeta}\rangle \in \widehat{T}(\alpha) \iff \langle \vec{t}, \vec{u}, f(\vec{\zeta})\rangle \in \widehat{T}(\beta)$$

for $\vec{t}, \vec{u} \in \mathbb{N}^{<\omega}$ and $\vec{\zeta} \in \alpha^{<\omega}$, which is immediate from the definition of $\widehat{T}(\alpha)$.

Then define $\varrho_\alpha \colon D^*_\alpha \to \mathbb{N}^{<\omega}$ by

- $\varrho_\alpha(\langle (t_0, u_0), \zeta_0, \cdots, (t_{m-1}, u_{m-1}), \zeta_{m-1}\rangle) = \langle u_i \mid i < m\rangle$, and
- $\varrho_\alpha(\langle (t_0, u_0), \zeta_0, \cdots, (t_{m-1}, u_{m-1}), \zeta_{m-1}, (t_m, u_m)\rangle) = \langle u_i \mid i \leq m\rangle$.

ϱ_α does not depend on the ordinal part of the input, and it implies (D, ϱ) is a semi-genedendron.

Now we claim that (D, ϱ) is a genedendron: Suppose that there is an infinite branch

$$B = \langle (t_0, u_0), \zeta_0, (t_1, u_1), \zeta_1, \cdots \rangle$$

of D_α. It means $\langle \vec{t}, \vec{u}, \vec{\zeta} \rangle$ is an infinite branch of $\hat{T}(\alpha)$. Hence by definition of $\hat{T}(\alpha)$, so the map $\mathbf{s}_i \mapsto \zeta_i$ is a function ranking the tree $T_{\vec{t},\vec{u}} = \{s \in \mathbb{N}^{<\omega} \mid \langle s, \vec{t} \restriction |s|, \vec{u} \restriction |s| \rangle \in T\}$. Hence $\forall^1 Z \langle Z, \vec{t}, \vec{u} \rangle \notin [T]$, which is equivalent to $\psi(\vec{t}, \vec{u})$. Thus $\exists^1 Y \psi(Y, \vec{u})$, so $R = \vec{u}$. By the definition of ϱ_α, $\bigcup_{n \in \mathbb{N}} \varrho_\alpha(b \restriction n) = \vec{u} = R$, as desired.

Lastly, it is clear that $\hat{T}(\omega)$ is recursive, so is D_ω. ϱ_ω is Δ^0_1-definable, so recursive. The local well-foundedness of D_α follows from that the term set $\mathbb{N}^2$ is well-ordered. □

Proposition 3.4.4 shows every Σ^1_2-singleton real is generated by a genedendron. It is easy to see that every real given by a genedendron is a Σ^1_2-singleton. However, the genedendron provided in the proof of Proposition 3.4.4 has little practical use. Genedendrons we can compute by hand come from β-preproofs over a suitable language and axioms.

Observe that for a locally well-founded prequasidendroid D, $D(\alpha)$ has an infinite branch iff $\mathrm{Lin}^*(D)(\alpha)$ is ill-founded. Hence the following is immediate:

Lemma 3.4.5. *Let (D, ϱ) be a locally well-founded semi-genedendron. Then $\mathsf{Clim}(\mathrm{Lin}^*(D))$ is the least ordinal α such that $D(\alpha)$ has an infinite branch.* □

We will frequently use the climax of the linearization of a genedendron to 'rank' a genedendron. Thus for a notational easiness, we use $\mathsf{Clim}(D)$ to denote $\mathsf{Clim}(\mathrm{Lin}^*(D))$ for a genedendron (D, ϱ). We can use the climax of a genedendron to rank a Σ^1_2-singleton real as follows:

Definition 3.4.6. Let R be a Σ^1_2-singleton real. Let us define the Σ^1_2-*altitude of* R by

$$\mathsf{Alt}_{\Sigma^1_2}(R) = \min\{\mathsf{Clim}(D) \mid (D, \varrho) \text{ is a locally well-founded recursive genedendron computing } R\}.$$

Let us compute $\mathsf{Alt}_{\Sigma^1_2}$ for some reals:

Example 3.4.7. Suppose that R is an arithmetical real, that is, there is an arithmetical formula $\phi(n)$ such that

$$\forall^0 n[R(n) = 0 \leftrightarrow \neg\phi(n)] \wedge [R(n) = 1 \leftrightarrow \phi(n)].$$

We claim that $\mathsf{Alt}_{\Sigma^1_2}(R) = 0$. Consider the ω-logic with the new extra unary predicate X, and consider the Schütte-styled ω-preproof for the sentence

$$\exists^0 n[(\phi(n) \vee n \in X) \wedge (\neg\phi(n) \vee n \notin X)]. \tag{3.8}$$

Then the ω-preproof has an infinite branch b, which encodes an ω-model satisfying the negation of (3.8). We can view the ω-preproof as a constant prequasidendroid D, and define ϱ_α by

$$\varrho_\alpha(\sigma) = \{(n, 0) \mid \ulcorner \underline{n} \notin X \urcorner \in \sigma\} \cup \{(n, 1) \mid \ulcorner \underline{n} \in X \urcorner \in \sigma\}.$$

Then $\varrho_0(b) = r$. Since $D = D_\alpha$ always has an infinite branch, $\mathsf{Clim}(D) = 0$.

3.4.2 Another example: Computing the Σ^1_2-altitude of the hyperjump of $\emptyset$

In this subsection, we prove that $\mathrm{rank}_{\Sigma^1_2}(\mathsf{HJ}(\emptyset)) = \omega_1^{\mathsf{CK}}$, where $\mathsf{HJ}(X)$ is the hyperjump of X. For technical convenience, we assume that $\mathsf{HJ}(X)$ has the form of a characteristic function.

One direction follows from an easy model-theoretic argument:

Lemma 3.4.8. $\mathsf{Alt}_{\Sigma^1_2}(\mathsf{HJ}(\emptyset)) \geq \omega_1^{\mathsf{CK}}$.

Proof. Suppose that M is a β-model of second-order arithmetic decoded from $\mathsf{HJ}(\emptyset)$ and (D, ϱ) is a locally well-founded recursive genedendron generating $\mathsf{HJ}(\emptyset)$. Then $(D, \varrho) \in M$, and M also thinks (D, ϱ) is a locally well-founded genedendron. However, M thinks (D, ϱ) does not generate any real (otherwise $\mathsf{HJ}(\emptyset) \in M$.) Hence M thinks D_α has no infinite branch for every well-order $\alpha \in M$, and every well-order in M has ordertype $< \omega_1^{\mathsf{CK}}$. This shows $\mathsf{Clim}(D) \geq \omega_1^{\mathsf{CK}}$. Since (D, ϱ) is arbitrary, we have the desired lower bound. □

$\mathsf{Alt}_{\Sigma^1_2}(\mathsf{HJ}(\emptyset)) \leq \omega_1^{\mathsf{CK}}$ is mildly challenging to establish. We will construct a locally well-founded recursive genedendron (D, ϱ) generating $\mathsf{HJ}(\emptyset)$ such that $\mathsf{Clim}(\mathrm{Lin}^*(D)) = \omega_1^{\mathsf{CK}}$, and here is where we cast β-logic.

Let us fix a Π^0_1 formula $P(X, x)$ defining a monotone operator whose fixpoint is $\mathsf{HJ}(\emptyset)$. That is, if we let $\Phi(X) = \{x \in \mathbb{N} \mid P(X, x)\}$, and define $I^\alpha_\Phi(X) = \bigcup_{\beta<\alpha} \Phi(I^\beta_\Phi(X))$, then $\mathsf{HJ}(\emptyset)$ is the characteristic function of $I^{\omega_1^{\mathsf{CK}}}_\Phi(\emptyset)$.

Now let us consider a variant of the inductive logic T, which is a variant of Girard's one [29, §11.3]. The language $\mathcal{L}$ of T comprises the language of *first order arithmetic* $\{0, S, +, \cdot, \leq\}$ of type $\mathbb{N}$, plus a new type Ord and a binary relation $\leq^{\mathrm{Ord}}$, a binary relation I_Φ of type $\mathrm{Ord} \times \mathbb{N}$, and a unary relation $\overline{I}_\Phi$ of type $\mathbb{N}$. For an ordinal α, the axioms and the rules for $T[\alpha]$ are the rules of β-logic plus the following,

1. The axioms and rules for ω-logic, that is,

$$\frac{}{\vdash t = t}\ =\mathrm{R}$$

$$\frac{}{t = u, A(t) \vdash A(u)}\ =\mathrm{L0} \qquad \frac{}{S^n 0 = S^m 0 \vdash}\ =\mathrm{L1}\ (n \neq m)$$

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall^{\mathbb{N}} x A(x) \vdash \Delta}\ \forall^{\mathbb{N}}\mathrm{L} \qquad \frac{\cdots \Gamma \vdash A(S^n 0), \Delta \cdots (n \in \mathbb{N})}{\Gamma \vdash \forall^{\mathbb{N}} x A(x), \Delta}\ \forall^{\mathbb{N}}\mathrm{R}$$

$$\frac{\cdots \Gamma, A(S^n 0) \vdash \Delta \cdots (n \in \mathbb{N})}{\Gamma, \exists^{\mathbb{N}} x A(x) \vdash \Delta}\ \exists^{\mathbb{N}}\mathrm{L} \qquad \frac{\Gamma \vdash A(t), \Delta}{\Gamma \vdash \exists^{\mathbb{N}} x A(x), \Delta}\ \exists^{\mathbb{N}}\mathrm{R}$$

Here t, u are $\mathbb{N}$-terms.

2. The rules for the 'operator iteration' I_Φ:

$$\frac{\cdots \Gamma, P(I_\Phi(\overline{\eta}, \cdot), t) \vdash \overline{\xi} \leq^{\mathrm{Ord}} \overline{\eta}, \Delta \cdots (\eta \in \alpha)}{\Gamma, I_\Phi(\overline{\xi}, t) \vdash \Delta}\ I_\Phi\mathrm{L}$$

$$\frac{\Gamma \vdash P(I_\Phi(\overline{\eta}, \cdot), t), \Delta}{\Gamma \vdash I_\Phi(\overline{\xi}, t), \Delta}\ I_\Phi\mathrm{R}\ (\eta < \xi)$$

Here I_ΦR can be applied only when $\eta < \xi \in \alpha$, and I_ΦL requires proofs for every $\eta < \xi \in \alpha$. $P(I_\Phi(\overline{\eta}, \cdot), t)$ is a formula obtained from $P(t, X)$ by replacing $X(x)$ with $I_\Phi(\overline{\eta}, x)$.

3. The rules for the 'operator fixedpoint' $\overline{I}_\Phi$:

$$\frac{\cdots \Gamma, P(I_\Phi(\overline{\eta}, \cdot), t) \vdash \Delta \cdots (\eta \in \alpha)}{\Gamma, \overline{I}_\Phi(t) \vdash \Delta}\ \overline{I}_\Phi\mathrm{L} \qquad \frac{\Gamma \vdash P(I_\Phi(\overline{\eta}, \cdot), t), \Delta}{\Gamma \vdash \overline{I}_\Phi(t), \Delta}\ \overline{I}_\Phi\mathrm{R0}\ (\eta \in \alpha)$$

$$\frac{\Gamma \vdash P(\overline{I}_\Phi(\cdot), t), \Delta}{\Gamma \vdash \overline{I}_\Phi(t), \Delta}\ \overline{I}_\Phi\mathrm{R1}\ (\eta \in \alpha)$$

Now let again M be the β-model (à la Girard) decoded from $\mathsf{HJ}(\emptyset)$. We can turn M to an $\mathcal{L}$-structure as follows: We drop the second-order part and Emb^M of M. Then define

$$I_\Phi^\xi = \bigcup\nolimits_{\eta<\xi} \Phi(I_\Phi^\eta),$$

where $\Phi(X) = \{x \in \mathbb{N} \mid P(X, x)\}$, and interpret $I_\Phi(\xi, n)$ iff $n \in I_\Phi^\xi$. Also, define $\overline{I}_\Phi(n)$ iff $n \in I_\Phi^{\omega_1^{\mathsf{CK}}}$.

Then we can prove the following lemma by induction on a proof:

Lemma 3.4.9. *Every sequent ω_1^{CK}-provable from T is valid over M in the following sense: If $\Gamma \vdash \Delta$ has an ω_1^{CK}-proof, then $M \vDash \bigwedge \Gamma \to \bigvee \Delta$.*

Now we modify the proof of the β-completeness theorem to construct a genededron: The essential part of the proof for the preproof property, giving a primitive recursive β-preproof for the given sequent. In each sequent $\Gamma \vdash \Delta$ appearing during the construction with level n, let us associate the stage number ς, the fixed set of ordinals $\mathcal{X}_{\Gamma\vdash\Delta} = \{\alpha_0, \cdots, \alpha_{q-1}\}$. Also, we add the following scheme to the construction:

1. We exclude $I_\Phi(\overline{\xi}, t)$ from the atomic formula scheme, and it instead follows the rule we will introduce.
2. Suppose that the sequent $\Gamma \vdash \Delta$ is not a weakening of an axiom, and the distinguished formula $B^\dagger$ appears in Δ.

   (i) $B^\dagger$ is $\forall^{\mathbb{N}} x B_0(x)$. The portion is

   $$\dfrac{\cdots \Gamma \vdash B_0(S^n 0), \Delta \cdots \text{ for all } n \in \mathbb{N}}{\Gamma \vdash \Delta}\, \forall^{\mathbb{N}}\mathrm{R}$$

   (j) $B^\dagger$ is $\exists^{\mathbb{N}} x B_0(x)$. The portion is

   $$\dfrac{\dfrac{\dfrac{\Gamma \vdash B_0(0), \cdots, B_0(S^{n-1}0), \Delta}{\Gamma \vdash B_0(S0), \cdots, B_0(S^{n-1}0), \Delta}\, \exists^{\mathbb{N}}\mathrm{R}}{\vdots}\, \exists^{\mathbb{N}}\mathrm{R}}{\dfrac{\Gamma \vdash B_0(S^{n-1}0), \Delta}{\Gamma \vdash \Delta}\, \exists^{\mathbb{N}}\mathrm{R}}$$

   Here n is the level of $\Gamma \vdash \Delta$.

   (k) $B^\dagger$ is $I_\Phi(\overline{\xi}, t)$. The portion is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\dagger) \qquad (\ddagger)}{\Gamma \vdash P(\overline{I}_\Phi, t), P(I_\Phi(\overline{\alpha}'_0, \cdot), t), \cdots, P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta}\,\mathrm{Cut}}{\Gamma \vdash P(I_\Phi(\overline{\alpha}'_0, \cdot), t), \cdots, P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta}\,I_\Phi\mathrm{R1}}{\Gamma \vdash P(I_\Phi(\overline{\alpha}'_1, \cdot), t), \cdots, P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta}\,I_\Phi\mathrm{R0}}{\vdots}\,I_\Phi\mathrm{R0}}{\Gamma \vdash P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta}\,I_\Phi\mathrm{R}}{\Gamma \vdash \Delta}$$

Here

($\dagger$) $\Gamma, C \vdash P(\overline{I}_\Phi, t), P(I_\Phi(\overline{\alpha}'_0, \cdot), t), \cdots, P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta$,

($\ddagger$) $\Gamma \vdash C, P(\overline{I}_\Phi, t), P(I_\Phi(\overline{\alpha}'_0, \cdot), t), \cdots, P(I_\Phi(\overline{\alpha}'_{q'-1}, \cdot), t), \Delta$,

and $\alpha'_0, \cdots, \alpha'_{q'-1}$ is the enumeration of the set of ordinals $\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}$ occurring in the ςth conclusion less than ξ, and C is the least $\mathcal{L}[\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}]$-formula not occurring in Γ, Δ.

(l) $B^\dagger$ is $\overline{I}_\Phi(t)$. The portion is the same as the previous case, except that the rule we apply is $\overline{I}_\Phi$R.

3. Now suppose that it is not the previous case, and $B^\dagger$ appears in Γ.

(i) $B^\dagger$ is $\forall^{\mathbb{N}} x B_0(x)$. The portion is

$$\dfrac{\dfrac{\dfrac{\Gamma, B_0(0), \cdots, B_0(S^{n-1}0) \vdash \Delta}{\Gamma, B_0(S0), \cdots, B_0(S^{n-1}0) \vdash \Delta}\,\forall^{\mathbb{N}}\mathrm{L}}{\vdots}\,\forall^{\mathbb{N}}\mathrm{L}}{\Gamma, B_0(S^{n-1}0) \vdash \Delta}\,\forall^{\mathbb{N}}\mathrm{L}}{\Gamma \vdash \Delta}$$

Again, n is the level of $\Gamma \vdash \Delta$.

(j) $B^\dagger$ is $\exists^{\mathbb{N}} x B_0(x)$. The portion is

$$\dfrac{\cdots \Gamma, B_0(S^n 0) \vdash \Delta \cdots (n \in \mathbb{N})}{\Gamma \vdash \Delta}\,\exists^{\mathbb{N}}\mathrm{L}$$

(k) $B^\dagger$ is $I_\Phi(\overline{\xi}, t)$. The portion is

$$\cdots \dfrac{\dfrac{(\dagger) \qquad (\ddagger)}{\cdots \Gamma, P(I_\Phi(\overline{\eta}, \cdot), t) \vdash \overline{\xi} \leq^{\mathrm{Ord}} \overline{\eta}, \Delta \cdots (\eta < \alpha)}\, I_\Phi \mathrm{L}}{\Gamma \vdash \Delta}\, \mathrm{Cut} \cdots$$

where

- $(\dagger)$ is $\Gamma, P(I_\Phi(\overline{\eta}, \cdot), t), C\overline{\xi} \leq^{\mathrm{Ord}} \overline{\eta}, \Delta$.
- $(\ddagger)$ is $\Gamma, P(I_\Phi(\overline{\eta}, \cdot), t)C, \overline{\xi} \leq^{\mathrm{Ord}} \overline{\eta}, \Delta$.
- C is the least $\mathcal{L}[\mathcal{X}_{\Gamma_\varsigma \vdash \Delta_\varsigma}]$-formula not occurring in the conclusion of the Cut rule in the conclusion of the Cut rule in the portion.

(l) $B^\dagger$ is $\overline{I}_\Phi(t)$. The portion is the same as the previous case, except that the rule we apply is $\overline{I}_\Phi \mathrm{L}$, and η ranges over α.

The reader can see that the portion for I_Φ is a combination of the cases for 'ordinal rules' and the atomic case; $I_\Phi(\overline{\xi}, t)$ is an atomic formula, so we added the atomic case rule. However, introducing $I_\Phi(\overline{\xi}, t)$ requires $I_\Phi(\overline{\eta}, t)$ for $\eta < \xi$, so we added ordinal quantifier-like portions. It is the same for $\overline{I}_\Phi(t)$.

The associated parameter sets are changed as follows:

1. If the rule was none of $\forall^{\mathrm{Ord}}\mathrm{R}$, $\exists^{\mathrm{Ord}}\mathrm{L}$, $I_\Phi \mathrm{L}$, or $\overline{I}_\Phi \mathrm{L}$, do not change the ordinal set.
2. If the rule was either $\forall^{\mathrm{Ord}}\mathrm{R}$ or $\exists^{\mathrm{Ord}}\mathrm{L}$, $I_\Phi \mathrm{L}$, or $\overline{I}_\Phi \mathrm{L}$, we add ξ to the set of distinguished ordinal parameters at the ξth sequent.

Now let π_α be the α-preproof for the empty sequent generated by the previous scheme. We can see that ${}^f\pi_\beta = \pi_\alpha$ for every increasing $f \colon \alpha \to \beta$, which guarantees the homogeneity of the resulting α-preproof π_α. Hence we get a β-preproof $\langle \pi_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ for the empty sequent.

We need to check that the proof for the preproof property also works for the new proof system. Lemma 3.3.21 and Remark 3.3.22 also applies to T. We also define the

rank of a formula as stated in Definition 3.3.23, and we can see that Lemma 3.3.24 and Lemma 3.3.25 also hold for T. Thus we can prove the variant of Lemma 3.3.26:

Lemma 3.4.10. *If B is a formula in $\mathcal{L}[\alpha]$ whose ordinal parameters belong to X, then there is n such that B appears in $\Gamma_n \vdash \Delta_n$ of an infinite branch, but not in both sides of the sequent.*

Proof. The proof is more or less identical to Lemma 3.3.26. The only difference is that the designated atomic formula may have the form $I_\Phi(\overline{\xi}, t)$ or $\overline{I}_\Phi(t)$, but our generation scheme ensures we finally get the formula C in some sequent. □

We are ready to define a genedendron for $\mathsf{HJ}(\emptyset)$. *Throughout the remaining subsection, let us fix π by the β-preproof for the sequent $0 = 0 \vdash S0 = 0$.* Then consider the following extraction scheme ϱ_α: If $\pi_\alpha = (D_\alpha, \phi_\alpha)$, then

$$\varrho_\alpha(\sigma) = \{(n, 1) \mid \overline{I}_\Phi(S^n 0) \text{ is an antecedent of } \mathrm{proj}_1(\phi_\alpha(\sigma))\} \cup \{(n, 0) \mid \overline{I}_\Phi \text{ is a succedent of } \mathrm{proj}_1(\phi_\alpha(\sigma))\}$$

(That is, $\varrho_\alpha(\sigma)$ is a finite partial function obtained from the σth sequent of π_α, by looking the position of the sentence $\overline{I}_\Phi$, which asserts 'n is in $I_\Phi^{<\alpha} = \bigcup_{\xi\in\alpha} I_\Phi^\xi(\emptyset)$.' Then $n \in I_\Phi^{<\alpha}$ if the sentence occurs in the antecedent, and $n \notin I_\Phi^{<\alpha}$ otherwise.)

It defines a semi-genedendron, which is clearly locally well-founded. Since π is recursive, so is (D, ϱ).

Lemma 3.4.11. *(D, ϱ) is a genedendron generating $\mathsf{HJ}(\emptyset)$.*

Proof. Suppose that D_α has an infinite branch, which gives an infinite branch $\langle \Gamma_n \vdash \Delta_n \mid n \in \mathbb{N}\rangle$ of π_α. By the proof of the preproof property of the β-logic, the infinite branch gives a β-model N of height $\leq \alpha$, which satisfies sentences of Γ_n and the negation

of sentences of Δ_n. Furthermore, the $\mathbb{N}$-part of N is the standard set of natural numbers since we can match natural numerals to standard natural numbers.

Now let $(I^{\xi}_{\Phi})^N := \{m \mid N \vDash I_{\Phi}(\overline{\xi}, S^m 0)\}$ and $\overline{I}^N_{\Phi} := \{m \mid N \vDash \overline{I}_{\Phi}(S^m 0)\}$. Let us make some observations for $(I^{\xi}_{\Phi})^N$:

- If $I_{\Phi}(\overline{\xi}, S^m 0)$ occurs in an antecedent of the infinite branch, then $P(I_{\Phi}(\overline{\eta}, \cdot), S^m 0)$ occurs in an antecedent of the infinite branch for some $\eta < \xi$. This shows $(I^{\xi}_{\Phi})^N \subseteq \bigcup_{\eta<\xi} \Phi((I^{\eta}_{\Phi})^N)$.
- Now suppose that $I_{\Phi}(\overline{\xi}, S^m 0)$ occurs in a succedent of the infinite branch. If X_n is the set of ordinals occurring in the nth sequent of the infinite branch, then $\bigcup_{n\in\mathbb{N}} X_n = \text{Ord}^N$. Moreover, for a fixed stage number ς, $I_{\Phi}(\overline{\xi}, S^m 0)$ occurs in a succedent of a sequent with the stage number ς.

  Fix $\eta < \xi \in \alpha$, and find $n \in \mathbb{N}$ such that $\eta \in X_n$. $I_{\Phi}(\overline{\xi}, S^m 0)$ can occur in a sequent of the stage number $n+1$. By proceeding along the branch further, we can find a sequent of the stage number n such that $I_{\Phi}(\overline{\xi}, S^m 0)$ is the distinguished formula in the succedent. Then the generation scheme adds a formula $I_{\Phi}(\overline{\eta}, S^m 0)$ in a succedent. This shows

$$m \notin (I^{\xi}_{\Phi})^N \implies m \notin \bigcup_{\eta<\xi} \Phi((I^{\eta}_{\Phi})^N),$$

  or in other words, $\bigcup_{\eta<\xi} \Phi((I^{\eta}_{\Phi})^N) \subseteq (I^{\xi}_{\Phi})^N$.

Hence for each $\xi \in \text{Ord}^N$, we have $\bigcup_{\eta<\xi} \Phi((I^{\eta}_{\Phi})^N) = (I^{\xi}_{\Phi})^N$. By a similar reasoning, we also have $\overline{I}^N_{\Phi} = \bigcup_{\xi\in\text{Ord}^N} \Phi((I^{\xi}_{\Phi})^N)$. Now we claim that $\Phi(\overline{I}^N_{\Phi}) \subseteq \overline{I}^N_{\Phi}$ by showing

$$m \notin \overline{I}^N_{\Phi} \implies m \notin \Phi(\overline{I}^N_{\Phi}) :$$

Suppose that $\overline{I}_{\Phi}(S^m 0)$ occurs in a succedent of a sequent on the infinite branch. Then the generation scheme guarantees $P(\overline{I}_{\Phi}(\cdot), S^m 0)$ also occurs in a succedent of another

sequent. It implies $N \not\models P(\overline{I}_\Phi(\cdot), S^m 0)$ holds. Since N is an ω-model, it implies $\neg P(\overline{I}_\Phi^N, m)$, i.e., $m \notin \Phi(\overline{I}_\Phi^N)$.

Combining all the facts we have proven, we have $I_\Phi^\xi = (I_\Phi^\xi)^N$ for all $\xi \in \mathrm{Ord}^N$, $\overline{I}_\Phi^N = I_\Phi^{\mathrm{Ord}^N}$, and $\overline{I}_\Phi^N$ is closed under Φ. This shows the characteristic function of $\overline{I}_\Phi^N$ is $\mathsf{HJ}(\emptyset)$. □

Lemma 3.4.12. *$D_{\omega_1^{\mathsf{CK}}}$ is ill-founded, so $\mathsf{Clim}(D) \leq \omega_1^{\mathsf{CK}}$.*

Proof. Suppose not, assume that $D_{\omega_1^{\mathsf{CK}}}$ is well-founded. Then $\pi_{\omega_1^{\mathsf{CK}}}$ is an ω_1^{CK}-proof for the empty sequent. Now let M be the β-model decoded from $\mathsf{HJ}(\emptyset)$. By Lemma 3.4.9, M satisfies $(0 = 0) \to (S0 = 0)$, a contradiction. □

Hence (D, ϱ) is a genedendron of climax ω_1^{CK} generating $\mathsf{HJ}(\emptyset)$. As a final remark, (D, ϱ) being a genedendron is a theorem of ACA_0, where (D, ϱ) is what is constructed from the β-preproof property. Proving $\mathsf{Clim}(D) = \omega_1^{\mathsf{CK}}$ is not precisely possible in ACA_0 since its statement uses ordinals. What we can prove from ACA_0 with the aforementioned arguments is the following:

1. If there is a (coded) β-model M of second-order arithmetic, and if α is a well-order embedding every well-order in M, then D_α is ill-founded for every recursive genedendron (D, ϱ) generating $\mathsf{HJ}(\emptyset)$. (It happens when α is the ordered sum of every well-order in M.)
2. If (D, ϱ) is the recursive genedendron provided by the β-preproof property, and if D_α is ill-founded for some well-order α, then $\varrho_\alpha(B)$ is the hyperjump of $\emptyset$ for every infinite branch B of D_α.

The reader might wonder why the resulting ordinal analysis in subsection 3.4.2 does not produce ω_2^{CK} but ω_1^{CK} since the Σ^1_2-proof-theoretic ordinal of ACA_0 is ω_1^{CK} while that

of ACA_0 + 'HJ($\emptyset$) exists' is ω_2^{CK}. The reason is that the fixed point $\overline{I}_\Phi$ of the theory in subsection 3.4.2 appears as a 'proper class' and not a set, and there is no way to produce new 'sets' from the fixed point $\overline{I}_\Phi$.

3.5 Connecting the proof-theoretic dilator and $\Pi^1_1[R]$-proof theory

In this section, we prove the main theorem of this chapter. Proving the main theorem requires a variant of *implicational predilator* defined by Aguilera-Pakhomov [2, Lemma 8] for two linear orders over $\mathbb{N}$. The implication predilator is recursive when the two linear orders are also recursive. We modify this construction to find a recursive predilator for two R-recursive linear orders for a Σ^1_2-singleton real R.

Let us recall the construction of an implicational predilator: For two linear orders A and B and a parameter linear order X (which should be a well-order), we construct a tree $T(X)$ simultaneously building

1. An infinite decreasing sequence over B, and
2. An embedding $A \to X$.

The infinite decreasing sequence tries to witness $\neg\mathsf{WO}(B)$, and the embedding tries to see $\mathsf{WO}(A)$ by embedding A to a well-order X. That is, the tree tries to simultaneously verify $\mathsf{WO}(A) \wedge \neg\mathsf{WO}(B)$. If $\mathsf{WO}(A) \to \mathsf{WO}(B)$ holds, then $T(X)$ will be well-founded for every well-order X, so we can turn $T(X)$ into a well-order. In general, we can turn $T(X)$ into a linear order uniformly, which gives a predilator.

Now let A and B be the two R-recursive linear orders. We can find two recursive functions $a(X)$ and $b(X)$ such that $A = a(R)$, $B = b(R)$. $a(X)$ or $b(X)$ itself may

not be a linear order for some X, but $a(R)$ and $b(R)$ must be linear orders. The idea is to construct an implicational dilator and the Σ^1_2-singleton real R simultaneously. We do not (and cannot) use the whole R as a parameter to ensure the resulting predilator is recursive. But we can still use the partial information for R to compute $a(R)$ and $b(R)$. By Proposition 3.4.4, we may assume that R is generated by a locally well-founded recursive genedendron (D, ϱ).

Proposition 3.5.1 (ACA_0). *Let (D, ϱ) be a locally well-founded recursive genedendron generating a real R, and $a(Z), b(Z)$ be two recursive functions such that $a(R)$, $b(R)$ are linear orders. Then we can construct a recursive predilator $\mathsf{Imp}_{(D,\varrho),a,b}$ such that the following holds: For every well-order X,*

(i) *$\mathsf{Imp}_{(D,\varrho),a,b}$ is a dilator iff "if there is a well-order α such that D_α is ill-founded, then for every infinite branch $B \subseteq D_\alpha$, we have $\mathsf{WO}(a(R)) \to \mathsf{WO}(b(R))$ for $R = \varrho_\alpha(B)$." This statement is Π^1_2, and we write it informally as*

$$\neg\mathsf{Dil}(\mathrm{Lin}^*(D)) \to [\mathsf{WO}(a(R)) \to \mathsf{WO}(b(R))].$$

(Recall that $\neg\mathsf{Dil}(\mathrm{Lin}^*(D))$ iff the real generated by (D, ϱ) exists.)

(ii) *If $\mathsf{Imp}_{(D,\varrho),a,b}$ is not a dilator, $\mathsf{Imp}_{(D,\varrho),a,b}(X)$ is illfounded iff both $a(R)$ and $\mathsf{Clim}(D)$ embed into X.*[11]

(iii) *If $X \geq a(R), \mathsf{Clim}(D)$, then there is an embedding $b(R) \to \mathsf{Imp}_{(D,\varrho),a,b}(X)$.*

Proof. Fix a well-order X. The tree $T(X)$ is the set of triples $\langle \sigma, f, g \rangle$ satisfying the following:

1. Every component of $\langle \sigma, f, g \rangle$ is a finite sequence of the same length.

[11] We understand $\mathsf{Clim}(D) \leq X$ as the assertion '$D(X)$ is ill-founded.'

2. $\sigma \in D_X$.

3. For every $i < |f|$, $b(\varrho_X(\sigma))$ does not decide[12] $f(i) \notin \text{field}(b(\varrho_X(\sigma)))$.

4. For every $i, j < |f|$, $b(\varrho_X(\sigma))$ does not decide

$$f(i), f(j) \in \text{field}(b(\varrho_X(\sigma))) \to f(i) \not>_{b(\varrho_X(\sigma))} f(j).$$

5. $g \in X^{<\omega}$.

6. For $i, j < |g|$, if $g(i) <_X g(j)$, then $a(\varrho_X(\sigma))$ does not decide

$$i, j \in \text{field}(a(\varrho_X(\sigma))) \wedge i \geq_{a(\varrho_X(\sigma))} j.$$

Here 's decides P_s' for a recursive predicate P means a partial oracle s has enough information to decide the statement 'P_s is valid', and P_s is really valid.

Informally, $\sigma \in D_X$ encodes partial information $\varrho_X(\sigma)$ for a real R. f tries to construct an infinite decreasing sequence over $b(R)$, and g tries to construct an embedding from $a(R)$ to X. We understand $a(\varrho_X(\sigma))$ and $b(\varrho_X(\sigma))$ as R-oracle Turing machines with partial information $\varrho_X(\sigma)$ for the oracle R, so the computation may not be finished until they get further information about the oracle.

Conditions for f and g take the double-negated form; For example, we have the current form of (3) instead of the positive form 'For every $i < |f|$, $b(\varrho_X(\sigma))$ decides $f(i) \in \text{field}(b(\varrho_X(\sigma)))$.' However, we take the double-negated form in place of the positive form, because the previous 'positive form' can fail not only because $f(i) \in \text{field}(b(R))$, but also because $\varrho_X(\sigma)$ does not have enough information about R. To exclude the pathology from the latter case, we exclude $\langle \sigma, f, g \rangle$ only when we clearly know $f(i) \in \text{field}(b(R))$ fails from the given information $\varrho_X(\sigma)$. The real R is

[12] $b(Z)$ is a recursive function, so some facts about $b(Z)$ are computable from a proper initial segment of $b(Z)$. $\varrho_X(\sigma)$ is a finite sequence, so '$b(\varrho_X(\sigma))$ does not decide P' means $\varrho_X(\sigma)$ is not long enough to decide P.

uniquely determined, so the further branching along the tree $T(X)$ gives more information about R, and that $b(R)$ being R-recursive guarantees we can eventually determine $f(i) \in \text{field}(b(\varrho_X(\sigma)))$ from a sufficiently large $\varrho_X(\sigma) \subseteq R$.

Then let us impose a variant of Kleene-Brouwer order over $T(X)$, given by $\langle \sigma, f, g\rangle \ \leq_{\mathsf{KB}} \ \langle \sigma', f', g'\rangle$ if and only if $\langle \sigma(0), f(0), g(0), \cdots, \sigma(|\sigma| - 1), f(|\sigma| - 1), g(|\sigma|-1)\rangle$ is Kleene-Brouwer less than or equal to $\langle \sigma'(0), f'(0), g'(0), \cdots, \sigma'(|\sigma'|-1), f'(|\sigma'|-1), g'(|\sigma'|-1)\rangle$. $\sigma(i)$ and $\sigma'(i)$ are compared under the underlying order of D, and all other components are compared under the order over $\mathbb{N}$. Let us define $\mathsf{Imp}_{(D,\varrho),a,b}(X)$ by the resulting linearization. Then we can see that $\mathsf{Imp}_{(D,\varrho),a,b}$ admits the support transformation, so $\mathsf{Imp}_{(D,\varrho),a,b}$ is a semidilator, and it is easy to see that $\mathsf{Imp}_{(D,\varrho),a,b}$ is a predilator.

Now let us check the equivalent condition for $\mathsf{Imp}_{(D,\varrho),a,b}$ being a dilator: If $\text{Lin}^*(D)$ is a dilator, then D_X is well-founded for every well-order X, so $\mathsf{Imp}_{(D,\varrho),a,b}(X)$ cannot have an infinite branch for every well-order X. Hence $\mathsf{Imp}_{(D,\varrho),a,b}$ is a dilator in this case. Now suppose that $\text{Lin}^*(D)$ is not a dilator, and divide the cases:

1. Suppose that $b(R)$ is a well-order. Then $\mathsf{Imp}_{(D,\varrho),a,b}(X)$ as a tree cannot have an infinite branch for a well-ordered X, so $\mathsf{Imp}_{(D,\varrho),a,b}$ is a dilator.
2. Suppose that $b(R)$ is ill-founded. Then $\mathsf{Imp}_{(D,\varrho),a,b}(X)$ as a tree has an infinite branch if and only if D_X has an infinite branch and there is an embedding $a(R) \to X$. Put it in other words,

$$\neg\mathsf{WO}(\mathsf{Imp}_{(D,\varrho),a,b}(X)) \iff \neg\mathsf{WO}(\text{Lin}^*(D_X)) \wedge a(R) \leq X. \tag{3.9}$$

Hence $\mathsf{Imp}_{(D,\varrho),a,b}$ is a dilator iff

$$\forall^1 X[\mathsf{WO}(X) \to \neg[a(R) \leq X \wedge \neg\mathsf{WO}(\text{Lin}^*(D_X))]],$$

which is equivalent to

$$\forall^1 X[[\mathsf{WO}(X) \wedge \neg\mathsf{WO}(\mathrm{Lin}^*(D_X))] \to a(R) \not\leq X]. \tag{3.10}$$

We claim that (3.10) is equivalent to $\neg\mathsf{WO}(a(R))$: Fix a well-order α such that $\mathrm{Lin}^*(D_\alpha)$ is ill-founded. If $a(R)$ is well-founded, then $x = \alpha + a(R)$ witnesses the negation of (3.10). Conversely, if the negation of (3.10) holds, then $a(R)$ is embedded into some well-order, so $a(R)$ is a well-order too.

It finishes the proof of (i). For (ii), observe that if $\mathsf{Imp}_{(D,\varrho),a,b}$ is not a dilator, then D_X has an infinite branch, $b(R)$ is ill-founded, and (3.9) holds. We can see that (3.9) is precisely (ii).

For (iii), we need to construct an embedding $e\colon b(R) \to \mathsf{Imp}_{(D,\varrho),a,b}(X)$. Let us fix an infinite branch $\langle \sigma_i \mid i \in \mathbb{N}\rangle$ of D_X. Then $\bigcup_{n\in\mathbb{N}} \varrho_X(\langle \sigma_i \mid i < n\rangle) = R$. Also, we can fix an embedding $g\colon a(R) \to X$ and extend g to a function of domain $\mathbb{N}$ in a silly manner (for example, by taking $g(i)$ to be the $a(R)$-least element when $i \notin \mathrm{field}(a(R))$.)

Let us enumerate $b(R) = \{\beta_i \mid i \in \mathbb{N}\}$ in an $\mathbb{N}$-increasing way. Then define $e\colon b(R) \to \mathsf{Imp}_{(D,\varrho),a,b}(X)$ inductively as follows: Assume that $e(\beta_j)$ is defined for every $j < i$.

1. Suppose that $\beta_i = \max_{b(R)}\{\beta_j \mid j \leq i\}$. Then define

$$e(\beta_i) = \langle\langle\sigma_0\rangle, \langle\beta_i\rangle, g\restriction 1\rangle.$$

2. Otherwise, let $\beta_l = \min_{b(R)}\{\beta_j >_{b(R)} \beta_i \mid j < i\}$, and suppose

$$e(\beta_l) = \langle \sigma\restriction m, f, g\restriction m\rangle.$$

Then define

$$e(\beta_i) = \langle \sigma\restriction(m+1), f^\frown\langle\beta_i\rangle, g\restriction(m+1)\rangle.$$

We can see that e is an order-preserving map as follows: Observe that the order between $e(\beta_i)$ and $e(\beta_j)$ only depends on the second component. Furthermore, the second component of $e(\beta_i)$ is an $b(R)$-decreasing sequence ending with β_i indexed by natural numbers less than or equal to i.

Now we claim by induction on i that $e \restriction \{\beta_0, \cdots, \beta_i\}$ is order-preserving: If $\beta_i = \max_{b(R)}\{\beta_j \mid j \leq i\}$, then clearly $\beta_j \leq \beta_i$ and $e(\beta_j) \leq e(\beta_i)$ for $j < i$ by definition of $e(\beta_i)$. Otherwise, let $\beta_l = \min_{b(R)}\{\beta_j >_{b(R)} \beta_i \mid j < i\}$. Then we have $\beta_i < \beta_l$ and $e(\beta_i) < e(\beta_l)$. For $j < i$, if $\beta_i < \beta_j$, then $\beta_l \leq \beta_j$, so $e(\beta_i) < e(\beta_l) \leq e(\beta_j)$ by the induction hypothesis. If $\beta_i \geq \beta_j$, then $\beta_l > \beta_j$, so $e(\beta_l) > e(\beta_j)$. This means that the following holds: If f_l and f_j are the second component of $e(\beta_l)$ and $e(\beta_j)$ respectively, then either

1. there is a least $m < |f_l|, |f_j|$ such that $f_l(m) > f_j(m)$, or
2. f_j is a proper extension of f_l.

If f_i is the second component of $e(\beta_i)$, then f_i is a proper extension of f_l. Hence if the first case holds, then $e(\beta_i) > e(\beta_j)$ as m is the least number such that $f_l(m) = f_i(m) > f_j(m)$. In the second case, suppose that $f_j = f_l^\frown \langle \beta_k, \cdots, \beta_j \rangle$ for some $k < j$. Then $\beta_k > \beta_i$ never holds; Otherwise, it contradicts the choice of β_l. Hence $\beta_k \leq \beta_i$, so $f_j \leq f_i$ under the lexicographic order. □

Theorem 3.5.2. *Let T be a Π^1_2-sound theory extending ACA_0 and (D, ϱ) be a recursive locally well-founded genedendron generating R. Furthermore, assume that T proves (D, ϱ) is a locally well-founded genedendron. If α is an R-recursive linear order such that D_α is ill-founded, then*

$$|T|_{\Pi^1_2}(\alpha) = |T[R] + \mathsf{WO}(\alpha)|_{\Pi^1_1[R]}.$$

Proof. Since α is R-recursive, we can find a recursive function a such that $\alpha = a(R)$. For one direction of the inequality, let us decompose $|T|_{\Pi^1_2}$ by the sum of recursive dilators $\sum_{n\in\mathbb{N}} D_n$. Since $T \vdash \mathsf{Dil}(D_i)$ for each i, we have

$$T[R] + \mathsf{WO}(a(R)) \vdash \mathsf{WO}((D_0 + \cdots D_n)(a(R))).$$

Hence $(D_0 + \cdots + D_n)(a(R)) \leq |T[R] + \mathsf{WO}(a(R))|_{\Pi^1_1[R]}$, so taking the supremum to n gives

$$|T|_{\Pi^1_2}(a(R)) \leq |T[R] + \mathsf{WO}(a(R))|_{\Pi^1_1[R]}.$$

For the remaining direction, suppose that b is a recursive function such that $b(R)$ is a well-order and

$$T[R] + \mathsf{WO}(a(R)) \vdash \mathsf{WO}(b(R)).$$

Hence we have

$$T \vdash \neg\mathsf{Dil}(D) \to (\mathsf{WO}(a(R)) \to \mathsf{WO}(b(R))).$$

Since T extends ACA_0, T proves Proposition 3.5.1. Also, we assumed that T proves (D, ϱ) is a locally well-founded genedendron. Hence we have $T \vdash \mathsf{Dil}(\mathsf{Imp}_{(D,\varrho),a,b})$, so $\mathsf{Imp}_{(D,\varrho),a,b} \leq |T|_{\Pi^1_2}$. Since $D_{a(R)}$ is ill-founded, we have $a(R) \geq \mathsf{Clim}(D)$, so by (iii) of Proposition 3.5.1, we have $b(R) \leq \mathsf{Imp}_{(D,\varrho),a,b}(a(R))$. In sum, we have

$$b(R) \leq \mathsf{Imp}_{(D,\varrho),a,b}(a(R)) \leq |T|_{\Pi^1_2}(a(R)).$$

Since every $T[R]$-provably R-recursive well-order takes the form $b(R)$ for some recursive b, we have

$$|T[R] + \mathsf{WO}(a(R))|_{\Pi^1_1[R]} \leq |T|_{\Pi^1_2}(a(R)). \qquad \square$$

The following corollary follows by applying Theorem 3.5.2 to the genedendron we found in subsection 3.4.2:

Corollary 3.5.3.

$$|\mathsf{ACA}_0 + \mathsf{HJ}(\emptyset)\ \mathit{exists}|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = |\mathsf{ACA}_0|_{\Pi^1_2}(\omega_1^{\mathsf{CK}}) = \varepsilon_{\omega_1^{\mathsf{CK}}+1}. \qquad \square$$

3.6 Final remarks

The main result in this chapter implies there is a systematic way to produce proof-theoretic information about $\Pi^1_1[R]$-consequences for a Σ^1_2-singleton real R by combining Π^1_2- and Σ^1_2-information of a theory. In this last section, we discuss what we can observe from the results in this chapter and directions for future work.

We may view $|T|_{\Pi^1_2}(\alpha)$ as a 'section' of $|T|_{\Pi^1_2}$, and we may ask if sections of $|T|_{\Pi^1_2}$ gives useful information about $|T|_{\Pi^1_2}$. For example, we may ask if the sections of a proof-theoretic dilator are enough to determine the Π^1_2-consequence comparison modulo true Σ^1_2-statements:

Question 3.6.1. Let S, T be Π^1_2-sound r.e. extensions of ACA_0. Define

$$S \subseteq^{\Sigma^1_2}_{\Pi^1_2} T \iff \text{For every } \Pi^1_2\text{-sentence } \sigma,\ S \vdash^{\Sigma^1_2} \sigma \implies T \vdash^{\Sigma^1_2} \sigma,$$

where $T \vdash^{\Sigma^1_2} \sigma$ means 'σ is proavable from T with the set of true Σ^1_2-sentences.' Do we have the following equivalence?

$$S \subseteq^{\Sigma^1_2}_{\Pi^1_2} T \iff \forall (D, \varrho)[|S|_{\Pi^1_2}(\mathsf{Clim}(D)) \leq |T|_{\Pi^1_2}(\mathsf{Clim}(D))],$$

where (D, ϱ) ranges over ACA_0-provably recursive locally well-founded genedendrons.

The example illustrated in subsection 3.4.2 can be understood as a Σ^1_2-ordinal analysis for Peano arithmetic plus the unary predicate for the hyperjump of $\emptyset$ (equivalently, Kleene's $\mathcal{O}$.) We obtained a genedendron by taking the β-preproof of the false sequent obtained from the β-preproof property. It resembles obtaining the proof-theoretic ordinal or dilator from cut elimination, so we may understand the role of the β-preproof property in Σ^1_2-proof theory is like that of the cut-elimination in Π^1_1- and Π^1_2-proof theory.

We may expect the same for the ω-logic: For a given ω-proof system for some theory T with the preproof property, we can generate an ω-model of T from an infinite branch

of a preproof of the empty sequent. It should provide the framework for the Σ^1_1-proof theory. Then we may ask which algebraic object captures the behavior of the ill-founded ω-preproof: In the Π^1_1- and Π^1_2- cases, each of them are proof-theoretic ordinal and a proof-theoretic dilator, respectively. In the Σ^1_2-case, it is the climax of a β-preproof of the empty sequent. In the Σ^1_1-case, pseudo-well-orders might behave as the algebraic object as foreshadowed in [97, §2.4].

One may ask if we can generalize the main theorem. One natural direction is pursuing a proof-theroetic meaning of $|T|_{\Pi^1_2}(\alpha)$ for a larger α. There are two main obstacles to this way of generalization. One is related to the limit of genedendrons and pseudodilators. We associated some ordinals below δ^1_2 with a recursive genedendron (D, ϱ), which generates a Σ^1_2-singleton real, and the climax of D is the ordinal. To extract a proof-theoretic meaning of $|T|_{\Pi^1_2}(\alpha)$ for larger α, we should associate α with a more sophisticated object than genedendrons. One of the reasons is that the underlying pseudodilator of a recursive genedendron must have the climax less than δ^1_2, so any recursive genedendron cannot be associated with ordinals larger than δ^1_2. The appropriate object should come from β_n-logic or its generalizations. The other obstacle, which is more serious than the previous one, is that the proof of Theorem 3.5.2 relies on the fact that every $\Pi^1_1[R]$-statement for a Σ^1_2-singleton real R is Π^1_2, so the proof-theoretic dilator already sees $\Pi^1_1[R]$-statements. For a more complex R, there is no guarantee that the proof-theoretic dilator sees $\Pi^1_1[R]$-statements. We might need to consider recursive genedendrons 'derived from' a higher object associated with $\alpha > \delta^1_2$ simultaneously to get the right generalization.

The other way of a generalization is asking the proof-theoretic meaning of $|T|_{\Pi^1_n}(P)$ for an $(n-1)$-ptyx P with reasonable definability constraint (like, implicit Σ^1_n-definability.) As an example, the author conjectures the following:

Conjecture 3.6.2. Working over ZFC with the existence of $0^\sharp$, let T be a Π^1_3-sound

extension of ACA_0. If we view $0^\sharp$ as a dilator (cf. [6]), we have

$$|T|_{\Pi^1_3}(0^\sharp) = |T + \exists 0^\sharp|_{\Pi^1_1[0^\sharp]}.$$

Σ^1_2-altitude appears as a byproduct of the main result, but it should be able to get its own attention. Σ^1_2-altitude defines a way to rank Σ^1_2-singleton reals and is computable by hand for some special cases. We may ask if the Σ^1_2-altitude has a recursion-theoretic or set-theoretic characterization, and the following conjecture suggests one possibility:

Conjecture 3.6.3. Let R be a Σ^1_2-singleton real. $\mathsf{Alt}_{\Sigma^1_2}(R)$ is equal to the least height of a transitive model M of $\mathsf{ATR}^{\mathsf{set}}_0$ on which R is a Σ_1-definable class in the language of set theory.

3.A Iterated hyperjumps

The main aim of this section is to provide the Π^1_1-singleton definition of the ξ-th iterated hyperjump of the empty set. We formulate the iterated hyperjump of the empty set as an ordinal notation system.

3.A.1 Inductive definition

Definition 3.A.1. Let $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be an operator (which is usually definable) that is *inclusive*, i.e., $X \subseteq \Gamma(X)$ for every set $X \subseteq \omega$. We define $\langle \Gamma^\xi \mid \xi \in \mathrm{Ord} \rangle$ recursively by $\Gamma^\xi = \bigcup_{\eta<\xi} \Gamma(\Gamma^\eta)$. The *closure point of* Γ, which is denoted by $|\Gamma|$, is the least ordinal λ such that $\Gamma^\lambda = \Gamma^{\lambda+1}$. We say Γ^λ for the closure point of Γ the *set defined by* Γ

We formulated the definition of Γ^ξ in set theory. However, we can formulate the same definition in a subsystem of second-order arithmetic by replacing ordinals with elements

of a sufficiently long well-order. We do not know in general that we have a sufficiently long well-order terminating the inductive definition. However, we can still state which set is defined by a given inclusive operator.

We want to make sure the set defined by an operator is ACA_0-provably unique even when it may not ACA_0-provably exist. If we work over ATR_0, the trichotomy of well-orders guarantees the sequence $\langle \Gamma^\xi \mid \xi \in \mathrm{field}(\alpha) \rangle$ given by an arithmetical inclusive operator Γ only depends on the ordertype of α. The next proposition says we can weaken ATR_0 to ACA_0 for an inclusive arithmetical operator:

Proposition 3.A.2 (ACA_0)**.** *Let Γ be an inclusive arithmetical operator. For a given well-order α, let us say a set X is a* chain of Γ-inductive definition along α *if for each $\xi \in \mathrm{field}(\alpha)$ and n,*

$$n \in (X)_\xi \iff \exists \eta <_\alpha \xi [n \in \Gamma((X)_\eta)],$$

where $(X)_\xi = \{m \mid \langle \xi, m \rangle \in X\}$. A chain of Γ-inductive definition X along α is exact *if α has the maximal element μ and its predecessor μ^-, and*

1. *$\eta <_\alpha \xi <_\alpha \mu$ implies $(X)_\eta \subsetneq (X)_\xi$.*
2. *$(X)_\mu = (X)_{\mu^-}$.*

Suppose that α and β are well-orders, X an exact chain of Γ-inductive definition along α, Y an exact chain of Γ-inductive definition along β. Then there is an isomorphism $f : \alpha \to \beta$ such that $(X)_\xi = (Y)_{f(\xi)}$ for each $\xi \in \mathrm{field}(\alpha)$.

Proof. Let μ_α, μ_α^- be the maximal element and its predecessor of α, and similarly for μ_β and μ_β^-. We prove the following assertion by arithmetical transfinite induction (which is a theorem of ACA_0; See [89, Lemma V.2.1]): For each $i \in \mathrm{field}(\alpha)$, either $i = \mu_\alpha$ or there is a unique $j \in \mathrm{field}(\beta)$ such that $j \neq \mu_\beta$, $(X)_i = (Y)_j$.

Suppose that $i \neq \mu_\alpha$ and the claim holds for $i' <_\alpha i$, so for each $i' <_\alpha i$ we have a unique $j' \in \mathrm{field}(\beta) \setminus \{\mu_\beta\}$ such that $(X)_{i'} = (Y)_{j'}$. Such j' is uniformly arithmetically definable from i', X, Y, so we can unambiguously denote such j' by $f(i')$ for some arithmetically definable function f. Let us define

$$j = \min{}_\beta\{k \in \mathrm{field}(\beta) \mid \forall i' <_\alpha i[f(i') < k]\}.$$

We claim $(X)_i = (Y)_j$: First,

$$n \in (X)_i \iff \exists i' <_\alpha i[n \in \Gamma((X)_{i'})] \implies \exists j' <_\beta j[n \in (Y)_{j'}]$$

so $(X)_i \subseteq (Y)_j$. Conversely, if $j' <_\beta j$, then there is $i' <_\alpha i$ such that $j' \leq_\beta f(i')$, so we have the reversed inclusion. The construction of the isomorphism $f \colon \alpha \to \beta$ is straightforward. □

Hence the following Σ^1_2-formula witnesses the set H defined by an arithmetical Γ is a Σ^1_2-singleton set: "There is a well-order α with the maximal element μ with its predecessor, and an exact chain of Γ-inductive definition, we have $H = (X)_\mu$." Moreover, ACA_0 proves the uniqueness of the real satisfying the Σ^1_2-singleton definition. (Note that ACA_0 does not prove the existence of the real in general.)

Remark 3.A.3. We could make the Σ^1_2-singleton definition into a Π^1_1-singleton definition by using linear orders with a Π^0_m-transfinite induction for a sufficiently large m in place of well-orders. More precisely, if Γ is Π^0_m-definable, the proof of Proposition 3.A.2 shows the following Π^1_1-formula defines the set H defined by Γ:

> Suppose that α is a linear order with the maximal element μ with its predecessor, such that Π^0_{m+99}-transfinite induction along α holds. For every exact chain of Γ-inductive definition, we have $H = (X)_\mu$.

ACA_0 proves there is at most one real satisfying the Π^1_1-formula.

3.A.2 Ordinal notation system

In this section, we follow notions in [84, §8]. First, let us fix the following notations:

- $\mathsf{sup}(a, b) = \langle 0, a, b\rangle$.
- $a \circledcirc b = \langle 1, a, b\rangle$.

Also, for a natural number a, x and a real number R, $[a](\cdot, R)$ denotes the ath R-primitive recursive function.

As an illustrative example for an ordinal notation system, the following ordinal notation resembles Kleene's $\mathcal{O}$, where $|a| = \min\{\alpha \mid a \in M_{\alpha+1}\}$:

1. $M_0 = \emptyset$.
2. $M_{\alpha+1} = M_\alpha \cup \{0\} \cup \{\mathsf{sup}(a, b) \mid a \in M_\alpha,\ \forall^0 x[[b](x, M_{|a|}) \in M_\alpha]\}$.
3. $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ if δ is limit.

Here we understand $\mathsf{sup}(a, b)$ as the notation for the ordinal

$$\max\{|a| + 1, \sup_{x<\omega}(|[b](x, M_{|a|})| + 1)\}.$$

Now let us use the notations for $X \subseteq \omega$:

- $\mathcal{F}(X) = \{x \mid \langle x, x\rangle \in X\}$, and
- $X_{<x} = \{y \mid \langle y, x\rangle \in X \wedge \langle x, y\rangle \notin X\}$.
- $M^*_\alpha = \{\langle x, y\rangle \mid x, y \in M_\alpha \wedge |x| \leq |y|\}$.

If we define

$$\Theta(X) = \{0\} \cup \{\mathsf{sup}(a, b) \mid a \in \mathcal{F}(X) \wedge \forall^0 x[[b](x, X_{<a}) \in M_\alpha]\},$$

then the previous ordinal notation system satisfies $M_{\alpha+1} = M_\alpha \cup \Theta(M_\alpha^*)$. Let us take it as a general definition for an ordinal notation system:

Definition 3.A.4. Let Θ be an inclusive operator. $\mathfrak{M}^\Theta = (M^\Theta, |\cdot|)$ is defined by

1. $M_0 = \emptyset$.
2. $M_{\alpha+1} = M_\alpha \cup \Theta(M_\alpha^*)$.
3. $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ if δ is limit.
4. $M^\Theta = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$.

$|M|$ is the least ordinal α such that $M_\alpha = M_{\alpha+1}$.

The definition of $M_{\alpha+1}$ depends not only M_α, but also on M_α^* for $a \in M_\alpha$, so the definition of M_α itself is not precisely the inductive definition. We get around this issue by defining M_α^* inductively:

Lemma 3.A.5 ([84, Lemma 8.13]). *For an operator Θ, if we define*

$$\Theta_\leq(X) = \{\langle x, y\rangle \mid x \in \mathcal{F}(X) \wedge y \in \Theta(X) \setminus \mathcal{F}(X)\} \cup \{\langle x, y\rangle \mid x, y \in \Theta(X) \setminus \mathcal{F}(X)\},$$

then $\Theta_\leq^\alpha = M_\alpha^*$ for every ordinal α.

Note that if Θ is Π^0_m or Σ^0_m, then so is $\Theta_\leq$ since $\mathcal{F}$ is primitive recursive. The following type of ordinal notation systems are guaranteed to have a nice behavior, which the coding lemma will characterize:

Definition 3.A.6. A notation system $\mathfrak{M} = (M, |\cdot|)$ is *Richterian*[13] if $\mathfrak{M} = \mathfrak{M}^{\Theta}$ where $\Theta(X) = \mathfrak{R}(X) \cup \Phi(X)$, where $\mathfrak{R}(X)$ is the *Richter operator*

$$\mathfrak{R}(X) = \{0\} \cup \{\mathsf{sup}(a,b) \mid a \in \mathcal{F}(X) \wedge \forall^0 x[[b](x, X_{<a}) \in \mathcal{F}(X)]\} \\ \cup \{a \circledcirc b \mid a \in \mathcal{F}(X) \vee b \in \mathcal{F}(X)\},$$

and $\Phi(X)$ is always disjoint from $\{0\} \cup \{\mathsf{sup}(a,b),\ a \circledcirc b \mid a, b \in \omega\}$.

For a Richterian notation system $\mathfrak{M}$, an ordinal $\lambda \leq |M|$ is $\mathfrak{M}$-*Richterian* if $\mathfrak{R}(M^*_\lambda) \subseteq M_\lambda$. (In some sense, an ordinal is $\mathfrak{M}$-Richterian if M_λ is closed under the Richter operator.) $|M|$ is $\mathfrak{M}$-Richterian, but in many cases, there are $\mathfrak{M}$-Richterian ordinals below $|M|$.

The role of $a \circledcirc b$ is rather technical, and the author does not know about its necessity. Richter also mentioned in [83] that he does not know if the $\circledcirc$ operator is necessary. Also, the hierarchy given by a Richerian operator increases under Turing reducibility in the following sense:

Lemma 3.A.7 ([84, Lemma A.3]). *If $|a| + 1 < \beta$, then $\mathsf{TJ}(M_{|a|}) \leq_{\mathsf{T}} M_\beta$ uniformly in a. Moreover, the reduction can be primitive recursive.*

Proof. Let e be a recursive function such that

$$[e(a,x)](t, M_{|a|}) = \begin{cases} a & \text{if } \neg T^{M_{|a|}}(x,x,t), \\ 1 & \text{otherwise.} \end{cases}$$

(Note that $1 \notin M$.) Then we can see that

$$x \notin \mathsf{TJ}(M_{|a|}) \iff \forall^0 t \neg T^{M_{|a|}}(x,x,t) \\ \iff \forall^0 t [e(a,x)](t, M_{|a|}) = a \in M_{|a|+1} \iff \mathsf{sup}(a, e(a,x)) \in M_\alpha,$$

[13] [84] used the word *good*, but the word 'good' is not good to use since it is used too much elsewhere.

so $\mathbb{N} \setminus \mathsf{TJ}(M_{|a|}) \leq_{\mathsf{T}} M_\alpha$ is witnessed by $x \mapsto \mathsf{sup}(a, e(a,x))$. Since Kleene's T-predicate is primitive recursive, we can choose e to be primitive recursive. □

The next proposition says $M_{|x|}$ and $M^*_{|x|}$ are Turing equivalent:

Proposition 3.A.8. *Let $\mathfrak{M} = (M, |\cdot|)$ be a Richterian notation system. For each $x \in M$, $M_{|x|} \equiv_{\mathsf{T}} M^*_{|x|}$.*

Proof. $M_{|x|} \leq_{\mathsf{T}} M^*_{|x|}$ is clear. (In fact, $M_{|x|} \leq_1 M^*_{|x|}$.) For $M^*_{|x|} \leq_{\mathsf{T}} M_x$, suppose that $\langle u, v\rangle \in M^*_{|x|}$ so $u, v \in M_{|x|}$. Let id and c_y be uniform primitive recursive codes for the identity function and the constant function of value y, respectively. First, let us observe that if $|u| < |x|$, then

$$|u| > |v| \iff \mathsf{sup}(u, c_v) \in M_{|x|}.$$

Also, $|u| < |x|$ iff $\mathsf{sup}(u, \mathsf{id}) \in M_{|x|}$ since $|\mathsf{sup}(u, \mathsf{id})| = |u| + 1$. Hence for *every* u and v,

$$\langle u, v\rangle \in M^*_{|x|} \iff u, v \in M_{|x|} \wedge \big[\mathsf{sup}(u, \mathsf{id}) \in M_{|x|} \to \mathsf{sup}(u, c_v) \in M_{|x|}\big]. \qquad \square$$

The following *coding lemma* (which should be called the coding theorem) appeared first in [83], and its full details appeared in [84, §A]. According to [83], its initial idea goes back to Gandy's unpublished result of $|\Pi^0_1| = \omega_1^{\mathsf{CK}}$.

Theorem 3.A.9 (Richter's Coding Lemma, [84, Lemma 8.17]). *Let $\mathfrak{M} = (M, |\cdot|)$ be a Richterian notation system and let $T_{\mathfrak{M}} = \{\langle x, \alpha\rangle \mid \alpha \in \mathrm{Ord} \wedge x \in M_\alpha\}$. Then*

1. *Every $\mathfrak{M}$-Richterian ordinal is admissible relative to $T_{\mathfrak{M}}$.*
2. *For every Σ_1-formula $\phi(v_0, \cdots, v_{n-1})$ in the language of primitive recursive set functions on ordinals with the extra predicate for $T_{\mathfrak{M}}$, there is a primitive recursive function h such that for every $\mathfrak{M}$-Richterian ordinal λ,*

$$a_0, \cdots a_{m-1} \in M_\lambda \wedge \lambda \vDash \phi(|a_0|, \cdots, |a_{m-1}|) \iff h(a_0, \cdots, a_{m-1}) \in M_\lambda.$$

3. If λ is $\mathfrak{M}$-Richterian, then for $X \subseteq \omega$, X is λ-r.e. in $T_{\mathfrak{M}} \restriction \lambda$ iff $X \leq_{\mathsf{m}} M_\lambda$. $\square$

We will only use the first clause of Richter's Coding Lemma in this chapter.

3.A.3 Ordinal notation system for the least recursively inaccessible ordinal

Let us consider the following Richterian notation system:

Definition 3.A.10. Let us define J as follows:

$$n \in J(X) \iff n \in \mathfrak{R}(X) \vee \big[\mathfrak{R}(X) \subseteq \text{field}(X) \wedge n \in \{\langle 2, x\rangle \mid x \in \mathcal{F}(X)\}\big].$$

Roughly, $x \mapsto \langle 2, x\rangle$ maps x to an ordinal notation denoting the next admissible ordinal greater than $|x|$. We can see that J is an arithmetical Richterain operator, so the closure point of $\mathfrak{M}^J = (M, |\cdot|)$ is admissible. From the informal description, we can suspect that $|\langle 2, x\rangle|$ for $x \in M$ is admissible, and the following proposition confirms it is:

Proposition 3.A.11. *For $x \in M$, $|\langle 2, x\rangle|$ is $\mathfrak{M}$-Richterian. In fact, $|\langle 2, x\rangle|$ is the least $\mathfrak{M}$-Richterian ordinal above $|x|$.*

Proof. $\lambda = |\langle 2, x\rangle|$ means λ is the least ordinal such that $\langle 2, x\rangle \in M_{\lambda+1} = M_\lambda \cup J(M_\lambda^*)$, which happens only when $\mathfrak{R}(M_\lambda^*) \subseteq M_\lambda$. Hence λ is $\mathfrak{M}$-Richterian. If $\lambda' > |x|$ is another $\mathfrak{M}$-Richterian ordinal, then $\langle 2, x\rangle \in M_{\lambda'+1}$. It shows $|\langle 2, x\rangle| \leq \lambda'$. $\square$

By Richter's coding lemma, $|M|$ is not only admissible but also limit admissible. Hence $|M|$ is recursively inaccessible. Now let us claim that $|M|$ is the least recursively

inaccessible ordinal. Observe that $J(X)$ works as follows: If X is not $\mathfrak{R}$-saturated, take $\mathfrak{R}(X)$. If X is $\mathfrak{R}$-saturated, take $\{\langle 2, x\rangle \mid x \in \mathcal{F}(X)\}$. Also, observe that $\mathfrak{R}$ is Π^0_1, and $X \mapsto \{\langle 2, x\rangle \mid x \in \mathcal{F}(X)\}$ is primitive recursive (equivalently, Π^0_0.) Let us generalize the definability complexity of J (and that of $J_{\leq}$) as follows:

Definition 3.A.12. A $[\Pi^0_1, \Pi^0_0]$-*operator* is an operator Γ of the form

$$n \in \Gamma(X) \iff n \in \Gamma_0(X) \vee [\Gamma_0(X) \subseteq X \implies n \in \Gamma_1(X)]$$

for a Π^0_1-operator Γ_0 and a primitive recursive operator Γ_1.

Then we have the following:

Proposition 3.A.13. *If Γ is a $[\Pi^0_1, \Pi^0_0]$-operator, then $|\Gamma|$ is less than or equal to the least recursively inaccessible ordinal.*

The proof of Proposition 3.A.13 depends on the proof of the following lemma appearing in [83, Lemma 2.2]. We include its proof for completeness.

Lemma 3.A.14. *Let Γ be a Π^0_1-operator and λ an admissible ordinal. Then $\Gamma^\lambda = \Gamma^{\lambda+1}$.*

Proof. First, since '$n \notin \Gamma(X)$' is an r.e. predicate on $\mathbb{N} \times \mathcal{P}(\mathbb{N})$, we can find a recursive predicate R on $\mathbb{N}^3$ such that

$$n \notin \Gamma(X) \iff \exists x \in \mathrm{Seq}(X) \exists y \in \mathrm{Seq}(\mathbb{N} \setminus X) R(n, x, y).$$

Here $\mathrm{Seq}(X)$ is the set of natural numbers coding a finite sequence over X. In particular,

$$n \in \Gamma(X) \iff \forall^0 x \in \mathrm{Seq}(X) \forall^0 y \big[R(n, x, y) \to y \in \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus X)\big].$$

Note that both $X \mapsto \mathrm{Seq}(X)$ and $X \mapsto \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus X)$ are monotone. Now let $\langle \Gamma^\xi \mid \xi \in \mathrm{Ord}\rangle$ be a sequence defined by Γ. We claim that if λ is admissible and $n \in \Gamma^{\lambda+1} = \Gamma(\Gamma^\lambda)$, then $n \in \Gamma^\lambda$.

Suppose that $n \in \Gamma^{\lambda+1} = \Gamma(\Gamma^\lambda)$, which is equivalent to

$$\forall^0 x \in \mathrm{Seq}(\Gamma^\lambda)\forall^0 y[R(n,x,y) \to y \in \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus \Gamma^\lambda)].$$

Now for an admissible set M of height λ, we claim that there is an Σ_1^M-definable normal function $f\colon \lambda \to \lambda$ such that for $\alpha < \lambda$,

$$\forall x \in \mathrm{Seq}(\Gamma^\alpha)\forall y[R(n,x,y) \implies y \in \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus \Gamma^{f(\alpha)})].$$

Note that $\langle \Gamma^\xi \mid \xi < \lambda\rangle$ is Σ_1-definable over M. For $x \in \mathrm{Seq}(\Gamma^\lambda)$ and $y \in \mathbb{N}$, define

$$g(x,y) = \begin{cases} \min\{\alpha \mid y \in \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus \Gamma^\alpha)\} & \text{If } R(n,x,y), \\ 0 & \text{otherwise.} \end{cases}$$

Then $g\colon \mathrm{Seq}(\Gamma^\lambda) \times \mathbb{N} \to \lambda$ is also Σ_1-definable over M. For $\alpha < \lambda$, define $h(\alpha) = \sup\{g(x,y) \mid x \in \mathrm{Seq}(\Gamma^\alpha) \wedge y \in \mathbb{N}\}$, and define f by Σ_1-recursion over M so that $f(0) = h(0)$, $f(\alpha+1) = f(\alpha) + h(\alpha)$, and $f(\delta) = \sup_{\alpha<\delta} f(\alpha)$ if δ limit. Then f is the desired function.

Since f is normal and Σ_1-definable over M, f has a fixed point $\beta < \lambda$. Hence we have

$$\forall x \in \mathrm{Seq}(\Gamma^\beta)\forall y[R(n,x,y) \to y \in \mathbb{N} \setminus \mathrm{Seq}(\mathbb{N} \setminus \Gamma^\beta)],$$

so $n \in \Gamma^{\beta+1} \subseteq \Gamma^\lambda$. □

Proof of Proposition 3.A.13. Suppose that Γ is a $[\Pi^0_1, \Pi^0_0]$-operator given by a Π^0_1-operator Γ_0 and a Π^0_0-operator Γ_1. Modifying the proof of Lemma 3.A.14 gives the following: If α is admissible, then $\Gamma(\Gamma^\alpha) = \Gamma_1(\Gamma^\alpha)$. Now let λ be a recursively inaccessible ordinal. Suppose that $n \in \Gamma^{\lambda+1} = \Gamma_1(\Gamma^\lambda)$. Since Γ_1 is (primitive) recursive, answering the query $n \in \Gamma_1(\Gamma^\lambda)$ uses finitely many questions about the membership questions on Γ^λ. In addition, λ is a limit, so we have that

$$n \in \Gamma_1(\Gamma^\lambda) \iff \exists \xi < \lambda \forall \eta < \lambda[\eta \geq \xi \to n \in \Gamma_1(\Gamma^\eta)].$$

Since admissible ordinals are cofinal in λ, we can find an admissible $\eta \geq \xi$ for ξ given in the previous statement. For such η, $n \in \Gamma_1(\Gamma^\eta) = \Gamma^{\eta+1} \subseteq \Gamma^\lambda$. □

Hence we have

Corollary 3.A.15. *For $\mathfrak{M}^J = (M, |\cdot|)$, $|M|$ is the least recursively inaccessible ordinal.*

Note that the proof of Proposition 3.A.13 (In fact, the modification of Lemma 3.A.14) also shows every admissible ordinal is $\mathfrak{M}^J$-Richterian. From the remaining part of this section, let us prove the following:

Theorem 3.A.16. *For $\mathfrak{M}^J = (M, |\cdot|)$, if $x \in M$ then $\mathcal{O}^{M_{|x|}} \equiv_{\mathsf{T}} M_{|\langle 2,x\rangle|}$ uniformly in x.*

Here $\mathcal{O}^X$ is the set of codes for X-primitive recursive functions coding a well-order; i.e., set of e such that $Z_e^X := \{\langle x, y\rangle \mid [e](\langle x, y\rangle, X) = 0\}$ is a well-order. By Kleene normal form theorem, $\mathcal{O}^X$ is Turing equivalent to a universal $\Pi^1_1[X]$-set.

Proof. Let us first prove $M_{|\langle 2,x\rangle|} \leq_{\mathsf{m}} \mathcal{O}^{M_{|x|}}$. We have two cases: If x is not of the form $\langle 2, y\rangle$, then $M_{|x|}$ is not closed under the Richter operator. By Proposition 3.A.11, $|\langle 2, x\rangle|$ is the least $\mathfrak{M}$-Richterian ordinal greater than $|x|$. This means that $M_{|\langle 2,x\rangle|}$ is the least fixed point of $\mathfrak{R}$ containing $M_{|x|}$, so $M_{|\langle 2,x\rangle|}$ is $\Pi^1_1[M_{|x|}]$. If x is of the form $\langle 2, y\rangle$, $M_{|x|}$ is closed under the Richter operator so the previous formula gives $M_{|x|}$ and not $M_{|\langle 2,x\rangle|}$. However, $M_{|\langle 2,x\rangle|}$ is the least fixed point of $\mathfrak{R}$ containing $M_{|x|} \cup \{\langle 2, y\rangle \mid y \in M_{|x|}\}$. Hence we still have $M_{|\langle 2,x\rangle|}$ is $\Pi^1_1[M_{|x|}]$.

For uniformity, let us find a Π^1_1-formula $\phi(X, n)$ equivalent to the following:

> n is in the least fixed point of $\mathfrak{R}$ containing X, *or if* $\mathfrak{R}(X) \subseteq X$, then n is in the least fixed point of $\mathfrak{R}$ containing $X \cup \{\langle 2, y\rangle \mid y \in X\}$.

Hence we can provide the reduction as follows: By Kleene normal form theorem, we can find a primitive recursive code e such that $\phi(X, n)$ iff $[e](\cdot, n, X)$ codes a *strict* well-order (so reflexivity fails). By smn-theorem, we can find e' such that $[e](x, n, X) = \{[e'](n)\}(x, X)$ for every $n, x \in \mathbb{N}$ and a real X. Then we have

$$n \in M_{|\langle 2,x\rangle|} \iff [e'](n) \in \mathcal{O}^{M_{|x|}}.$$

For $\mathcal{O}^{M_{|x|}} \leq_{\mathsf{T}} M_{|\langle 2,x\rangle|}$, we follow the proof in [85, §I.4]. First, let us consider the primitive recursive function h, taking an index e for an X-primitive recursive well-order Z_e^X and $n \in \mathbb{N}$ returning an X-primitive recursive index $h(e, n)$, so it satisfies

$$[h(e,n)](\langle u, v\rangle, X) = \begin{cases} 0 & \text{if } [e](z, X) = 0 \text{ for } z = \langle u, v\rangle, \langle u, n\rangle, \langle v, n\rangle, \\ 1 & \text{otherwise.} \end{cases}$$

We can find a *primitive recursive* h satisfying the above equation by effectively finding a primitive recursive code for the function

$$e, n \mapsto (\langle u, v\rangle, X \mapsto \min(\{1\} \cup \{[e](z, X) \mid z = \langle u, v\rangle, \langle u, n\rangle, \langle v, n\rangle\})).$$

Intuitively, $h(e, n)$ is an index for an initial segment of Z_e^X below n, if n is in the field of the well-order. (Otherwise, $h(e, n)$ codes the empty order.) By Kleene's recursion theorem, there is a partial recursive function f such that for every e on which $f(e)$ is defined,

$$f(e) = \mathsf{sup}(x, \lambda n. f(h(e, n))).$$

and the left-hand side is defined iff the right-hand side is defined. Now we claim f is the desired reduction; i.e.,

$$e \in \mathcal{O}^{M_{|x|}} \iff f(e) \in M_{|\langle 2,x\rangle|}.$$

We prove the left-to-right implication by induction on the ordertype of $Z_e^{M_{|x|}}$: Note that $Z_{h(e,n)}^{M_{|x|}}$ is a proper initial segment of $Z_e^{M_{|x|}}$ for every n. Hence by the inductive hypothesis, $f(h(e, n)) \in M_{|\langle 2,x\rangle|}$ for every $n \in \mathbb{N}$. Since $M_{|\langle 2,x\rangle|}$ is closed under the Richter

operator $\mathfrak{R}$ and $\mathsf{sup}(x, \mathsf{id}) \in M_{|\langle 2,x\rangle|}$, we have $f(e) = \mathsf{sup}(x, \lambda n.f(h(e,n))) \in M_{|\langle 2,x\rangle|}$. For the right-to-left implication, we apply induction on $|f(e)|$: If $f(e) \in M_{|\langle 2,x\rangle|}$, then $f(h(e,n)) \in M_{|\langle 2,x\rangle|}$ and $|f(h(e,n))| < |f(e)|$ for every n. By the inductive hypothesis, $f(h(e,n)) \in M_{|\langle 2,x\rangle|}$ implies $Z_{h(e,n)}^{M_{|x|}}$ is a well-order for every n, so $Z_e^{M_{|x|}}$ is also a well-order. □

Now let us *define* the iterated hyperjump of the empty set with the help of the J-hierarchy:

Definition 3.A.17. Let α be an ordinal less than the least recursively inaccessible ordinal. For $x \in M$ such that $|x|$ is the αth admissible ordinal[14], we define $\mathsf{HJ}^\alpha(\emptyset)$ by $M_{|x|}^*$.

Let us analyze the previous arguments to see some of them carries over ACA_0 with a minimal assumption:

Remark 3.A.18. We have a Π^1_1-formula defining the singleton $\{\mathsf{HJ}^\alpha(\emptyset)\}$ such that ACA_0 proves there is at most one real X satisfying the Π^1_1-formula. To get the desired Π^1_1-formula, let us fix a notation $x \in M$ such that $|x| = \omega_\alpha^{\mathsf{CK}}$, and modify the definition of J as follows:

$$n \in J_x(X) \iff n \in \mathfrak{R}(X) \vee \big[\mathfrak{R}(X) \subseteq \mathrm{field}(X) \\ \wedge\, n \in \{\langle 2, y\rangle \mid y \in \mathcal{F}(X)\} \cup \{k \in \mathbb{N} \mid x \in \mathrm{field}\, X\}\big]$$

Then J-hierarchy and J_x-hierarchy agree upon level $|x|$, but the $|x|+1$th level of the J_x-hierarchy becomes trivial. Hence $\mathsf{HJ}^\alpha(\emptyset)$ is equal to the last non-trivial level of the J_x-hierarchy. Hence with the help of the terminologies in Proposition 3.A.2, the following formula defines a singleton for $\mathsf{HJ}^\alpha(\emptyset)$:

[14] More precisely, the αth ordinal that is either admissible or a limit of admissible ordinals. For technical convenience, we take 0 as the zeroth admissible ordinal.

For every well-order γ with a maximal element μ and its immediate predecessor μ^-, and an exact chain of J_x-inductive definition A along γ, $(A)_\mu = \mathbb{N} \neq (A)_{\mu^-}$ and $X = (A)_{\mu^-}$.

The previous formula is Σ^1_2 due to the universal quantifier over a well-order. Employing the trick in Remark 3.A.3, we obatin a Π^1_1-formula $\phi_x(X)$ defining a singleton for $\mathsf{HJ}^\alpha(\emptyset)$. By the proof of Proposition 3.A.2, ACA_0 proves there is at most one X satisfying $\phi_x(X)$.

Remark 3.A.19. We can see that the proof of Theorem 3.A.16 works over ACA_0 with the assumption $M_{|\langle 2,x\rangle|}$ exists. We may use elements of well-orders in place of ordinals at the beginning of the proof of Theorem 3.A.16, and Kleene normal form theorem is a theorem of ACA_0. The last part of the proof can be seen the application of arithmetical transfinite induction, which is a theorem of ACA_0. Proposition 3.A.8 is also ACA_0-provable.

Now for a set X, let $\mathsf{HJ}(X)$ be the universal $\Pi^1_1[X]$-set. ACA_0 does not prove it exists, but we can still express the definition of $\mathsf{HJ}(X)$ in the language of second-order arithmetic. Kleene normal form theorem says ACA_0 proves $\mathcal{O}^X \equiv_\mathsf{T} \mathsf{HJ}(X)$ if both exist, and in fact, the existence of one of $\mathcal{O}^X$ or $\mathsf{HJ}(X)$ implies the existence of the other. In sum, ACA_0 proves the following: If $M_{|\langle 2,x\rangle|}$ exists, then $\mathsf{HJ}(M_{|x|}) \equiv_\mathsf{T} M_{|\langle 2,x\rangle|}$.

3.A.4 Iterated hyperjumps and iterated Spector classes

The following theorem shows the connection between iterated hyperjumps of the empty set and the iterated Spector class over $\mathbb{N}$:

Theorem 3.A.20. *For every ξ less than the least recursively inaccessible ordinal,* $\mathsf{SP}^{\xi+1}_\mathbb{N} = \Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]$.

Proof. The case $\xi = 0$ is clear. Now let us prove the equality by induction on ξ. First, combining [66, Theorem 6B.5, 8A.1] and [66, Corollary 9A.3] gives that for each real X, the set of $\Pi^1_1[X]$-relations (equivalently, Π^1_1-relations over the structure $(\mathbb{N}; X)$) is precisely the least Spector class over $\mathbb{N}$ containing X and $\mathbb{N} \setminus X$. We will use this fact in the remaining part of the proof. Hence if $\mathfrak{M}$ is a Spector class over $\mathbb{N}$ such that $X, \mathbb{N} \setminus X \in \mathfrak{M}$ and $\phi(x, X)$ is a $\Pi^1_1[X]$-formula, then the set $\{x \in \mathbb{N} \mid \phi(x, X)\}$ is in $\mathfrak{M}$. Moreover, it is a standard fact that the class of $\Pi^1_1[X]$-sets is always a Spector class for a real X.

Let us consider the case $\xi = \eta + 1$ first. By the inductive hypothesis, $\mathsf{SP}^\xi_\mathbb{N} = \Pi^1_1[\mathsf{HJ}^\eta(\emptyset)]$, which means that every set in $\mathsf{SP}^\xi_\mathbb{N}$ is Turing reducible into $\mathsf{HJ}(\mathsf{HJ}^\eta(\emptyset)) \equiv_\mathsf{T} \mathsf{HJ}^\xi(\emptyset)$. This shows

$$\mathsf{SP}^\xi_\mathbb{N} \subsetneq \mathsf{SP}^\xi_\mathbb{N} \cup \{\mathsf{HJ}^\xi(\emptyset), \mathbb{N} \setminus \mathsf{HJ}^\xi(\emptyset)\} \subseteq \Pi^1_1[\mathsf{HJ}^\xi(\emptyset)],$$

so $\mathsf{SP}^{\xi+1}_\mathbb{N} \subseteq \Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]$. Conversely, if $\mathfrak{M}$ is a Spector class over $\mathsf{SP}^\xi_\mathbb{N}$ and $\mathsf{HJ}^\xi(\emptyset), \mathbb{N} \setminus \mathsf{HJ}^\xi(\emptyset) \in \mathfrak{M}$, then $\Pi^1_1[\mathsf{HJ}^\xi(\emptyset)] \subseteq \mathfrak{M}$ by the minimality of $\Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]$. This shows $\Pi^1_1[\mathsf{HJ}^\xi(\emptyset)] \subseteq \mathsf{SP}^{\xi+1}_\mathbb{N}$.

Now consider the case when ξ is limit. By Lemma 3.A.7, every $\mathsf{HJ}^\eta(\emptyset)$ is Turing reducible to $\mathsf{HJ}^\xi(\emptyset)$ for $\eta < \xi$. Hence by the inductive hypothesis, every set in $\mathsf{SP}^\xi_\mathbb{N} = \bigcup_{\eta<\xi} \mathsf{SP}^\eta_\mathbb{N}$ is Turing reducible to $\mathsf{HJ}^\xi(\emptyset)$. This shows $\mathsf{SP}^{\xi+1}_\mathbb{N} \subseteq \Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]$.

Conversely, suppose that $\mathfrak{M} \supseteq \mathsf{SP}^\xi_\mathbb{N}$ is a Spector class over $\mathbb{N}$. We work with an admissible companion $A_\mathfrak{M}$ (cf. [66, §9E]) instead of $\mathfrak{M}$, so $A_\mathfrak{M}$ is an admissible set such that

1. $A_\mathfrak{M}$ is the least admissible set containing Δ-sets in $\mathfrak{M}$; i.e., a set $X \in \mathfrak{M}$ such that $\mathbb{N} \setminus X \in \mathfrak{M}$.

2. There is a relation $R_{\mathfrak{M}} \in A_{\mathfrak{M}}$ such that a relation $P \subseteq \mathbb{N}^n$ is in $\mathfrak{M}$ iff P is $\Sigma_1(R_{\mathfrak{M}})$ on $A_{\mathfrak{M}}$.

3. Every Δ-set in $\mathfrak{M}$ is rud definable in $R_{\mathfrak{M}}$ over $A_{\mathfrak{M}}$.

Since $\mathsf{SP}^{\xi}_{\mathbb{N}} \subseteq \mathfrak{M}$, we have $\mathsf{SP}^{\xi}_{\mathbb{N}} \subseteq A_{\mathfrak{M}}$. Now fix $a, b \in \mathbb{N}$ such that $|\mathsf{sup}(a,b)| = \omega_{\xi}^{\mathsf{CK}}$, so the sequence $\langle |[b](n, M_{|a|})| : n \in \omega \rangle$ is a cofinal sequence below $\omega_{\xi}^{\mathsf{CK}}$. Then observe that for each $n < \omega$, $x \in M^*_{|[b](n,M_{|a|})|}$ if and only if

$$A_{\mathfrak{M}} \vDash \exists F \exists \gamma \in \mathrm{Ord} \Big[F \colon (\gamma+2) \to \mathcal{P}(\mathbb{N}) \\ \wedge \forall \xi < \gamma + 2 \Big(F(\xi) = \textstyle\bigcup_{\eta<\xi} J_{[b](n,M_{|a|})}(F(\eta)) \Big) \\ \wedge F(\gamma+1) = \mathbb{N} \neq F(\gamma) \wedge x \in F(\gamma) \Big].$$

Now consider the set H such that $x \in H$ if and only if

$$A_{\mathfrak{M}} \vDash \exists n \in \omega \exists F \in Y \exists \gamma \in \mathrm{Ord} \cap Y \Big[F \colon (\gamma+2) \to \mathcal{P}(\mathbb{N}) \\ \wedge \forall \xi < \gamma + 2 \Big(F(\xi) = \textstyle\bigcup_{\eta<\xi} J_{[b](n,M_{|a|})}(F(\eta)) \Big) \\ \wedge F(\gamma+1) = \mathbb{N} \neq F(\gamma) \wedge x \in F(\gamma) \Big].$$

By Σ_1-collection, the above formula is equivalent to a Σ_1-formula with parameter $R_{\mathfrak{M}}$ originated from $M_{|a|}$. Hence H is $\Sigma_1(R_{\mathfrak{M}})$-defined over $A_{\mathfrak{M}}$, so $H \in \mathfrak{M}$. Moreover, $H = \bigcup_{n<\omega} M^*_{|[b](n,M_{|a|})|} = M^*_{|\mathsf{sup}(a,b)|} = \mathsf{HJ}^{\xi}(\emptyset)$. Similarly, $x \notin H$ if and only if

$$A_{\mathfrak{M}} \vDash \forall n \in \omega \exists F \in Y \exists \gamma \in \mathrm{Ord} \cap Y \Big[F \colon (\gamma+2) \to \mathcal{P}(\mathbb{N}) \\ \wedge \forall \xi < \gamma + 2 \Big(F(\xi) = \textstyle\bigcup_{\eta<\xi} J_{[b](n,M_{|a|})}(F(\eta)) \Big) \\ \wedge F(\gamma+1) = \mathbb{N} \neq F(\gamma) \wedge x \in \mathbb{N} \setminus F(\gamma) \Big].$$

The above formula is also equivalent to a $\Sigma_1(R_{\mathfrak{M}})$-formula, so $\mathbb{N} \setminus H \in \mathfrak{M}$. Hence $\mathfrak{M}$ is a Spector class containing $\mathsf{HJ}^{\xi}(\emptyset)$ and its complement, so $\Pi^1_1[\mathsf{HJ}^{\xi}(\emptyset)] \subseteq \mathfrak{M}$. $\square$

Theorem 3.A.21. *For ξ less than the least recursively inaccessible ordinal, $\mathsf{HJ}^\xi(\emptyset)$ is a Σ^1_2-singleton real; Furthermore,*

1. *We have a Σ^1_2-formula $\phi(X)$ with a unique free variable X such that ACA_0 proves there is at most one X satisfying $\phi(X)$.*
2. *Every set in $\mathsf{SP}^\xi_\mathbb{N}$ is Turing reducible into $\mathsf{HJ}^\xi(\emptyset)$. Hence every definable set over $\mathsf{SP}^\xi_\mathbb{N}$ is definable over $(\mathbb{N}; \mathsf{HJ}^\xi(\emptyset))$, and vice versa.*
3. *If ξ is a successor, then $\mathsf{HJ}^\xi(\emptyset) \in \mathsf{SP}^\xi_\mathbb{N}$.*

Proof. The first clause of the statement follows from Proposition 3.A.2 and Remark 3.A.18. The proof of the second clause of the statement is implicit in the proof of Theorem 3.A.20.

Now let us prove the third clause. Suppose $\xi = \eta+1$, then $\mathsf{HJ}^\eta(\emptyset), \mathbb{N}\setminus\mathsf{HJ}^\eta(\emptyset) \in \mathsf{SP}^\xi_\mathbb{N}$ and $\mathsf{HJ}^\xi(\emptyset)$ is defined by a monotone inductive definition by the Richter operator starting from $\mathsf{HJ}^\eta(\emptyset)$. Hence $\mathsf{HJ}^\xi(\emptyset)$ is in $\mathsf{SP}^\xi_\mathbb{N}$ since $\mathsf{SP}^\xi_\mathbb{N}$ is closed under monotone inductive definition whose initial set and its complement are in $\mathsf{SP}^\xi_\mathbb{N}$. □

Now let us prove Proposition 3.2.8. Before restating the statement of the proposition, let us recall that T is a first-order theory sound to $\mathsf{SP}^\xi_\mathbb{N}$. We understand a set in $\mathsf{SP}^\xi_\mathbb{N}$ as a predicate, so the language of T is the language of first-order arithmetic plus new predicates corresponding to sets in $\mathsf{SP}^\xi_\mathbb{N}$. The author guesses the next proposition holds for every ξ less than the least recursively inaccessible ordinal, but we prove it only for successor ordinals. If ξ is a limit, then $\mathsf{HJ}^\xi(\emptyset)$ is not in $\mathsf{SP}^\xi_\mathbb{N}$, causing technical complications.

Proposition 3.A.22. *Let ξ be a successor ordinal less than the least recursively inaccessible ordinal. Suppose that T is a strictly acceptable axiomatization of $\mathsf{SP}^\xi_\mathbb{N}$ containing*

$\mathrm{Th}(\mathsf{SP}^{\xi}_{\mathbb{N}})$. *Then we have*

$$\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(T) = |\mathsf{ACA}_0 + \text{`}\mathsf{HJ}^{\xi}(\emptyset) \textit{ exists'} + T \upharpoonright \mathsf{HJ}^{\xi}(\emptyset)|_{\Pi^1_1[\mathsf{HJ}^{\xi}(\emptyset)]},$$

where $T \upharpoonright R$ is the theory obtained from T by restricting its language to that of first-order arithmetic with an extra unary predicate for R.

Proof. For a notational convenience, let $H = \mathsf{HJ}^{\xi}(\emptyset)$. We first claim that $\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(T) = \delta^{(\mathbb{N};H)}(T \upharpoonright H)$: $\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(T) \geq \delta^{(\mathbb{N};H)}(T \upharpoonright H)$ is clear. For $\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(T) \leq \delta^{(\mathbb{N};H)}(T \upharpoonright H)$, suppose that $\prec$ is an arithmetical-in-$\mathsf{SP}^{\xi}_{\mathbb{N}}$ well-order such that $T \vdash \mathsf{TI}(\prec, X)$. Now let us find arithmetical-in-H definition of $\prec$: Suppose that $\sigma(x, y)$ is a sentence defining $\prec$. For each $A \in \mathsf{SP}^{\xi}_{\mathbb{N}}$, let us fix a partial H-recursive function f_A witnessing $A \leq_{\mathsf{T}} H$, so $f_X(x) = 1$ iff $x \in A$. (If $A = H$, take the characteristic function of H.) Then consider the formula $\sigma'(x, y)$ obtained from $\sigma(x, y)$ by replacing every occurrence of $n \in A$ for $A \in \mathsf{SP}^{\xi}_{\mathbb{N}}$ with $f_A(n) = 1$. Then σ and σ' define the same binary relation, i.e., $\mathsf{SP}^{\xi}_{\mathbb{N}}$ satisfies $\forall x, y[\sigma(x, y) \leftrightarrow \sigma'(x, y)]$. Since T satisfies every true first-order statement over $\mathsf{SP}^{\xi}_{\mathbb{N}}$, we have that $T \vdash \forall x, y[\sigma(x, y) \leftrightarrow \sigma'(x, y)]$. If $\prec'$ is a binary relation defined by σ', then $\prec$ and $\prec'$ are the same and $T \vdash \mathsf{TI}(\prec, X) \leftrightarrow \mathsf{TI}(\prec', X)$. This completes the proof of the desired inequality.

Now let us claim $\delta^{(\mathbb{N};H)}(T \upharpoonright H) = |\mathsf{ACA}_0 + \text{`}H \text{ exists'} + T \upharpoonright H|_{\Pi^1_1[H]}$. For a technical convenience, let us consider the extension $\mathsf{ACA}_0[H]$ of ACA_0: The language of $\mathsf{ACA}_0[H]_T$ is the language of second-order arithmetic with a constant set symbol H, and it comprises the following axioms:

1. Axioms of ACA_0.
2. For the Π^1_1-formula $\phi(X)$ defining a Π^1_1 singleton $\{\mathsf{HJ}^{\xi}(\emptyset)\}$ presented in Remark 3.A.18, we have $\phi(H)$.

3. Every theorem in T in the language of the first-order arithmetic (possibly with a free variable, allowing to express pseudo Π^1_1-sentences) with a unary predicate H corresponding to $\mathsf{HJ}^{\xi}(\emptyset)$.

We can see that $\mathsf{ACA}_0[H]_T$ is a conservative extension of ACA_0 + 'H exists' + $T \upharpoonright H$. By an argument for the conservativity of ACA_0 over PA (cf. [89, Lemma IX.1.3]), $\mathsf{ACA}_0[H]_T$ is a conservative extension of $T \upharpoonright H$. Hence

$$\delta^{(\mathbb{N};H)}(T \upharpoonright H) = \sup\{\mathrm{opt}(\prec) \mid \prec \text{ is arithmetical-in-}H \text{ and } \mathsf{ACA}_0[H]_T \vdash \forall^1 X\, \mathsf{TI}(\prec, X)\}. \tag{3.11}$$

ACA_0 proves arithmetical transfinite induction on $\prec$ is equivalent to the well-foundness of $\prec$, so Proposition 3.1.11 implies the right-hand-side of (3.11) is equal to $|\mathsf{ACA}_0[H]_T|_{\Pi^1_1[H]}$. □

CHAPTER 4

MEASURABLE DILATOR

O, it is excellent

To have a giant's strength, but it is tyrannous

To use it like a giant.

William Shakespeare, *Measure for Measure*

Act II, Scene II.

In this chapter, we prove the existence of a measurable dilator from a rank-into-rank embedding. Every result in this chapter appears [40] in verbatim.

4.1 Elementary embeddings

In this section, we review facts about rank-into-rank embedding. We mostly focus on notions introduced by Martin [58] with additional details from [18]. *We will avoid using the full axiom of choice in the rest of the chapter unless specified,* although we may use weaker variants, like the axioms of countable or dependent choice.

4.1.1 Rank-into-rank embedding

In this section, we review facts about rank-into-rank embeddings. Recall that an I_3-embedding is an elementary embedding $j\colon V_\lambda \to V_\lambda$ for $\lambda = \sup_{n<\omega} \kappa_n$, where $\kappa_0 = \operatorname{crit} j$ and $\kappa_{n+1} = j(\kappa_n)$. Let us define the following notions:

Definition 4.1.1. 1. $M_0 = V_\lambda, j_0 = j$.

2. $M_{\alpha+1} = \bigcup_{\xi \in \mathrm{Ord}^{M_\alpha}} j_\alpha(V_\xi^{M_\alpha})$, $j_{\alpha+1} = j_\alpha \cdot j_\alpha$.

3. $j_{\alpha,\alpha}$ is the identity, $j_{\alpha,\beta+1} = j_\beta \circ j_{\alpha,\beta}$ for $\alpha \leq \beta$.

4. If $\alpha > 0$ is a limit ordinal, define $((M_\alpha, j_\alpha), j_{\beta,\alpha})_{\beta<\alpha}$ is the direct limit of $((M_\beta, j_\beta), j_{\beta,\gamma})_{\beta \leq \gamma < \alpha}$.; More precisely,

$$M_\alpha = \bigcup_{\beta<\alpha} \{(\beta, x) \mid \beta < \alpha, x \in M_\beta\}/\sim,$$

where $(\beta, x) \sim (\gamma, y)$ iff there is $\delta < \alpha$ such that $\beta, \gamma \leq \delta$ and $j_{\beta,\delta}(x) = j_{\gamma,\delta}(y)$. Then for $x \in M_\beta$,

$$j_{\beta,\alpha}(x) := [\beta, x]_\sim, \quad j_\alpha([\beta, x]_\sim) = [\beta, j_\beta(x)]_\sim = j_{\beta,\alpha}(j_\beta(x)).$$

where $[\beta, x]_\sim$ is a $\sim$-equivalence class given by (β, x). We also take $M_{\alpha+n} = M_\alpha$.

M_α may not be well-founded for a limit α. If M_α is well-founded, then we say j is *α-iterable*. We identify M_α with its transitive collapse if M_α is well-founded.

Note that $M_\alpha = M_{\alpha+1}$, and in particular, $M_n = V_\lambda$ for every natural number n. $M_\omega \neq V_\lambda$ since $\lambda + 1 \subseteq M_\omega$ when M_ω is well-founded.

Lemma 4.1.2. *Let α, β be an ordinal and $n < \omega$.*

1. $j_{\alpha,\alpha+n} = j_\alpha^n := \underbrace{j_\alpha \circ \cdots \circ j_\alpha}_{n \text{ times}}$.
2. $j_\alpha \cdot j_{\alpha+n} = j_{\alpha+n+1}$.
3. $j_\alpha \circ j_{\alpha+n} = j_{\alpha+n+1} \circ j_\alpha$.
4. $j_{\alpha,\beta} = j_{\alpha+1,\beta} \circ j_\alpha$ *for* $\alpha < \beta$.
5. $j_{\alpha,\beta} \circ j_{\alpha+n} = j_{\beta+n} \circ j_{\alpha,\beta}$ *for* $\alpha \leq \beta$.
6. j_α *is well-defined elementary embedding from* (M_α, j_α) *to* $(M_\alpha, j_{\alpha+1})$ *and* $M_\alpha = M_{\alpha+1}$.

Proof. We prove it by induction on $\max(\alpha, \beta)$.

1. We can prove it by induction on n.
2. The case $n = 0$ is clear by definition. For the successor case,

$$j_\alpha \cdot j_{\alpha+n+1} = (j_\alpha \cdot j_{\alpha+n}) \cdot (j_\alpha \cdot j_{\alpha+n}) = j_{\alpha+n+1} \cdot j_{\alpha+n+1} = j_{\alpha+n+2}.$$

3. $(j_\alpha \circ j_{\alpha+n})(x) = j_\alpha(j_{\alpha+n}(x)) = (j_\alpha \cdot j_{\alpha+n})(j_\alpha(x)) = (j_{\alpha+n+1} \circ j_\alpha)(x)$.
4. We can prove it by induction on β.
5. The case $\alpha = \beta$ is trivial. Also,

$$j_{\alpha,\beta+1} \circ j_{\alpha+n} = j_\beta \circ j_{\alpha,\beta} \circ j_{\alpha+n} = j_\beta \circ j_{\beta+n} \circ j_{\alpha,\beta} = j_{\beta+n+1} \circ j_\beta \circ j_{\alpha+\beta} = j_{\beta+n+1} \circ j_{\alpha,\beta+1}.$$

   For a limit β, let us prove it by induction on n:

$$j_{\alpha,\beta}(j_{\alpha+n+1}(\xi)) = j_{\alpha,\beta}((j_\alpha \cdot j_{\alpha+n})(\xi)) = (j_{\alpha,\beta} \cdot (j_\alpha \cdot j_{\alpha+n}))(j_{\alpha,\beta}(\xi))$$

   and

$$\begin{aligned} j_{\alpha,\beta} \cdot (j_\alpha \cdot j_{\alpha+n}) &= j_{\alpha,\beta}\left(j_\alpha\left(\textstyle\bigcup_{\eta\in \mathrm{Ord}^{M_\alpha}} j_{\alpha+n} \restriction V_\eta^{M_\alpha}\right)\right) \\ &= j_\beta\left(j_{\alpha,\beta}\left(\textstyle\bigcup_{\eta\in \mathrm{Ord}^{M_\alpha}} j_{\alpha+n} \restriction V_\eta^{M_\alpha}\right)\right) = j_\beta\left(\textstyle\bigcup_{\eta\in \mathrm{Ord}^{M_\alpha}} j_{\alpha,\beta}(j_{\alpha+n} \restriction V_\eta^{M_\alpha})\right) \\ &= j_\beta\left(\textstyle\bigcup_{\eta\in \mathrm{Ord}^{M_\beta}} j_{\beta+n} \restriction V_\eta^{M_\beta}\right) = j_\beta \cdot j_{\beta+n} = j_{\beta+n+1}. \end{aligned}$$

6. We prove it in the following order:
   (a) j_α is well-defined.
   (b) j_α is ordinal-cofinal: For every $\xi \in \mathrm{Ord}^{M_\alpha}$ there is $\eta \in \mathrm{Ord}^{M_\alpha}$ such that $M_\alpha \vDash \xi < j_\alpha(\eta)$.
   (c) If $j_\alpha \colon M_\alpha \to M_\alpha$ is elementary for formulas over the language $\{\in\}$, then $j_\alpha \colon (M_\alpha, j_\alpha) \to (M_\alpha, j_{\alpha+1})$.

(d) $j_{0,\alpha} \colon (V_\lambda, j_0) \to (M_\alpha, j_\alpha)$ is elementary.

(e) $M_\alpha = M_{\alpha+1}$.

(f) $j_{\alpha+1} \colon M_\alpha \to M_\alpha$ is elementary.

(a) First, j_α is clearly well-defined if $\alpha = 0$ or $\alpha = \gamma + 1$ for some $\gamma < \alpha$. For a limit α, the issue is if $(\gamma, x) \sim (\delta, y)$ for $\gamma, \delta < \alpha$, $x \in M_\gamma$, $y \in M_\delta$ ensures $j_\alpha([\gamma, x]_\sim) = j_\alpha([\delta, y]_\sim)$ as we defined $j_\alpha([\gamma, x]_\sim) = [\gamma, j_\gamma(x)]_\sim$. Fix $\zeta < \alpha$ such that $\gamma, \delta < \zeta$ and $j_{\gamma,\zeta}(x) = j_{\delta,\zeta}(y)$.

$$\begin{aligned} j_\alpha([\gamma, x]_\sim) = [\gamma, j_\gamma(x)]_\sim = [\zeta, j_{\gamma,\zeta} \circ j_\gamma(x)]_\sim = [\zeta, j_\zeta \circ j_{\gamma,\zeta}(x)]_\sim \\ = [\zeta, j_\zeta \circ j_{\delta,\zeta}(y)]_\sim = [\zeta, j_{\delta,\zeta} \circ j_\delta(y)]_\sim = [\delta, j_\delta(x)]_\sim = j_\alpha([\delta, y]_\sim). \end{aligned}$$

Thus j_α is always well-defined.

(b) The case $\alpha = 0$ follows from the assumption $\lambda = \sup_{n<\omega} \kappa_n$. If $\alpha = \gamma + 1$, the inductive hypothesis gives $M_\gamma = M_{\gamma+1}$. Fix $\xi \in \mathrm{Ord}^{M_\gamma} = \mathrm{Ord}^{M_\alpha}$, then we can find $\eta \in \mathrm{Ord}^{M_\alpha}$ such that $\xi < j_\gamma(\eta)$. Hence

$$\xi \leq j_\gamma(\xi) < j_\gamma(j_\gamma(\eta)) = j_{\gamma+1}(j_\gamma(\eta)) = j_\alpha(j_\gamma(\eta)),$$

as desired. If α is limit, then every ordinal in M_α has the form $[\gamma, \xi]_\sim$ for some $\gamma < \alpha$ and $\xi \in \mathrm{Ord}^{M_\gamma}$. We can find $\eta \in \mathrm{Ord}^{M_\gamma}$ such that $M_\gamma \vDash \xi < j_\gamma(\eta)$, so

$$[\gamma, \xi]_\sim < [\gamma, j_\gamma(\eta)]_\sim = j_\alpha([\gamma, \eta]_\sim).$$

(c) Now suppose that $j_\alpha \colon M_\alpha \to M_\alpha$ is elementary for formulas over the language $\{\in\}$. Then let us employ the following general fact:

Lemma 4.1.3. *Suppose that N is a model of $\mathsf{Z} + \Sigma_1$-Collection + '$\xi \mapsto V_\xi$ is well-defined,' $j \colon N \to N$ is Δ_0-elementary. If $A \subseteq N$ is* amenable*, i.e., $x \cap A \in N$ for every $x \in N$, then $j \colon (N, A) \to (N, j[A])$ is Δ_0-elementary over the language $(\in, A)$, where $j[A] = \bigcup_{\xi \in \mathrm{Ord}^N} j(A \cap V_\xi^N)$.*

Its proof follows from the proof of [44, Lemma 4.14(1)]. Since $j_\alpha \colon M_\alpha \to M_\alpha$ is ordinal cofinal, we can prove that $j_\alpha \colon (M_\alpha, j_\alpha) \to (M_\alpha, j_{\alpha+1})$ is fully elementary by induction on the quantifier complexity of a formula as presented in [44, Lemma 4.14(2)].

(d) The case $\alpha = 0$ is easy, and the successor case follows from the induction hypothesis and the previous item. The limit case follows from the definition of $(M_\alpha, j_\alpha, j_{\beta,\alpha})_{\beta<\alpha}$.

(e) $j_\alpha(V_\xi^{M_\alpha}) \subseteq M_\alpha$ gives $M_{\alpha+1} \subseteq M_\alpha$. For $M_\alpha \subseteq M_{\alpha+1}$, observe that

$$V_\lambda \vDash \forall \xi \in \mathrm{Ord}[j_0(V_\xi) = V_{j_0(\xi)}].$$

Since $j_{0,\alpha} \colon (V_\lambda, j_0) \to (M_\alpha, j_\alpha)$ is elementary, we have

$$M_\alpha \vDash \forall \xi \in \mathrm{Ord}[j_\alpha(V_\xi) = V_{j_\alpha(\xi)}].$$

Furthermore, in M_α, for each x we can find $\xi \in \mathrm{Ord}$ such that $\operatorname{rank} x < j_\alpha(\xi)$. Hence $x \in V_{j_\alpha(\xi)} = j_\alpha(V_\xi)$. It proves $M_\alpha \subseteq M_{\alpha+1}$.

(f) For the elementarity of $j_{\alpha+1}$, observe that for a given formula ϕ we have

$$M_\alpha \vDash \forall \xi \in \mathrm{Ord} \forall \vec{x} \in V_\xi[\phi(\vec{x}) \leftrightarrow \phi((j \restriction V_\xi)(\vec{x}))].$$

Fix ξ and apply j_α. Then we get

$$M_\alpha \vDash \forall \vec{x} \in j_\alpha(V_\xi)[\phi(\vec{x}) \leftrightarrow \phi(j_\alpha(j \restriction V_\xi)(\vec{x}))].$$

Since ξ is arbitrary, we have

$$M_\alpha \vDash \forall \vec{x} \in M_{\alpha+1}[\phi(\vec{x}) \leftrightarrow \phi(j_{\alpha+1}(\vec{x}))]. \qquad \square$$

The following is an easy corollary of the previous proposition:

Corollary 4.1.4. *$j_{\alpha,\beta} \circ j_{\alpha+n,\alpha+m} = j_{\beta+n,\beta+m} \circ j_{\alpha,\beta}$ for $\alpha \leq \beta$ and $n \leq m < \omega$.* $\square$

We will use the convention $\kappa_\alpha = j_{0,\alpha}(\kappa_0)$ for a general α. It is an ordinal if j is α-iterable, but it can be ill-founded otherwise. Now, let us state a lemma about critical points whose proof is straightforward:

Lemma 4.1.5. *Let α, β be ordinals and $n < \omega$.*

1. $\mathrm{crit}\, j_\alpha = \kappa_\alpha$.
2. $j_{\alpha,\beta}(\kappa_{\alpha+n}) = \kappa_{\beta+n}$ *for* $\alpha \leq \beta$. □

We will use the following large cardinal notion to construct a measurable dilator:

Definition 4.1.6. Let $j\colon V_\lambda \to V_\lambda$ be an I_3-embedding. We say j is *iterable* if M_α defined from j is well-founded for every α. We say λ is *iterable* if it has an iterable embedding $j\colon V_\lambda \to V_\lambda$.

Although unnecessary in this chapter, it is worthwhile to note that an I_3-embedding is iterable if and only if M_α is well-founded for every $\alpha < \omega_1$. (See Dimonte [18] for the details.) We may compare it with the fact that a countable semidilator D is a dilator if and only if $D(\alpha)$ is well-founded for every $\alpha < \omega_1$. It gives a clue that elementary embedding iteration may have the structure of a dilator.

4.1.2 β-embedding

We will consider a measure over κ_n for an iterable embedding $j\colon V_\lambda \to V_\lambda$. The measure will be a projection of a measure over the set of β-*embeddings* defined as follows:

Definition 4.1.7. An embedding $k\colon V_{\alpha+\beta} \to V_{\alpha'+\beta}$ is a β-*embedding* if $\mathrm{crit}\, k = \alpha > \beta$.

β-embeddings are first introduced in Martin's proof of $\mathbf{\Pi}^1_2$-determinacy [58] from an iterable embedding. We can see that the following holds:

Lemma 4.1.8. *If $k\colon V_{\alpha+\beta} \to V_{\alpha'+\beta}$ is a β-embedding, then $k(\alpha) + \beta = \alpha' + \beta$ and $k(\alpha) \leq \alpha'$*

Proof. Observe that

$$V_{\alpha+\beta} \vDash \forall \xi \in \mathrm{Ord}(\xi < \alpha \vee \exists \eta < \beta(\xi = \alpha + \eta)).$$

Hence by elementarity,

$$V_{\alpha'+\beta} \vDash \forall \xi \in \mathrm{Ord}(\xi < k(\alpha) \vee \exists \eta < \beta(\xi = k(\alpha) + \eta)).$$

This shows $\alpha' + \beta = k(\alpha) + \beta$. Since α is a critical point, it is inaccessible. This shows $k(\alpha)$ is also inaccessible, so if $\alpha' < k(\alpha)$ then $\alpha' + \beta < k(\alpha)$, a contradiction. □

It is not true that $k(\alpha) = \alpha'$ holds (It fails when, for example, $\alpha' = k(\alpha) + 1$ and $\beta = \omega$.) However, by replacing α' if necessary, we may assume that $\alpha' = k(\alpha)$, and we will assume throughout this chapter that every β-embedding $k \colon V_{\alpha+\beta} \to V_{\alpha'+\beta}$ maps α to α'.

We can define a measure for β-embeddings:

Definition 4.1.9. Let k be a β-embedding and $\gamma + 1 < \beta$. Define

$$\mathsf{Emb}^k_\gamma = \{k' \colon V_{\mathrm{crit}\, k'+\gamma} \to V_{k'(\mathrm{crit}\, k')+\gamma} \mid \mathrm{crit}\, k' < \mathrm{crit}\, k,\ k'(\mathrm{crit}\, k') = \mathrm{crit}\, k\}$$

and a measure μ^k_γ by

$$X \in \mu^k_\gamma \iff k \restriction V_{\mathrm{crit}\, k+\gamma} \in k(X).$$

It can be easily shown that $\mathsf{Emb}^k_\gamma \in \mu^k_\gamma$, so we can think of μ^k_γ as a measure over Emb^k_γ. Also, the following facts are easy to verify:

Lemma 4.1.10. *Let k be a β_1-embedding and $\gamma + 1 < \beta_0 \leq \beta_1$. Then $\mathsf{Emb}^k_\gamma = \mathsf{Emb}^{k \restriction V_{\mathrm{crit}\, k+\beta_0}}_\gamma$ and $\mu^k_\gamma = \mu^{k \restriction V_{\mathrm{crit}\, k+\beta_0}}_\gamma$.* □

Lemma 4.1.11. *Let k be a β-embedding and $\gamma_0 \leq \gamma_1 < \gamma_1 + 1 < \beta$. Consider the projection map $\pi^k_{\gamma_0,\gamma_1} \colon \mathsf{Emb}^k_{\gamma_1} \to \mathsf{Emb}^k_{\gamma_0}$, $\pi^k_{\gamma_0,\gamma_1}(k') = k' \restriction V_{\mathrm{crit}\, k'+\gamma_0}$. Then*

$$X \in \mu^k_{\gamma_0} \iff (\pi^k_{\gamma_0,\gamma_1})^{-1}[X] = \{z \in \mathsf{Emb}^k_{\gamma_1} \mid z \restriction V_{\mathrm{crit}\, z+\gamma_0} \in X\} \in \mu^k_{\gamma_1}.$$

Also,

$$Y \in \mu^k_{\gamma_1} \implies \pi^k_{\gamma_0,\gamma_1}[Y] \in \mu^k_{\gamma_0}. \qquad \square$$

The following proof is a modification of Schlutzenberg's answer on MathOverflow [88]:

Lemma 4.1.12. *Let $k\colon V_\alpha \to V_{\alpha'}$ and $\xi < \delta < \alpha$. If $\xi \notin \operatorname{ran} k$, then $k(k \restriction V_\delta)(\xi) < k(\xi)$.*

Proof. Let us choose the least $\gamma \leq \delta$ such that $k(\gamma) > \xi$. (Such γ exists since $k(\delta) \geq \delta > \xi$.) Note that $k(\gamma) > \xi + 1$ also holds, otherwise $k(\gamma) = \xi + 1$ implies $\gamma = \gamma' + 1$ for some $\gamma' < \gamma$ and $k(\gamma') = \xi$.

Clearly we have $\sup k[\gamma] \leq \xi$, so $k(\sup k[\gamma]) \leq k(\xi)$. Also,

$$k(\sup k[\gamma]) = k(\sup_{\zeta<\gamma}(k \restriction V_\delta)(\zeta)) = \sup_{\zeta<k(\gamma)} (k(k \restriction V_\delta))(\zeta) > (k(k \restriction V_\delta))(\xi).$$

Putting everything together, we have $k(k \restriction V_\delta)(\xi) < k(\sup k[\gamma]) \leq k(\xi)$. $\square$

Corollary 4.1.13. *For an elementary embedding $k\colon V_\lambda \to V_\lambda$ we have $(k \cdot k)(\xi) \leq k(\xi)$ for every $\xi < \lambda$. In particular, $j_{n+1}(\xi) \leq j_n(\xi)$ holds for every $\xi < \lambda$ and $n \in \mathbb{N}$.*

Proof. If $\xi \in \operatorname{ran} k$, so if $\xi = k(\eta)$ for some $\eta < \lambda$, then

$$(k \cdot k)(\xi) = (k \cdot k)(k(\eta)) = k(k(\eta)) = k(\xi).$$

If $\xi \notin \operatorname{ran} k$, then Lemma 4.1.12 implies $(k \cdot k)(\xi) = k(k \restriction V_{\xi+\omega})(\xi) < k(\xi)$. $j_{n+1}(\xi) \leq j_n(\xi)$ follows from $j_{n+1} = j_n \cdot j_n$. $\square$

Most ultrafilters induced from an elementary embedding are normal. The ultrafilter μ^k_γ is also 'normal' in the following sense:

Lemma 4.1.14. *Let $\gamma_0 < \gamma_1 < \beta$ be limit ordinals and k a β-embedding. If $\{Y_{k'} \mid k' \in \mathsf{Emb}^k_{\gamma_0}\} \subseteq \mu^k_{\gamma_1}$ is a family of sets, then*

$$\triangle_{k' \in \mathsf{Emb}^k_{\gamma_0}} Y_{k'} := \{k'' \in \mathsf{Emb}^k_{\gamma_1} \mid \forall k' \in \operatorname{ran} k'' \cap \mathsf{Emb}^k_{\gamma_0} (k'' \in Y_{k'})\} \in \mu^k_{\gamma_1}.$$

Proof. We want to show that $k \restriction V_{\mathrm{crit}\, k+\gamma_1} \in k(\triangle_{k' \in \mathsf{Emb}^k_{\gamma_0}} Y_{k'})$, which is equivalent to

$$\forall k' \in \operatorname{ran}(k \restriction V_{\mathrm{crit}\, k+\gamma_1}) \cap k(\mathsf{Emb}^k_{\gamma_0})[k \restriction V_{\mathrm{crit}\, k+\gamma_1} \in k(Y)_{k'}].$$

For each $k' \in \operatorname{ran}(k \restriction V_{\mathrm{crit}\, k+\gamma_1}) \cap k(\mathsf{Emb}^k_{\gamma_0})$, we can find $\hat{k}' \in V_{\mathrm{crit}\, k+\gamma_1} \cap \mathsf{Emb}^k_{\gamma_0}$ such that $k' = k(\hat{k}')$. Since $Y_{\hat{k}'} \in \mu^k_{\gamma_1}$, we have $k \restriction V_{\mathrm{crit}\, k+\gamma_1} \in k(Y_{\hat{k}'}) = k(Y)_{k'}$, as desired. □

We will frequently use 'for μ-almost all' throughout this chapter, so it is convenient to introduce measure quantifier notation:

Definition 4.1.15. Let μ be an ultrafilter over D. Let us define

$$\forall(\mu)x \in D\phi(x) \iff \{x \in D \mid \phi(x)\} \in \mu.$$

Measure quantifier can be iterated, so for example, if μ_i is a measure over D_i for $i = 0, 1$,

$$\forall(\mu_0)x_0 \in D_0 \forall(\mu_1)x_1 \in D_1 \phi(x_0, x_1) \iff \{x_0 \in D_0 \mid \{x_1 \in D_1 \mid \phi(x_0, x_1)\} \in \mu_1\} \in \mu_0.$$

The reader should be careful that we cannot switch the order of two quantifiers in the above definition. Switching the order of measure quantifier is impossible even when the same measure quantifier repeats: For example, consider the following statement for a κ-complete μ over a measurable cardinal κ.

$$\forall(\mu)\alpha < \kappa \forall(\mu)\beta < \kappa[\alpha < \beta] :\iff \{\alpha < \kappa \mid \{\beta < \kappa \mid \alpha < \beta\} \in \mu\} \in \mu.$$

However, we can 'delete' unused measure quantifiers:

Lemma 4.1.16. *Let $\phi(x, y)$ be a formula with no z as a free variable. If μ_i is a measure over D_i for $i = 0, 1$ and ν a measure over D, then we have*

$$\forall(\mu_0)x \in D_0 \forall(\nu)z \in D \forall(\mu_1)y \in D_1 \phi(x, y) \iff \forall(\mu_0)x \in D_0 \forall(\mu_1)y \in D_1 \phi(x, y).$$

□

The following lemma says an 'upper diagonal' over the set $\mathsf{Emb}^k_{\gamma_0} \times \mathsf{Emb}^k_{\gamma_1}$ for $\gamma_0 < \gamma_1$ is large, like the set $\{(\alpha, \beta) \mid \alpha < \beta < \kappa\}$ is large under a normal ultrafilter on a measurable cardinal κ:

Lemma 4.1.17. *Let $\gamma_0 < \gamma_1 < \beta$ be limit ordinals and k be a β-embedding. Then*

$$\forall(\mu^k_{\gamma_0})k^0 \in \mathsf{Emb}^k_{\gamma_0} \forall(\mu^k_{\gamma_1})k^1 \in \mathsf{Emb}^k_{\gamma_1} [k^0 \in \operatorname{ran} k^1].$$

Proof. The claim is equivalent to

$$\forall(\mu^k_{\gamma_0})k^0 \in \mathsf{Emb}^k_{\gamma_0} [\{k^1 \in \mathsf{Emb}^k_{\gamma_1} \mid k^0 \in \operatorname{ran} k^1\} \in \mu^k_{\gamma_1}]$$

and by the definition of $\mu^k_{\gamma_1}$, it is equivalent to

$$\forall(\mu^k_{\gamma_0})k^0 \in \mathsf{Emb}^k_{\gamma_0} [k(k^0) \in \operatorname{ran} k \restriction V_{\operatorname{crit} k+\gamma_1}],$$

which holds since $\mathsf{Emb}^k_{\gamma_0} \subseteq V_{\operatorname{crit} k+\gamma_1}$. □

Now let us fix limit ordinals $\gamma_0 < \cdots < \gamma_{m-1} < \beta$ and a β-embedding k. We define the product of measures $\mu = \mu^k_{\gamma_0} \times \cdots \times \mu^k_{\gamma_{m-1}}$ over $\mathsf{Emb}^k_{\gamma_0} \times \cdots \times \mathsf{Emb}^k_{\gamma_{m-1}}$ by

$$X \in \mu \iff \forall(\mu^k_0)k^0 \in \mathsf{Emb}^k_{\gamma_0} \cdots \forall(\mu^k_{m-1})k^{m-1} \in \mathsf{Emb}^k_{\gamma_{m-1}} [\langle k^0, \cdots, k^{m-1} \rangle \in X].$$

As an application of Lemma 4.1.17, we have

$$\Delta^k_m := \{\langle k^0, \cdots, k^{m-1}\rangle \in \mathsf{Emb}^k_{\gamma_0} \times \cdots \times \mathsf{Emb}^k_{\gamma_{m-1}} \mid$$
$$k^0 \in \operatorname{ran} k^1 \wedge \cdots \wedge k^{m-2} \in \operatorname{ran} k^{m-1}\} \in \mu.$$

The following proposition says the product measure μ is generated by an intersection of Δ^k_m and a cube whose each component is large:

Proposition 4.1.18. *For each $X \in \mu$ we can find $Y \in \mu^k_{\gamma_{m-1}}$ such that*

$$\Delta^k_m \cap ((\pi^k_{\gamma_0,\gamma_{m-1}})[Y] \times \cdots \times (\pi^k_{\gamma_{m-2},\gamma_{m-1}})[Y] \times Y) \subseteq X.$$

Proof. We prove this proposition for $m = 3$; The general case follows from a similar argument. Define

$$Y_0 = \{k^0 \in \mathsf{Emb}^k_{\gamma_0} \mid \forall (\mu^k_{\gamma_1}) k^1 \in \mathsf{Emb}^k_{\gamma_1} \forall (\mu^k_{\gamma_2}) k^2 \in \mathsf{Emb}^k_{\gamma_2} [\langle k^0, k^1, k^2\rangle \in X]\}.$$

Then $X \in \mu$ implies $Y_0 \in \mu^k_{\gamma_0}$. Next, we define

$$Y_1 = (\pi^k_{\gamma_0,\gamma_1})^{-1}[Y_0] \cap \{k^1 \in \mathsf{Emb}^k_{\gamma_1} \mid \forall k^0 \in Y_0 \cap \operatorname{ran} k^1 \forall (\mu^k_{\gamma_2}) k^2 \in \mathsf{Emb}^k_{\gamma_2}$$
$$[\langle k^0, k^1, k^2\rangle \in X]\}.$$

We claim $Y_1 \in \mu^k_{\gamma_1}$: To see this, for each $k^0 \in \mathsf{Emb}^k_{\gamma_0}$ let us take

$$Y_{1,k^0} := \{k^1 \in \mathsf{Emb}^k_{\gamma_1} \mid k^0 \in Y_0 \to \forall (\mu^k_{\gamma_2}) k^2 \in \mathsf{Emb}^k_{\gamma_2} [\langle k^0, k^1, k^2\rangle \in X]\}.$$

Then $Y_{1,k^0} \in \mu^k_{\gamma_1}$ for every $k^1 \in \mathsf{Emb}^k_{\gamma_1}$. Then by Lemma 4.1.14,

$$\triangle_{k^0 \in \mathsf{Emb}^k_{\gamma_0}} Y_{1,k^0}$$
$$= \{k^1 \in \mathsf{Emb}^k_{\gamma_1} \mid \forall k^0 \in Y_0 \cap \operatorname{ran} k^1 \forall (\mu^k_{\gamma_2}) k^2 \in \mathsf{Emb}^k_{\gamma_2} [\langle k^0, k^1, k^2\rangle \in X]\} \in \mu^k_{\gamma_1}.$$

Combining with Lemma 4.1.11, we have $Y_1 \in \mu^k_{\gamma_1}$. Lastly, let us define

$$Y_2 = (\pi^k_{\gamma_1,\gamma_2})^{-1}[Y_1] \cap$$
$$\{k^2 \in \mathsf{Emb}^k_{\gamma_2} \mid \forall k^1 \in Y_1 \cap \operatorname{ran} k^2 \forall k^0 \in Y_0 \cap \operatorname{ran} k^1 [\langle k^0, k^1, k^2\rangle \in X]\}.$$

Then we can show $Y_2 \in \mu^k_{\gamma_2}$. It is straightforward to see $\Delta^k_3 \cap (Y_0 \times Y_1 \times Y_2) \subseteq X$ and $\pi^k_{\gamma_i,\gamma_2}[Y_2] \subseteq \pi^k_{\gamma_i,\gamma_2}[(\pi^k_{\gamma_i,\gamma_2})^{-1}[Y_i]] \subseteq Y_i$, so we have a desired result. □

Let us observe that for limit ordinals $\gamma_0 < \gamma_1 < \beta$, a β-embedding k, and $k_i \in \mathsf{Emb}^k_{\gamma_i}$ for $i = 0, 1$, the relation $k_0 \in \operatorname{ran} k_1$ is similar to the Mitchell order. It can be easily seen that for $\gamma_0 < \gamma_1 < \gamma_2 < \beta$ and $k_i \in \mathsf{Emb}^k_{\gamma_i}$ for $i < 3$, $k_0 \in \operatorname{ran} k_1$ and $k_1 \in \operatorname{ran} k_2$ imply $k_0 \in \operatorname{ran} k_2$. The next proposition tells us there is an arbitrarily long countable sequence of elementary embeddings increasing under the Mitchell order:

Proposition 4.1.19. *Let $\gamma < \beta$ be limit ordinals, k a β-embedding, $\langle \gamma_\xi \mid \xi < \alpha\rangle$ a countable increasing sequence of limit ordinals below γ, and $X \in \mu^k_\gamma$. Then we can find $\{k_\xi \mid \xi < \alpha\} \subseteq X$ such that for each $\eta < \xi < \alpha$, $k_\eta \restriction V_{\operatorname{crit} k_\eta + \gamma_\eta} \in \operatorname{ran} k_\xi \restriction V_{\operatorname{crit} k_\xi + \gamma_\xi}$.*

Proof. For $\delta \leq \alpha$, define a *δ-chain* as a sequence $\vec{k} = \langle k_\xi \mid \xi < \delta\rangle$ of members of X such that for each $\eta < \xi < \delta$, $k_\eta \restriction V_{\operatorname{crit} k_\eta + \gamma_\eta} \in \operatorname{ran} k_\xi \restriction V_{\operatorname{crit} k_\xi + \gamma_\xi}$. We first claim that every δ-chain extends to a $(\delta+1)$-chain: Suppose that $\vec{k}$ is a δ-chain. Note that $\vec{k} \in V_{\operatorname{crit} k + \beta}$, so the following is witnessed by $k' = k \restriction V_{\operatorname{crit} k + \gamma}$:

$$V_{k(\operatorname{crit} k)+\beta} \vDash \exists k' [k' \in k(X) \wedge k(\vec{k}) \in \operatorname{ran}(k' \restriction V_{\operatorname{crit} k' + \gamma_\delta})].$$

Then by elementarity of k, we have

$$V_{\operatorname{crit} k + \beta} \vDash \exists k' [k' \in X \wedge \vec{k} \in \operatorname{ran}(k' \restriction V_{\operatorname{crit} k' + \gamma_\delta})].$$

(Note that $\gamma_\delta < \beta < \operatorname{crit} k$.) Take any k' witnessing the previous claim, then $\vec{k}^\frown\langle k'\rangle$ is a $(\delta+1)$-chain.

Now let us prove by induction on $\delta \leq \alpha$ that for every $\delta' < \delta$, a δ'-chain extends to a δ-chain. The case when δ is 0 or a successor is clear. For the limit case, fix a cofinal sequence $\langle \delta_n \mid n < \omega \rangle$ of δ with $\delta_0 = \delta'$. We know that every δ_n-chain extends to a δ_{n+1}-chain, so by the axiom of dependent choice, we can find a δ-chain extending a given δ'-chain. (This is why we require $\alpha < \omega_1$; Otherwise, we need a choice axiom stronger than the dependent choice.) □

Note that β-embedding in this subsection is irrelevant to Girard's β-logic.

4.2 Recap on dendrograms

A dendroid is a tree-like structure representing $D(\alpha)$ for a dilator D and a well-order α. The original notation of a dendroid given by Girard [26] is a mixture of a dilator D and a well-order α, and its definition does not directly allow its pre- notion. A dendrogram can be viewed as separating the dilator part from Girard's dendroid, thereby capturing only the structure of a dilator. It appears during the construction of a measure family for the Martin flower, and I believe a dendrogram is the best way to construct dilators by hand. The following definition of a predendrogram is equivalent to what is presented in Appendix D, but more suitable for this chapter.

Definition 4.2.1. A *predendrogram* is a structure $C = (C, \lessdot, \multimap, \mathbf{e})$ such that the following holds:

1. $(C, \multimap)$ is a forest with the immediate successor relation $\multimap$. Moreover, if $\multimap^*$ is the transitive closure of $\multimap$, then for each $x \in C$, $\{y \in C \mid y \multimap^* x\}$ is finite and well-ordered by $\multimap^*$. We call the size of $\{y \in C \mid y \multimap^* x\}$ the *length of* x and denote it by $\mathrm{lh}(x)$.

2. For every $x \in C$, either x is a terminal point (i.e., x has no immediate successor) or there is y such that $x \multimap^* y$ and y is a terminal point.
3. $\lessdot$ is a partial order over C. Moreover, the following sets are linearly ordered by $\lessdot$:
   - The set of *roots* of C, i.e. elements with no immediate predecessor.
   - The set immediate successors of x for each $x \in C$.
4. $\mathbf{e}$ is a partial function from C to $\mathbb{N}$ such that $\mathbf{e}(x)$ is defined if and only if x is not a terminal point, and $\mathbf{e}(x) \leq \mathrm{lh}(x)$.

We denote the set of terminal points of C by $\mathrm{term}(C)$. We also define the *sequence* $\mathrm{pred}(x)$ *of predecessors of* x as the $\multimap$-increasing enumeration of $\{y \in C \mid y \multimap^* x\} \cup \{x\}$. $\mathrm{pred}(x)$ is a sequence of length $\mathrm{lh}(x) + 1$.

Each predendrogram induces a predilator in the following way:

Definition 4.2.2. Let C be a predendrogram. We define the *functorialization* $\mathrm{Fun}(C)$ of C as follows: For a linear order X, we define the $\mathrm{Fun}(C)(X)$ by the set of all $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle$ such that there is $x = x_m \in \mathrm{term}(C)$ such that $\mathrm{pred}(x) = \langle x_0, \cdots, x_m \rangle$ and for each $i < m$, ξ_i is the $\mathbf{e}(i)$th least element over $\{\xi_0, \cdots, \xi_i\}$. We impose $\mathrm{Fun}(C)(X)$ the Kleene-Brouwer order, where we compare x_i by the C-order, and ξ_i by the α-order. For $f : \alpha \to \beta$, we define $\mathrm{Fun}(C)(f)$ by

$$\mathrm{Fun}(C)(f)(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle) = \langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle.$$

We can see that for a predendrogram C, $\mathrm{Fun}(C)$ is a predilator (more precisely, an F-predilator.) We can turn $\mathrm{Fun}(C)$ into a denotation system through the category equivalence $\mathfrak{f} : \mathsf{SDil}_\mathsf{F} \to \mathsf{SDil}$. Throughout the remaining part of the chapter, we conflate $\mathrm{Fun}(C)$ with its denotation system $\mathfrak{f}(\mathrm{Fun}(C))$.

Like predendroids, predendrograms also admit morphisms:

Definition 4.2.3. For two predendrograms C and D, a function $f\colon C \to D$ is a *predendrogram morphism* if for $x, y \in C$,

1. $C \vDash x \multimap y$ iff $D \vDash f(x) \multimap f(y)$.
2. $C \vDash x \lessdot y$ iff $D \vDash f(x) \lessdot f(y)$, and
3. $f(\mathbf{e}^C(x)) = \mathbf{e}^D(f(x))$.

Conversely, from a predilator D, we can get the corresponding predendrogram $\mathrm{Cell}(D)$, called the *cell decomposition* of D. Its construction is similar to that of the Branching functor in [26], which is given as follows. First, let us find the field of $\mathrm{Cell}(D)$:

Definition 4.2.4. For a predilator D and $n \in \mathbb{N}$, let us define a equivalence relation $\equiv_n^D$ over $\mathrm{field}(D)$ as follows:

$$s \equiv_n^D t \iff \mathbf{p}^D(s,t) > n \vee s = t.$$

Let $<^D$ be a linear order derived from the abstract construction of predilators. Then we can see that $\equiv_n^D$ is an interval over $(D, <^D)$:

Lemma 4.2.5. *An $\equiv_n^D$-equivalence class is an interval on $(D, <^D)$.*

Proof. Let $s, t, u \in \mathrm{field}(D)$, be such that $s <^D t <^D u$ and $\mathbf{p}^D(s,u) > n$. From $\min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u)) = \mathbf{p}^D(s,u) > n$, we have that s, t, u are all $\equiv_n^D$-equivalent. □

Hence we can define $(D, <^D)/\equiv_n^D$. Now, let us define the cell decomposition of D as follows:

Definition 4.2.6. For a predilator D, $\mathrm{Cell}(D)$ is the set of $[t]_{\equiv_m^D}$ for every $t \in \mathrm{field}(D)$ and $m \leq \mathrm{arity}(t)$. We define relations over $\mathrm{Cell}(D)$ as follows: For $x, y \in \mathrm{Cell}(D)$,

1. $x \multimap y$ iff there is $t \in \text{field}(D)$ and $m < \text{arity}(t)$ such that $x = [t]_{\equiv^D_m}$ and $y = [t]_{\equiv^D_{m+1}}$.
2. $x \lessdot y$ iff there are $s, t \in \text{field}(D)$ and $m \leq \text{arity}\, s, \text{arity}\, t$ such that $x = [s]_{\equiv^D_m}$, $y = [t]_{\equiv^D_m}$, and $s <^D t$.
3. $\mathbf{e}([t]_{\equiv^D_m})$ is the natural e number such that $\Sigma^D_t(m)$ is the eth least member over the set $\{\Sigma^D_t(0), \cdots, \Sigma^D_t(m)\}$.

For an embedding $f : D \to E$, let us define $\text{Cell}(f)$ by $\text{Cell}(f)([t]_{\equiv^D_m}) = [f(t)]_{\equiv^E_m}$.

Then we can see that Cell and Fun form category equivalences:

Theorem 4.2.7. Cell *and* Fun *are category equivalences between the category of predendrograms and the category of predilators.*

Let us provide the characterization of preflowers in terms of predendrograms:

Proposition 4.2.8. *A predilator D is a preflower if and only either every $x \in \text{Cell}(D)$ has length 0, or if there is $x^* \in \text{Cell}(D)$ of length 0 such that for every $x \in \text{Cell}(D)$, we have either*

1. $\text{lh}(x) = 0$ *and* $\text{Cell}(D) \vDash x \lessdot x^*$, *or*
2. $\text{lh}(x) > 0$, x^* *occurs in the sequence of predecessors of x, and* $\mathbf{e}^{\text{Cell}(D)}(x) < \text{lh}(x)$ *if* $\mathbf{e}^{\text{Cell}(D)}(x)$ *is defined.*

Sketch of proof. We only consider the case when D has a non-nullary term. For one direction, suppose that D is a preflower and $t \in \text{field}(D)$. If $\text{arity}\, t > 0$, then $\Sigma^D_t(0) = \text{arity}\, t - 1$. Hence $\Sigma^D_t(i)$ is an $< i$th least element of $\{\Sigma^D_t(j) \mid j \leq i\}$. Also, if $\text{arity}\, s = 0$ and $s \in \text{field}(D)$, then $D \vDash s <_{Д} t$ for every arity diagram Д between s and t, which

implies $[s]_{\equiv_0^D} < [t]_{\equiv_0^D}$ by Corollary C.4.10. If $t' \in \text{field}(D)$ is another term of arity > 0, then again Corollary C.4.10 implies $\mathbf{p}^D(t, t') > 0$, so we can take $x^* = [t]_{\equiv_0^D}$. For the other direction, from the given assumption, we have that $\text{Cell}(\text{Fun}(D)) \cong D$ satisfies the following:

1. If $\text{arity}^{\text{Cell}(\text{Fun}(D))}(x) = 0 < \text{arity}^{\text{Cell}(\text{Fun}(D))}(y)$, then $x <^{\text{Cell}(\text{Fun}(D))} y$.
2. If $\text{arity}^{\text{Cell}(\text{Fun}(D))}(x) > 0$, then $\Sigma_s^{\text{Cell}(\text{Fun}(D))}(0) = \text{arity}(x) - 1$ for every $x \in \text{Cell}(\text{Fun}(D))$.

Hence D is a preflower by Corollary C.4.10. □

In the latter part of the chapter, we iterate a measure over a finite dendrogram. Dendrograms are not linear, so we need to specify the order over the dendrogram before the iteration. It turns out that the following type of dendrogram includes a correct iteration order:

Definition 4.2.9. A predendrogram D is *trekkable* if

1. The field of D is an ordinal.
2. For each $x, y \in D$, if $D \vDash x \lessdot y$ or $D \vDash x \multimap y$, then x is less than y as ordinals. That is, the predecessor relation or a comparison relation over D respects an ordinal order.

A function $f : D \to D'$ between two trekkable predendrogram is a *trekkable predendrogram morphism* if f is an ordinal order-preserving predendrogram morphism.

In particular, the domain of a finite dendrogram is a natural number. Most trekkable predendrograms we care about are finite, although we will see a countable trekkable dendrogram in subsection 4.5.6.

Let us recall the completion dendrogram defined in Definition D.3.14. We will later iterate measures along a finite dendrogram, which requires considering the intermediate nodes of the dendrogram. For technical reasons, we also need to explicate intermediate nodes of a dendrogram to construct a measure family, and this is why we need the completion of a dendrogram. However, it will turn out that we iterate measures not over an arbitrary finite dendrogram, but over a finite dendrogram representing a flower with no nullary terms. For this reason, we do not need to unravel the unique root node in the dendrogram representing a flower with no nullary terms. Hence, let us consider the following variant of the completion dendrogram.

Definition 4.2.10. Let d be a predendrogram. We define $d^\bullet$ by the disjoint union of $\{x^\bullet \mid x \in d\}$ and the set of all non-terminal points of d. We define $\multimap$, $<$, and $\mathbf{e}$ over $d^\bullet$ as follows:

1. For $x, y \in d$, we have $d^\bullet \vDash x \multimap y$ (if y is not terminal in d) and $d^\bullet \vDash x \multimap y^\bullet$.
2. For $x, y \in d$ with $d \vDash x \lessdot y$, we have $d^\bullet \vDash x \lessdot x^\bullet \lessdot y \lessdot y^\bullet$. We ignore undefined elements from the defining inequality.
3. For a non-terminal $x \in d$, $\mathbf{e}^{d^\bullet}(x) = \mathbf{e}^d(x)$.

We can describe the construction of $d^\bullet$ as follows: Starting from d, we put all terminal nodes with a bullet. Then, for every intermediate node x, let us add a new node $x^\bullet$ with the same immediate predecessor just to the right side of x. The new nodes represent unraveled intermediate nodes in d. See Figure 4.1 for an example of the construction. $x^\bullet$ is always a terminal in d, so $\mathbf{e}(x^\bullet)$ is undefined. $d^\bullet$ is obtained from the construction of d^* by not adding copies of root nodes, and marking every terminal node a bullet, either added or already present in d. (See Definition D.3.14 for the definition of d^*.)

The following theorem says we can decompose comparison relations over a dilator

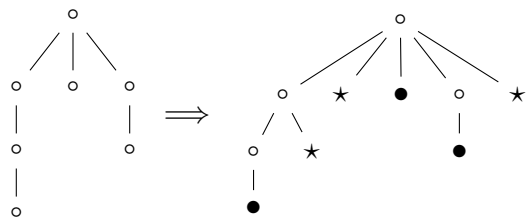


Figure 4.1: The construction of $d^\bullet$ from d. Filled circle nodes represent terminal nodes in d, and filled starred nodes represent unraveled intermediate nodes in d.

into simpler ones.

Theorem 4.2.11 (Elementary comparison decomposition theorem). *Let d be a predendrogram. Every comparison relation $s <_{\text{Д}} t$ over $\mathrm{Fun}(d^\bullet)$ is decomposed into the following* elementary comparison rules*: If we fix $\mathrm{pred}(s) = \langle s_0, \cdots, s_{\mathrm{lh}\, s}\rangle$ and $\mathrm{pred}(t) = \langle t_0, \cdots, t_{\mathrm{lh}\, t}\rangle$,*

(A) $t <_{\mathrm{Diag}_{\mathbb{N}}(b,a)} s$ for $s \multimap t$ and $a \subseteq b \subseteq \mathbb{N}$.

(B) $s <_{\mathrm{Diag}_{\mathbb{N}}(a,a)} t$ when s and t have the same predecessor and $s \lessdot t$.

(C) $s <_{\mathrm{Diag}_{\mathbb{N}}(a,b)} t$ when s and t have the same predecessor, $a_i = b_i$ for every $i < \mathrm{lh}\, s - 1$, and $a_{\mathrm{lh}\, s-1} \lessdot b_{\mathrm{lh}\, t-1}$ (Note that $\mathrm{lh}\, s = \mathrm{lh}\, t$.)

(D) $s <_{\mathrm{Diag}_{\mathbb{N}}(a,b)} t$ when $\mathrm{lh}\, s + 1 = \mathrm{lh}\, t$, $\mathrm{pred}(t) \restriction \mathrm{lh}\, s = \mathrm{pred}\, s$, $a \subseteq b \subseteq \mathbb{N}$, $a_i = b_i$ for $i < \mathrm{lh}\, s$, and $s \lessdot t_{\mathrm{lh}\, s}$.

Here we enumerate $a = \{a_0, \cdots, a_{\mathrm{lh}\, s-1}\}$ with respect to $\prec_s$, i.e., in the way that $a_i < a_j$ iff $i \prec_s j$ and similar to $b = \{b_0, \cdots, b_{\mathrm{lh}\, t-1}\}$. In particular, if the map $f\colon \mathrm{Fun}(d^\bullet) \to \Omega$ preserves every elementary comparison relation, then f preserves every comparison relation.

Proof. Suppose that $\mathrm{pred}(s) = \langle s_0, s_1, \cdots, s_{\mathrm{lh}\, s}\rangle$ and $\mathrm{pred}(t) = \langle t_0, t_1, \cdots, t_{\mathrm{lh}\, t}\rangle$, and

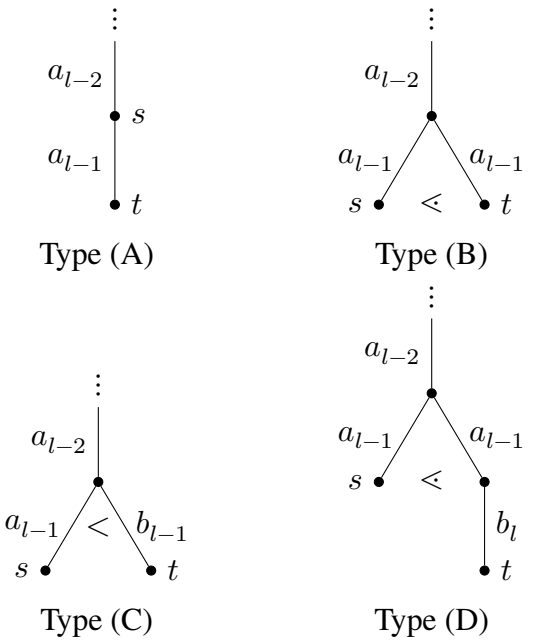


Figure 4.2: Elementary comparison relations

$a \in [\mathbb{N}]^{\mathrm{lh}\, s}$, $b \in [\mathbb{N}]^{\mathrm{lh}\, t}$. Now suppose that $\mathrm{Fun}(d^\bullet)(\mathbb{N}) \vDash s(a) < t(b)$: We have the following three possible cases:

1. $\mathrm{pred}(s) \supsetneq \mathrm{pred}(t)$ and $a \supsetneq b$.
2. There is $m \leq \min(\mathrm{lh}\, s, \mathrm{lh}\, t)$ such that for every $i < m$, $a_i = b_i$ and $s_i = t_i$, but $s_m \prec t_m$.
3. There is $m \leq \min(\mathrm{lh}\, s, \mathrm{lh}\, t)$ such that for every $i < m-1$, $a_i = b_i$ and $s_i = t_i$, $s_{m-1} = t_{m-1}$, but $a_{m-1} < b_{m-1}$.

The first case is easily decomposed into a series of Type (A) comparison relations. In the latter two cases, let us observe that $s(a) < s_m(\{a_0, \cdots, a_{m-1}\}) < t(b)$, and the first comparison is Type (A). Hence, we may assume $m = \mathrm{lh}\, s$ in the latter two cases.

Now let us consider the second case with $m = \mathrm{lh}\, s$. We have Type (B) if $m = \mathrm{lh}\, t$. We claim that if $m < \mathrm{lh}\, t$, then we can decompose the comparison $s(a) < t(b)$

into comparisons of Type (D) by induction on $\mathrm{lh}\, t \geq m$: The case $\mathrm{lh}\, t = m+1$ is Type (C). Now consider the case $\mathrm{lh}\, t > m+1$. Since a and b describe the relative position of parameters, we may assume that every component of a and b is a non-zero even number. If we take $b' = \{b_0, b_1, \cdots, b_{\mathrm{lh}\, t-2} - 1)\}$, then $s(a) < t_{\mathrm{lh}\, t-1}(b') < t(b)$. The first comparison $s(a) < t_{\mathrm{lh}\, t-1}(b')$ is the second case with $\mathrm{lh}(t_{\mathrm{lh}\, t-1}) < \mathrm{lh}(t)$, which is decomposed into comparisons of type (D) by the inductive hypothesis. The second comparison $t_{\mathrm{lh}\, t-1}(b') < t(b)$ is of Type (D).

Similarly, let us consider the third case with $m = \mathrm{lh}\, s$. We have Type (C) if $m = \mathrm{lh}\, t$. For $m < \mathrm{lh}\, t$, we claim that we can decompose $s(a) < t(b)$ into Type (D) and (C) by induction on $\mathrm{lh}\, t \geq m$: We may also assume that every component of a and b is a non-zero even number. Let us take $b' = \{b_0, \cdots, b_{m-1} - 1\}$. Then $s(a) < t_{\mathrm{lh}\, s}(b') < t(b)$, and the first comparison $s(a) < t_{\mathrm{lh}\, s}(b')$ is of Type (C). If $\mathrm{lh}\, t = m+1$, the second comparison $t_{\mathrm{lh}\, s}(b') < t(b)$ is of type (D). If $\mathrm{lh}\, t > m+1$, we can further decompose $t_{\mathrm{lh}\, s}(b') < t(b)$ into a comparison of type (D) and (C) by the inductive hypothesis. □

We finish this section with the following lemma, which we will apply:

Lemma 4.2.12. *Suppose that d is a dendrogram and $x, y \in d$. If $\mathrm{lh}\, x = \mathrm{lh}\, y = m$ and $\mathrm{Fun}(d^\bullet) \vDash x(m) < y(m)$, then either $(x' \multimap y \wedge x \lessdot y)$ or $\mathrm{Fun}(d^\bullet)(\omega) \vDash x'(m \setminus \{\mathbf{e}(x')\}) \lessdot y(m)$.*

Proof. Let $\mathrm{pred}(x) = \langle x_0, \cdots, x_m \rangle$ and $\mathrm{pred}(y) = \langle y_0, \cdots, y_m \rangle$. If $q < m$ is the least number such that $x_q \neq y_q$, then $q = \mathbf{p}^{\mathrm{Fun}(d^\bullet)}(x, y)$ and one of the following holds by Theorem C.3.11:

1. There is $p < q$ such that $\Sigma_x^{\mathrm{Fun}(d^\bullet)}(p) < \Sigma_y^{\mathrm{Fun}(d^\bullet)}(p)$.
2. $\Sigma_x^{\mathrm{Fun}(d^\bullet)}(p) = \Sigma_y^{\mathrm{Fun}(d^\bullet)}(p)$ for every $p < q$ and $x_q < y_q$.

If $q = m$, then x and y have the same immediate predecessor, and the second case holds, so $x < y$. Otherwise, if we let $a = m \setminus \{\mathbf{e}(x')\}$, then $a(\Sigma^{\mathrm{Fun}(d^\bullet)}_{x'}(p)) = \Sigma^{\mathrm{Fun}(d^\bullet)}_{x}(p)$ for every $p < m - 1$. Hence, each case implies the following

1. There is $p < q$ such that $a(\Sigma^{\mathrm{Fun}(d^\bullet)}_{x'}(p)) < \Sigma^{\mathrm{Fun}(d^\bullet)}_{y}(p)$, or
2. $a(\Sigma^{\mathrm{Fun}(d^\bullet)}_{x'}(p)) = \Sigma^{\mathrm{Fun}(d^\bullet)}_{y}(p)$ for every $p < q$ and $x_q < y_q$.

But in either case, we have $\mathrm{Fun}(d^\bullet)(\omega) \vDash x'(a) < y(m)$. □

4.3 Measurable dilator

A measurable dilator is a dilator analogue of a measurable cardinal defined by Kechris [49]. Like the existence of a measurable cardinal proves $\mathbf{\Pi}^1_1$-determinacy, the existence of a measurable dilator proves $\mathbf{\Pi}^1_2$-determinacy. In this section, we define and examine the properties of a measurable dilator.

4.3.1 Universal dilator and measurable dilator

Let us start with the following question: Do we have a dilator embedding every countable dilator? It is like asking if there is a dilator analogue of ω_1, a well-order that embeds every countable well-order.

Definition 4.3.1. A dilator D is *universal* if D embeds every countable dilator.

ZFC proves there is a universal dilator: Let us enumerate $\{D_\alpha \mid \alpha < \mathfrak{c}\}$ of every dilator whose field is $\mathbb{N}$, and take the ordered sum $\sum_{\alpha<\mathfrak{c}} D_\alpha$. However, the resulting

universal dilator is far from being definable. Indeed, Kechris [49] proved that ZFC does not prove there is an ordinal definable universal dilator:

Proposition 4.3.2 (Kechris [49]). ZFC *does not prove there is an ordinal definable universal dilator.*

Proof. Suppose that ZFC proves there is an ordinal definable universal dilator Ω, and let us work over ZFC. Suppose that D_x is a recursive dilator with a real parameter x. Now let us consider the tree T trying to construct a real and an embedding $D_x \to \Omega$ as follows: T is a tree over $\omega \times \text{field}(\Omega)$, and

$$\langle (s_0, t_0), \cdots, (s_{m-1}, t_{m-1}) \rangle \in T$$

if the following holds: For $i, j < m$, suppose that $\vec{s} = \langle s_0, \cdots, s_{m-1} \rangle$ is long enough to determine $i, j \in \text{field}(D_{\underline{\vec{s}}})$, the arity of i and j as $D_{\underline{\vec{s}}}$-terms. Furthermore, assume that Д is an arity diagram between i and j and $D_{\underline{\vec{s}}}$ can also see $i <_{\text{Д}} j$ holds. Then $t_i <_{\text{Д}} t_j$.

It is clear that if $(x, \vec{t})$ is an infinite branch, then $D_x \leq \Omega$. Conversely, we can turn an embedding $D_x \leq \Omega$ into an infinite branch of T. Hence $D_x \leq \Omega$ if and only if

$$T_x = \{ \langle t_0, \cdots, t_{m-1} \rangle \mid \langle (x(0), t_0), \cdots, (x(m-1), t_{m-1}) \rangle \in T \}$$

has an infinite branch. Since Ω is universal and D_x is always countable, we have $\mathsf{Dil}(D_x)$ if and only if D_x embeds Ω. Therefore $\mathsf{Dil}(D_x)$ if and only if $x \in p[T]$, and T is ordinal definable since Ω is. Hence, every Π^1_2-set has an ordinal definable element by taking the leftmost branch of T.

However, the previous statement consistently fails over a generic extension of L obtained by adding a Cohen real since the set

$$X = \{ r \in \mathbb{R} \mid \forall M [M \text{ is a transitive model of } \mathsf{ZFC}^- + (V = L) \\ \to r \text{ is Cohen over } M] \}$$

is a Π^1_2 set of reals without an ordinal definable element. $\square$

Meanwhile, we can find an ordinal definable universal dilator under 'every real has a sharp.' We sketch its construction in the next subsection. The next definition is what we promised at the beginning of the section:

Definition 4.3.3. A universal dilator Φ is *measurable* if for each finite dilator d there is a countably complete measure μ_d over Φ^d satisfying the following:

1. (Coherence) For each $f : d \to d'$, let $f^* : \Phi^{d'} \to \Phi^d$ be $f^*(p) = p \circ f$. (Φ^d is the set of embeddings from d to Φ.) Then we have
$$X \in \mu_d \iff (f^*)^{-1}[X] \in \mu_{d'}.$$
2. (ω_1-completeness) For a given countable dilator D and a countable family $\{d_n \mid n < \omega\}$ of finite subdilators of D, if we have $X_n \subseteq \Phi^{d_n}$ and $X_n \in \mu_{d_n}$ for each $n < \omega$, then we can find an embedding $e : D \to \Phi$ such that $e \restriction d_n \in X_n$ for every $n < \omega$.

We say Φ is *half-measurable* if the measure μ_d is defined only for sets in $\bigcup_{x \in \mathbb{R}} \mathcal{P}(\Phi^d) \cap A_x$, where A_x is the least admissible set containing Φ and x.

ω_1-completeness has the following equivalent formulation:

Proposition 4.3.4. *Let Φ be a universal dilator with a measure family $\{\mu_d\}_d$ satisfying coherence. Then the following are equivalent:*

1. *Φ satisfies ω_1-completeness.*
2. *For a given family $\{d_n \mid n < \omega\}$ of finite dilators, embeddings $f_n : d_n \to D$, and $X_n \in \Phi^{d_n}$, we can find $e : D \to \Phi$ such that $e \circ f_n \in X_n$ for each $n < \omega$.*

Proof. For one direction, let d'_n be the range of f_n. Then $\langle d'_n \mid n < \omega \rangle$ is the sequence of subdilators of D. Then by the ω_1-completeness, we can find $e \colon D \to \Phi$ such that $e \restriction d'_n \in (f_n^*)^{-1}[X_n]$ for every $n < \omega$. Hence $e \circ f_n \in X_n$ for each n, as desired. The other direction follows by taking f_n to be the inclusion map. □

4.3.2 From a measurable flower to a measurable dilator

It will turn out that the most natural way to define a universal dilator is by iterating an ultrapower. The resulting 'dilator' is a flower, so it cannot embed every countable dilator. However, the resulting flower still embeds every countable flower, so we can think of it as a *universal flower*. We can extract a universal dilator from it by 'differentiating' the universal flower.

Proposition 4.3.5. *Let Ω be a universal flower, i.e., Ω is a flower and embeds every countable flower. Then $\partial\Omega$ is a universal dilator.*

Proof. Let D be a countable dilator, so $\int D$ is a countable flower. By universality, there is an embedding $f \colon \int D \to \Omega$, and we have $\partial f \colon \partial(\int D) \to \partial\Omega$. Since $D \cong \partial(\int D)$, we have the desired result. □

From the previous proposition, let us sketch how to construct an ordinal definable universal dilator from sharps of reals:

Example 4.3.6. Let us work over ZFC with 'every real has a sharp.' Let F_x be a sharp flower for $x^\sharp$ defined in [6]. By [6, Proposition 12], every countable flower in $L[x]$ embeds into F_x. Now let us consider the system of flowers $\{F_x \mid x \in \mathbb{R}\}$ with a natural choice of embeddings provided in [6, Lemma 11], and consider its direct limit. The re-

sulting flower F is ordinal definable and embeds every countable flower. Then consider ∂F, which is an ordinal definable universal dilator.

The next example does not precisely give an ordinal definable universal flower without an additional assumption (like $V = L[U]$), but let us include it to illustrate how the iterated ultrapower reveals the structure of a universal flower.

Example 4.3.7. We work over ZFC with a measurable cardinal in this example. Let κ be a measurable cardinal with a normal measure U. Then we can define the αth iterate $\mathrm{Ult}^\alpha(V,U)$ with an embedding $j_\alpha\colon V \to \mathrm{Ult}^\alpha(V,U)$. Then let us define the flower F by $F(\alpha) = \kappa_\alpha := j_\alpha(\kappa)$. To define $F(f)$ for an increasing $f\colon \alpha \to \beta$, let us observe the following fact [46, Lemma 19.6]: For every ordinal α and $x \in \mathrm{Ult}^\alpha(V,U)$, we have $m < \omega$, $h\colon [\kappa]^m \to V$, and $\gamma_0 < \cdots < \gamma_{m-1} < \alpha$ such that $x = j_\alpha(h)(\kappa_{\gamma_0}, \cdots, \kappa_{\gamma_{m-1}})$. Here, we think of $\mathrm{Ult}^\alpha(V,U)$ as a 'dilator-like structure' with input α, and its element x is represented by a 'term' $h\colon [\kappa]^m \to V$ and 'parameters' $\gamma_0 < \cdots < \gamma_{m-1}$. Combining with [46, Lemma 19.9], we can see that if we define $\mathrm{Ult}^f(V,U)\colon \mathrm{Ult}^\alpha(V,U) \to \mathrm{Ult}^\beta(V,U)$ by

$$\mathrm{Ult}^f(V,U)(j_\alpha(h)(\kappa_{\gamma_0}, \cdots, \kappa_{\gamma_{m-1}})) = j_\beta(h)(\kappa_{f(\gamma_0)}, \cdots, \kappa_{f(\gamma_{m-1})})$$

then $\mathrm{Ult}^f(V,U)$ is well-defined. Then set $F(f) = \mathrm{Ult}^f(V,U) \restriction F(\alpha)$.

We need an appropriate support transformation to turn F into a dilator. For $\xi < \kappa_\alpha$, let m be the least natural number such that there are $h\colon [\kappa]^m \to V$ and $\gamma_0 < \cdots < \gamma_{m-1} < \alpha$ such that $\xi = j_\alpha(h)(\kappa_{\gamma_0}, \cdots, \kappa_{\gamma_{m-1}})$. From [46, Lemma 19.9] we can prove that $\gamma_0 < \cdots < \gamma_{m-1}$ are uniquely determined from ξ, so we can define $\mathrm{supp}^F_\alpha(\xi) = \{\gamma_0, \cdots, \gamma_{m-1}\}$. The support condition is easy to verify.

We have defined F only for ordinals, but we can extend F to other linear orders by Corollary B.4.14. To see F is a flower, observe that $F(\alpha)$ is an initial segment of $F(\beta)$ if $\alpha < \beta$. To see F is universal, let us prove the following stronger claim: If $D \in V_\kappa$ is a

flower, then D embeds into F. If $D \in V_\kappa$, then $D(\kappa)$ is isomorphic to κ. (See the proof of Proposition 4.4.10 for the reason.) Fix an isomorphism $g\colon D(\kappa) \to \kappa$, and consider $\iota_\alpha\colon D(\alpha) \to \kappa_\alpha$ given by

$$\iota_\alpha(t(\xi_0, \cdots, \xi_{m-1})) = j_\alpha(g)(t(\kappa_{\xi_0}, \cdots, \kappa_{\xi_{m-1}})).$$

Then we can see that $\iota\colon D \to F$ is a natural transformation.

So far, we know that a universal flower induces a universal dilator. Is it the same for a 'measurable flower?' The answer is affirmative:

Proposition 4.3.8. *Let us say a universal flower Ω is a* measurable flower *if, for every finite flower d with no nullary terms, there is a countably complete measure μ_d over Ω^d satisfying the coherence and σ-completeness for flowers. Then $\partial\Omega$ is a measurable dilator.*

Proof. Let d be a finite dilator. Then $\int d$ is a finite flower with no nullary terms. Then let us define a measure ν_d over $(\partial\Omega)^d$ by

$$X \in \nu_d \iff \{p \in \Omega^{\int d} \mid \partial p \circ \phi_d \in X\} \in \mu_{\int d},$$

where $\phi_d\colon d \to \partial(\int d)$ is the isomorphism natural in d. For coherence, suppose that d, d' are finite dilators and $f\colon d \to d'$ is a dilator embedding. Then for $X \subseteq (\partial\Omega)^d$,

$$\begin{aligned}
X \in \nu_d &\iff \{p \in \Omega^{\int d} \mid \partial p \circ \phi_d \in X\} \in \mu_d \\
&\iff (\textstyle\int f^*)^{-1}[\{p \in \Omega^{\int d} \mid \partial p \circ \phi_d \in X\}] \in \mu_{d'} \\
&\iff \{q \in \Omega^{\int d'} \mid \partial(q \circ \textstyle\int f) \circ \phi_d \in X\} \\
&\qquad = \{q \in \Omega^{\int d'} \mid \partial q \circ \phi_{d'} \circ f \in X\} \in \mu_{\int d'} \\
&\iff \{q \in \Omega^{\int d'} \mid \partial q \circ \phi_{d'} \in \{r \in (\partial\Omega)^{d'} \mid r \circ f \in X\}\} \in \mu_{\int d'} \\
&\iff \{r \in (\partial\Omega)^{d'} \mid r \circ f \in X\} \in \nu_{d'} \iff (f^*)^{-1}[X] \in \nu_{d'}.
\end{aligned}$$

Note that $\partial(q \circ \int f) \circ \phi_d = (\partial q) \circ \partial(\int f) \circ \phi_d = \partial q \circ \phi_{d'} \circ f$. For σ-completeness, suppose that D is a countable dilator and $\{d_n \mid n < \omega\}$ is a countable family of finite subdilators of D. Then $\int d_n \subseteq \int D$ is a finite flower with no nullary terms for each $n < \omega$. Now suppose that we are given $X_n \in \nu_{d_n}$ for each $n < \omega$, so

$$Y_n := \{p \in \Omega^{\int d_n} \mid \partial p \circ \phi_{d_n} \in X_n\} \in \mu_{\int d_n}.$$

Hence by the σ-completeness of Ω, there is an embedding $e \colon \int D \to \Omega$ such that $e \restriction \int d_n \in Y_n$ for each $n < \omega$. Now observe that the following diagram commutes:

$$\begin{array}{ccccc} d_n & \xrightarrow{\phi_{d_n}} & \partial(\int d_n) & & \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} & \searrow^{\partial(e\restriction \int d_n)} & \\ D & \xrightarrow[\phi_D]{} & \partial(\int D) & \xrightarrow[\partial e]{} & \partial\Omega \end{array}$$

Hence $e \restriction \int d_n \in Y_n$ implies $(\partial e \circ \phi_D) \restriction d_n = \partial e \circ \phi_{d_n} \in X_n$. □

Hence, we have a measurable dilator if we construct a universal flower with a measure family μ_d for a finite flower with no nullary terms satisfying coherence and σ-completeness. Thus, we construct a measurable flower instead of constructing a measurable dilator directly.

4.3.3 Measurable cardinal and $\mathbf{\Pi}^1_1$-determinacy

In this subsection, we review a proof of $\mathbf{\Pi}^1_1$-determinacy from the existence of a measurable cardinal. We will see later that almost the same proof carries over to the proof of $\mathbf{\Pi}^1_1$-determinacy from the existence of a measurable dilator. Let us start with the following lemma, which follows [48, §1.1] but apparently folklore.

Lemma 4.3.9. *For every $\Pi^1_1[R]$-formula $\phi(x)$ we can find an R-recursive family of linear orders $\{\prec_s \mid s \in \omega^{<\omega}\}$ such that*

1. $\prec_s$ *is a linear order over* $|s|$,
2. $s \subseteq t \implies \prec_s \subseteq \prec_t$,
3. $\phi(x)$ *iff* $\prec_x = \bigcup_{n<\omega} \prec_{x\restriction n}$ *is well-ordered.*

Proof. Let T be an R-recursive tree over $\omega \times \omega$ such that $\neg\phi(x)$ iff there is a real y such that $\langle x, y\rangle$ form an infinite branch of T. Fix a recursive enumeration $\langle \mathbf{s}_i \mid i < \omega\rangle$ of $\omega^{<\omega}$ such that if $\mathbf{s}_i \subsetneq \mathbf{s}_j$ then $i < j$. (It implies $|\mathbf{s}_i| \leq i$ for every i.) Then let us define $\prec_s$ of field $|s|$ as follows: We have $i \prec_s j$ if and only if either

1. $\langle s \restriction |\mathbf{s}_i|, \mathbf{s}_i\rangle \in T$, $\langle s \restriction |\mathbf{s}_j|, \mathbf{s}_j\rangle \in T$, and $\mathbf{s}_i <_{\mathsf{KB}} \mathbf{s}_j$, or
2. $\langle s \restriction |\mathbf{s}_i|, \mathbf{s}_i\rangle \notin T$ and $\langle s \restriction |\mathbf{s}_j|, \mathbf{s}_j\rangle \in T$, or
3. $\langle s \restriction |\mathbf{s}_i|, \mathbf{s}_i\rangle \notin T$, $\langle s \restriction |\mathbf{s}_j|, \mathbf{s}_j\rangle \notin T$, and $i < j$.

Then we can see that the first two conditions hold. Furthermore, $\prec_x$ satisfies the definition of $\prec_s$ in which s is replaced by x, so $\prec_x$ has ordertype $\alpha + (T_x, <_{\mathsf{KB}})$ for some $\alpha \leq \omega$, where $T_x = \{t \in \omega^{<\omega} \mid (x \restriction |t|, t) \in T\}$. Hence $\prec_x$ is well-founded iff $(T_x, <_{\mathsf{KB}})$ is well-founded iff T_x has no infinite branch iff $\phi(x)$. □

Let κ be a measurable cardinal with a normal measure $\mathcal{U}$. By iterating a measure, we have

Lemma 4.3.10. *Suppose that c is a countable well-order such that $c = \bigcup_{n<\omega} c_n$ for finite $c_n \subseteq c$. Then we can find an ω_1-complete measure $\mathcal{U}_{c_n}$ over the set κ^{c_n} of embeddings $c_n \to \kappa$ such that if $\{X_n \mid n < \omega\}$ is a family of sets such that $X_n \in \mathcal{U}_{c_n}$ for each $n < \omega$, then there is $f : c \to \kappa$ such that $f \restriction c_n \in X_n$ for each $n < \omega$.*

Proof. Let us consider

$$\mathcal{U}_{c_n} = \{X \subseteq \kappa^{c_n} \mid \epsilon_n^*[X] := \{p \circ \epsilon_n : |c_n| \to \kappa \mid p \in X\} \in \mathcal{U}^{|c_n|}\}$$

where $\epsilon_n : |c_n| \to c_n$ is the unique order isomorphism. We claim that $\mathcal{U}_{c_n}$ is the desired ultrafilter. Suppose that $X_n \in \mathcal{U}_{c_n}$ for each n. Then we can find $Y_n \in \mathcal{U}$ such that

$$[Y_n]^{|c_n|} \subseteq \epsilon_n^*[X_n] = \{p \circ \epsilon_n : |c_n| \to \kappa \mid p \in X_n\}.$$

Then let $Y = \bigcap_{n<\omega} Y_n \in \mathcal{U}$ and choose an embedding $f : c \to Y$. Then clearly

$$(f \restriction c_n) \circ \epsilon_n \in [Y]^{|c_n|} \subseteq \epsilon_n^*[X_n],$$

so $f \restriction c_n \in X_n$. □

For a coanalytic game G, let us associate the family $\{\prec_s | \ s \in \omega^{<\omega}\}$ such that

$$\text{Player I wins } G \text{ in the play } x \iff \mathsf{WO}(\prec_x).$$

Then consider the subsidiary game G'

$$\begin{array}{lccccc} \text{I} & x_0, \eta_0 & & x_2, \eta_1 & & \cdots \\ \text{II} & & x_1 & & x_3 & \cdots \end{array}$$

for $\eta_i < \kappa$. Player I wins G' iff

$$\forall i, j < \omega (i \prec_x j) \iff \eta_i < \eta_j.$$

Clearly, if Player I has a winning strategy in G', then taking the projection gives a winning strategy for Player I in G.

Proposition 4.3.11. *If Player II has a winning strategy in G', then Player II has a winning strategy in G.*

Proof. Let σ' be a winning strategy for Player II in G'. For each partial play $s = \langle x_0, x_1, \cdots, x_{2n-2} \rangle$ and $p \in \kappa^{\prec_s}$, define

$$f_s(p) = \sigma' \begin{pmatrix} x_0, p(0) & & \cdots & x_{2n-2}, p(n-1) \\ & x_1 & \cdots & \end{pmatrix}$$

Then define

$$\sigma(s) = a \iff \{p \in \kappa^{\prec_s} \mid f_s(p) = a\} \in \mathcal{U}^n.$$

It is well-defined because $f_s \colon \kappa^{\prec_s} \to \omega$ and $\mathcal{U}_{\prec_s}$ is ω_1-complete. Then take $Z_s = \{p \in [\kappa]^n \mid f_s(p) = \sigma(s)\}$.

We claim that σ is a winning strategy for Player II in G. Suppose not, let x be a play respecting σ but Player I wins. By the lemma, we can find an embedding $e \colon (\omega, \prec_x) \to \kappa$ such that $e \restriction (2n-1, \prec_{x\restriction(2n-1)}) \in Z_{x\restriction(2n-1)}$ for each n. Hence we have

$$f_{x\restriction(2n-1)}(e \restriction d_{x\restriction(2n-1)}) = \sigma(x \restriction (2n-1)) = x_{2n-1},$$

so the game

$$\begin{array}{ccccc} x_0, e(0) & & \cdots & x_{2n-2}, e(n-1) & \\ & x_1 & \cdots & & x_{2n-1} \end{array}$$

becomes a valid play respecting σ'. Since the play respects σ', Player II wins. However, $e \colon (\omega, \prec_x) \to \kappa$ is an embedding, so Player I wins, a contradiction. □

4.3.4 Measurable dilator and $\mathbf{\Pi}^1_2$-determinacy

In this subsection, we prove $\mathbf{\Pi}^1_2$-determinacy from the existence of a measurable dilator. We can also see that the following proof is more or less similar to that of $\mathbf{\Pi}^1_1$-determinacy from a measurable dilator.

Lemma 4.3.12. *Let $\phi(x)$ be a $\Pi^1_2[R]$-formula. Then there is a recursive family of finite dilators $\{d_s \mid s \in \omega^{<\omega}\}$ such that*

1. $|s| = \text{field}(d_s)$.
2. $s \subseteq t \implies d_s \subseteq d_t$.

3. $\phi(x)$ *iff* $d_x = \bigcup_{n<\omega} d_{x\restriction n}$ *is a dilator.*

Proof. Let $\phi(x) \equiv \forall y \in \mathbb{R} \neg\psi(x,y)$ for some $\Pi^1_1[R]$-formula $\psi(x,y)$. By modifying Lemma 4.3.9, for a $\Pi^1_1[R]$-formula $\psi(x,y)$ we can find an R-recursive family of linear orders $\{\prec_{s,t} |\ s,t \in \omega^{<\omega},\ |s| = |t|\}$ such that

1. $\prec_{s,t}$ is a linear order over $|s|$,
2. $s \subseteq s',\ t \subseteq t' \implies \prec_{s,t} \subseteq \prec_{s',t'}$,
3. $\psi(x,y)$ iff $\prec_{x,y} = \bigcup_{n<\omega} \prec_{x\restriction n, y\restriction n}$ is well-ordered.

Again, fix a recursive $\langle \mathbf{s}_i \mid i < \omega\rangle$ of $\omega^{<\omega}$ provided in the proof of Lemma 4.3.9. Note that $\mathbf{s}_0 = \langle\rangle$, which we will exclude in constructing a dilator family.

Then let us define a dilator d_s of the field $|s|$. Before starting the main construction, let us state the motivation: We want to define a family of R-recursive finite dilators approximating the tree

$$\hat{T}_x(\alpha) = \Big\{ \langle r_0, \xi_0, \cdots, r_{m-1}, \xi_{m-1}\rangle \ \Big|\ r_0, \cdots, r_{m-1} \in \omega,\ \xi_0, \cdots, \xi_{m-1} \in \alpha \\ \wedge\, i \mapsto \xi_i \text{ is an increasing map } \prec_{x\restriction m, \langle r_0, \cdots, r_{m-1}\rangle} \to \alpha \Big\}.$$

equipped with the Kleene-Brouwer order. This tree occurs in the proof of the Shoenfield absoluteness theorem, and we can see that $\neg\phi(x)$ holds iff there is $\alpha < \omega_1$ such that $\hat{T}_x(\alpha)$ has an infinite branch. We can also see that $\alpha \mapsto (\hat{T}_x(\alpha), <_{\mathsf{KB}})$ is a predilator, so $\hat{T}_x$ can be seen as a functorial Shoenfield tree.

Let us consider the following finite F-semidilator $\hat{d}_s$ with the following field equipped with the Kleene-Brouwer order:

$$\hat{d}_s(\alpha) = \Big\{ \langle \mathbf{s}_{i+1}(0), \xi_0, \cdots, \mathbf{s}_{i+1}(|\mathbf{s}_{i+1}|-1), \xi_{|\mathbf{s}_{i+1}|-1} \rangle \;\Big|\; \xi_0, \cdots, \xi_{|\mathbf{s}_{i+1}|-1} \in \alpha, \\ i < |s| \wedge k \mapsto \xi_k \text{ is an increasing map } \prec_{s\restriction|\mathbf{s}_{i+1}|,\mathbf{s}_{i+1}} \to \alpha \Big\}.$$

We also define for $f \colon \alpha \to \beta$,

- $\hat{d}_s(f)(\langle s_0, \xi_0, \cdots, s_m, \xi_m \rangle) = \langle s_0, f(\xi_0), \cdots, s_m, f(\xi_m) \rangle$,
- $\mathrm{supp}_\alpha^{\hat{d}_s}(\langle s_0, \xi_0, \cdots, s_m, \xi_m \rangle) = \{\xi_0, \cdots, \xi_{m-1}\}$.

We use $\mathbf{s}_{i+1}$ instead of $\mathbf{s}_i$ to exclude the empty sequence, which will be the topmost of the linear order. It also makes $\hat{d}_s$ a finite flower, but this observation is unnecessary in this proof. Then we can see that $\hat{d}_s$ is an F-predilator and

$$\mathrm{Tr}(\hat{d}_s) = \Big\{ \langle \mathbf{s}_{i+1}(0), \sigma(0), \cdots, \mathbf{s}_{i+1}(|\mathbf{s}_{i+1}|-1), \sigma(|\mathbf{s}_{i+1}|-1) \rangle \;\Big|\; \\ i < |s| \wedge \sigma \colon \prec_{s\restriction|\mathbf{s}_{i+1}|,\mathbf{s}_{i+1}} \to |\mathbf{s}_{i+1}| \text{ increasing} \Big\}$$

Note that for each $i < |s|$ there is a unique increasing map $\prec_{s\restriction|\mathbf{s}_{i+1}|,\mathbf{s}_{i+1}} \to |\mathbf{s}_{i+1}|$. For notational convenience, write

$$\tau_i = \langle \mathbf{s}_{i+1}(0), \sigma(0), \cdots, \mathbf{s}_{i+1}(|\mathbf{s}_{i+1}|-1), \sigma(|\mathbf{s}_{i+1}|-1) \rangle$$

for the unique increasing map $\sigma \colon \prec_{x\restriction|\mathbf{s}_{i+1}|,\mathbf{s}_{i+1}} \to |\mathbf{s}_{i+1}|$. Then $\mathfrak{f}(\hat{d}_s)$ is a finite dilator satisfying

1. $\mathrm{field}(\mathfrak{f}(\hat{d}_s)) = \{\tau_i \mid i < |s|\}$.
2. For each $i < |s|$, $\mathrm{arity}^{\mathfrak{f}(\hat{d}_s)}(\tau_i) = |\mathbf{s}_{i+1}|$.
3. $\Sigma_{\tau_i}^{\mathfrak{f}(\hat{d}_s)}(j) = k$ when j is the kth least element over $(|\mathbf{s}_{i+1}|, \prec_{s\restriction\mathbf{s}_{i+1},\mathbf{s}_{i+1}})$.
4. $\mathbf{p}^{\mathfrak{f}(\hat{d}_s)}(i,j)$ is the least natural m such that $\mathbf{s}_{i+1}(m) \neq \mathbf{s}_{j+1}(m)$.

We want to turn $\mathfrak{f}(\widehat{d}_s)$ into a dilator of the field $|s|$ with all desired properties pertaining. Thus we define a new dilator d_s from $\widehat{d}_s$ by 'replacing τ_i with i'; More precisely, we define d_s in a way that

1. $\mathrm{field}(d_s) = |s|$.
2. For each $i < |s|$, $\mathrm{arity}^{d_s}(i) = |\mathbf{s}_{i+1}|$.
3. $\Sigma_i^{d_s}(j) = k$ when j is the kth least element over $(|\mathbf{s}_{i+1}|, \prec_{s\restriction \mathbf{s}_{i+1}, \mathbf{s}_{i+1}})$.
4. $\mathbf{p}^{d_s}(i,j)$ is the least natural m such that $\mathbf{s}_{i+1}(m) \neq \mathbf{s}_{j+1}(m)$.

Then clearly $d_s \cong \mathfrak{f}(\widehat{d}_s)$, and it is easy to see that d_s satisfies the first two conditions. For the last condition, observe that $\widehat{T}_x(\alpha) = \bigcup_{n<\omega} \widehat{d}_{x\restriction n}(\alpha)$. $\square$

Let us fix a measurable dilator Ω with a family of measures $\{\mathcal{U}_d \mid d \in \mathsf{Dil}_{<\omega}\}$. Similarly, let G be a $\mathbf{\Pi}^1_2$-game, and fix a family of finite dilators $\{d_s \mid s \in \omega^{<\omega}\}$ such that

1. $\mathrm{field}(d_s) = |s|$.
2. $s \subseteq t \implies d_s \subseteq d_t$.
3. Player I wins in G by the play x iff $d_x = \bigcup_{n<\omega} d_{x\restriction n}$ is a dilator.

Then consider the subsidiary game G'

$$\begin{array}{lccccc} \mathrm{I} & x_0, \tau_0 & & x_2, \tau_1 & & \cdots \\ \mathrm{II} & & x_1 & & x_3 & \cdots \end{array}$$

where $\tau_i \in \mathrm{field}(\Omega)$. Player I wins G' iff $i \mapsto \tau_i$ forms an embedding, i.e.,

$$\forall Д(d_x \vDash i <_Д j \iff \Omega \vDash \tau_i <_Д \tau_j).$$

Proposition 4.3.13. *If Player II has a winning strategy in G', then Player II has a winning strategy in G.*

Proof. Again, let σ' be a winning strategy for Player II in G'. For each partial play $s = \langle x_0, x_1, \cdots, x_{2n-2} \rangle$ and $p \in \Omega^{d_s}$, define

$$f_s(p) = \sigma' \begin{pmatrix} x_0, p(0) & & \cdots & x_{2n-2}, p(n-1) \\ & x_1 & \cdots & \end{pmatrix}$$

Then define

$$\sigma(s) = a \iff \{p \in \Omega^{d_s} \mid f_s(p) = a\} \in \mathcal{U}_{d_s}.$$

Again, take $Z_s = \{p \in [\kappa]^n \mid f_s(p) = \sigma(s)\} \in \mathcal{U}_{d_s}$. Now suppose the contrary that σ is not a winning strategy for Player II, and let x be a play on G respecting σ but Player I wins. Since Ω is measurable, we can find an embedding $e \colon d_x \to \Omega$ such that $e \restriction d_{x\restriction(2n-1)} \in Z_{x\restriction(2n-1)}$ for each n. It means for each n,

$$f_{x\restriction(2n-1)}(e \restriction d_{x\restriction(2n-1)}) = \sigma(x \restriction (2n-1)) = x_{2n-1},$$

so the game

$$\begin{array}{cccc} x_0, e(0) & & \cdots & x_{2n-2}, e(n-1) \\ & x_1 & \cdots & \qquad\qquad x_{2n-1} \end{array}$$

becomes a valid play respecting σ'. Since the play respects σ', Player II wins. However, $e \colon d_x \to \Omega$ is an embedding, so Player I wins, a contradiction. □

4.4 The Martin Flower

In this section, we define a flower from an iterable cardinal that will be a measurable flower, which will be called the *Martin flower* Ω^1_{M} (M denotes Martin.) We define the Martin flower only for ordinals, which is enough to establish its properties, and we can

extend the Martin flower to every linear order with known machinery. We define the Martin flower for natural numbers and increasing maps between them first, then we define the Martin flower for ordinals. We will see that the two definitions cohere. Lastly, we prove that the Martin flower embeds every countable flower.

4.4.1 The Martin flower for natural numbers

We will define the Martin flower Ω^1_{M} from $j\colon V_\lambda \to V_\lambda$, and let us define Ω^1_{M} for natural numbers first:

$$\Omega^1_{\mathsf{M}}(n) = \kappa_n := \operatorname{crit} j_n.$$

and for $f\colon m \to n$,

$$\Omega^1_{\mathsf{M}}(f)(x) = j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)}(x).$$

For notational convenience, let us write

$$j_f(x) = j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)}(x).$$

Note that we will show that Ω^1_{M} is a flower. So far, we only defined Ω^1_{M} over the category of natural numbers with strictly increasing maps. To see Ω^1_{M} is a functor, we need it to preserve the function composition:

Lemma 4.4.1. *Let $f\colon m \to n$, $g\colon n \to k$ for $m \leq n \leq k$. Then $\Omega^1_{\mathsf{M}}(g \circ f) = \Omega^1_{\mathsf{M}}(g) \circ \Omega^1_{\mathsf{M}}(f)$.*

Proof. We always assume $\xi \in \Omega^1_{\mathsf{M}}(m) = \kappa_m$ throughout this proof. To illustrate the idea of the proof, let us consider the case $m = 0$ first: Then we can see that

$$j_g \circ j_{0,f(0)}(\xi) = j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(0)+1,g(1)} \circ j_{0,g(0)} \circ j_{0,f(0)}(\xi)$$

$$
\begin{aligned}
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(0)+1,g(1)} \circ j_{g(0),g(0)+f(0)} \circ j_{0,g(0)}(\xi) \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(0)+1,g(1)} \circ j_{g(0)+1,g(0)+f(0)} \circ j_{g(0)} \circ j_{0,g(0)}(\xi) \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(1),g(1)-1+f(0)} \circ j_{g(0)+1,g(1)} \circ j_{g(0)} \circ j_{0,g(0)}(\xi) \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(1),g(1)-1+f(0)} \circ j_{0,g(1)}(\xi) \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(2),g(2)-2+f(0)} \circ j_{0,g(2)}(\xi) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(f(0)),g(f(0))-f(0)+f(0)} \circ j_{0,g(f(0))}(\xi) \\
&= j_{g(n-2)+1,g(n-1)} \circ \cdots \circ j_{g(f(0))+1,g(f(0)+1)} \circ j_{0,g(f(0))}(\xi)
\end{aligned}
$$

and $j_{0,g(f(0))}(\xi) < \kappa_{g(f(0))+1}$ since $\xi < \kappa_1$. Hence all other elementary embeddings do not change $j_{0,g(f(0))}(\xi)$, so we have $j_g \circ j_{0,f(0)}(\xi) = j_{0,g(f(0))}(\xi)$.

For a general case, let us start from

$$j_f(\xi) = j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)}(\xi).$$

Applying $j_{0,g(0)}$ gives

$$j_{g(0)+f(m-2)+1,g(0)+f(m-1)} \circ \cdots \circ j_{g(0)+f(0)+1,g(0)+f(1)} \circ j_{g(0),g(0)+f(0)}(j_{0,g(0)}(\xi)),$$

which is equal to

$$j_{g(0)+f(m-2)+1,g(0)+f(m-1)} \circ \cdots \circ j_{g(0)+f(0)+1,g(0)+f(1)} \circ j_{g(0)+1,g(0)+f(0)}(j_{0,g(0)+1}(\xi)).$$

Applying $j_{g(0)+1,g(1)}$ gives

$$j_{g(1)+f(m-2),g(1)+f(m-1)-1} \circ \cdots \circ j_{g(1)+f(0),g(1)+f(1)-1} \circ j_{g(1),g(1)+f(0)-1}(j_{0,g(1)}(\xi)).$$

Applying $j_{g(1)+1,g(2)}$ then gives

$$j_{g(2)+f(m-2)-1,g(2)+f(m-1)-2} \circ \cdots \circ j_{g(2)+f(0)-1,g(2)+f(1)-2} \circ j_{g(2),g(2)+f(0)-2}(j_{0,g(2)}(\xi)).$$

By repetition, we have

$$j_{g(f(0))+f(m-2)-f(0)+1,g(f(0))+f(m-1)-f(0)} \circ \\ \cdots \circ j_{g(f(0))+f(0)-f(0)+1,g(f(0))+f(1)-f(0)} \circ j_{g(f(0)),g(f(0))+f(0)-f(0)}(j_{0,g(f(0))}(\xi)).$$

$j_{g(f(0)),g(f(0))+f(0)-f(0)}$ is the identity, so the above is equal to

$$j_{g(f(0))+f(m-2)-f(0)+1,g(f(0))+f(m-1)-f(0)} \circ \cdots \circ j_{g(f(0))+1,g(f(0))+f(1)-f(0)}(j_{0,g(f(0))}(\xi)).$$

Then apply $j_{g(f(0))+1,g(f(0)+1)}$, so we get

$$j_{g(f(0)+1)+f(m-2)-f(0),g(f(0)+1)+f(m-1)-f(0)-1} \circ \cdots \\ \circ j_{g(f(0)+1),g(f(0)+1)+f(1)-f(0)-1}(j_{g(f(0))+1,g(f(0)+1)}(j_{0,g(f(0))}(\xi))).$$

Applying $j_{g(f(0)+1)+1,g(f(0)+2)}$, $\cdots$, $j_{g(f(1)-1)+1,g(f(1))}$ consecutively gives

$$j_{g(f(1))+f(m-2)-f(1)+1,g(f(1))+f(m-1)-f(1)} \circ \cdots \\ \circ j_{g(f(1)),g(f(1))+f(1)-f(1)}(j_{g(f(0))+1,g(f(1))}(j_{0,g(f(0))}(\xi))),$$

which is equal to

$$j_{g(f(1))+f(m-2)-f(1)+1,g(f(1))+f(m-1)-f(1)} \circ \cdots (j_{g(f(0))+1,g(f(1))}(j_{0,g(f(0))}(\xi))).$$

Then we can see that applying $j_{g(f(1))+1,g(f(1)+1)}$, $\cdots$, $j_{g(f(m-1)-1)+1,g(f(m-1))}$ gives

$$j_{g(f(m-2))+1,g(f(m-1))}\left(\cdots\left(j_{g(f(0))+1,g(f(1))}(j_{0,g(f(0))}(\xi))\right)\cdots\right) = j_{g\circ f}(\xi). \tag{4.1}$$

Since $\xi < \kappa_m$, (4.1) is $< \kappa_{g(f(m-1))+1}$. Hence, applying $j_{g(f(m-1))+1,g(f(m-1)+1)}$ or all other remaining embeddings of j_g does not change the value of (4.1). In sum, we have $j_g(j_f(\xi)) = j_{g\circ f}(\xi)$. □

Now, let us define the support function to ensure a semidilator structure. For a strictly increasing $f\colon m \to n$, define

$$I_f = \{j_f(\xi) \mid \xi < \kappa_m\}.$$

Throughout this chapter, we identify an increasing map $f\colon m \to n$ to a finite subset $a \subseteq n$ of size m. Under this convention, we can identify f with its range. We want to understand I_a as members of $\Omega^1_{\mathsf{M}}(\omega)$ whose support is a subset of a. To see this idea working, we need a lemma:

Lemma 4.4.2. *For two finite subsets $a, b \subseteq \omega$, we have*

1. *$a \subseteq b$ implies $I_a \subseteq I_b$.*
2. *$I_a \cap I_b = I_{a\cap b}$.*

Proof. To prove the first statement, it suffices to show it when $|b| = |a| + 1$. Let $\{a(0), \cdots, a(m-1)\}$ be an increasing enumeration of a, $b = a \cup \{l\}$.

1. Consider the case $l < a(0)$. We have

$$j_a(\xi) = j_{a(m-2)+1,a(m-1)} \circ \cdots \circ j_{a(0)+1,a(1)} \circ j_{l+1,a(0)} \circ j_{0,l+1}(\xi)$$

   Also, $j_{0,l+1} = j_0^{l+1} = j_0^l \circ j_0 = j_{0,l} \circ j_0$. Hence we get

$$j_a(\xi) = j_{a(m-2)+1,a(m-1)} \circ \cdots \circ j_{a(0)+1,a(1)} \circ j_{l+1,a(0)} \circ j_{0,l}(j_0(\xi)).$$

   If $\xi < \kappa_m$, then $j_0(\xi) < \kappa_{m+1}$. Hence $j_a(\xi) = j_{a\cup\{l\}}(j_0(\xi)) \in I_{a\cup\{l\}}$.

2. Now let $a(i) < i < a(i+1)$ for some $i < i+1 < m$. We have

$$j_a(\xi) = j_{a(m-2)+1,a(m-1)} \circ \cdots j_{l+1,a(i+1)} \circ j_l \circ j_{a(i)+1,l} \circ \cdots \circ j_{0,a(0)}(\xi)$$

   Observe that $j_l \circ j_{a(i)+1,l} = j_{a(i)+1,l} \circ j_{a(i)+1}$, so we have

$$j_a(\xi) = j_{a(m-2)+1,a(m-1)} \circ \cdots j_{l+1,a(i+1)} \circ j_{a(i)+1,l} \circ j_{a(i)+1} \circ \cdots \circ j_{0,a(0)}(\xi)$$

By repeating a similar computation, we have

$$\begin{aligned}
j_a(\xi) &= j_{a(m-2)+1,a(m-1)} \circ \cdots j_{l+1,a(i+1)} \circ \boxed{j_l} \circ j_{a(i)+1,l} \circ \cdots \circ j_{0,a(0)}(\xi) \\
&= j_{a(m-2)+1,a(m-1)} \circ \cdots j_{l+1,a(i+1)} \circ j_{a(i)+1,l} \circ \boxed{j_{a(i)+1}} \\
&\qquad\qquad \circ j_{a(i-1)+1,a(i)} \circ \cdots \circ j_{0,a(0)}(\xi) \\
&= j_{a(m-2)+1,a(m-1)} \circ \cdots \circ j_{a(i)+1,l} \circ j_{a(i-1)+1,a(i)} \circ \boxed{j_{a(i-1)+2}} \circ \cdots \circ j_{0,a(0)}(\xi) \\
&\qquad\qquad \vdots \\
&= j_{a(m-2)+1,a(m-1)} \circ \cdots \circ \boxed{j_{a(0)+(i+1)}} \circ j_{0,a(0)}(\xi) \\
&= j_{a(m-2)+1,a(m-1)} \circ \cdots \circ j_{0,a(0)}(\boxed{j_{i+2}}(\xi)) = j_{a\cup\{l\}}(j_{i+2}(\xi)).
\end{aligned}$$

Since $\xi < \kappa_m$, we have $j_{i+2}(\xi) < \kappa_{m+1}$. Hence $j_a(\xi) = j_{a\cup\{l\}}(j_{i+2}(\xi)) \in I_{a\cup\{l\}}$.

3. Consider the case $l > a(m-1)$. By definition, every member of I_a has the form

$$j_a(\xi) = j_{a(m-2)+1,a(m-1)} \circ \cdots \circ j_{a(0)+1,a(1)} \circ j_{0,a(0)}(\xi)$$

for some $\xi < \kappa_m$. Then

- $j_{a(0),0}(\xi) < \kappa_{m+a(0)}$,
- $j_{a(0)+1,a(1)} \circ j_{0,a(0)} < j_{a(0)+1,a(1)}(\kappa_{m+a(0)}) = \kappa_{(m-1)+a(1)}$,

and so on. Repeating this type of argument gives $j_a(\xi) < \kappa_{1+a(m-1)}$. This implies

$$j_a(\xi) = j_{a(m-1)+1,l}(j_a(\xi)) = j_{a\cup\{l\}}(\xi) \in I_{a\cup\{l\}}.$$

It implies the first clause of the lemma, and we immediately have $I_{a\cap b} \subseteq I_a \cap I_b$. For the remaining direction, it suffices to show the following: Suppose that l is the largest element of $(a \setminus b) \cup (b \setminus a)$, and assume that $l \in a$. Then $I_a \cap I_b \subseteq I_{a\setminus\{l\}} \cap I_b$. Suppose that $a = a' \cup \{l\} \cup c$, $b = b' \cup c$ for some a', b', c such that $\max a', \max b' < l < \min c$. Also, assume that we are given $\xi < \kappa_{|a|}$, $\eta < \kappa_{|b|}$ such that $j_a(\xi) = j_b(\eta)$. Now let us divide the cases:

1. $c = \emptyset$: Then we have $j_{a' \cup \{l\}}(\xi) = j_{b'}(\eta) = j_b(\eta) < \kappa_{\max b+1}$. Also, we have $\max b + 1 \leq l$. Hence

$$j_{l-1} \circ j_{\max a'+1,l-1} \circ j_{a'}(\xi) = j_{\max a'+1,l} \circ j_{a'}(\xi) < \kappa_l.$$

However, $\operatorname{ran} j_{l-1} \restriction \mathrm{Ord} \subseteq [0, \kappa_{l-1}) \cup [\kappa_l, \lambda)$, so we get

$$j_{l-1}(j_{\max a'+1,l-1} \circ j_{a'}(\xi)) < \kappa_{l-1}.$$

This is possible only when $j_{l-1}(j_{\max a'+1,l-1} \circ j_{a'}(\xi)) = j_{\max a'+1,l-1} \circ j_{a'}(\xi)$. By repeating the same argument sufficiently many times, we have

$$j_{\max a'+1,l} \circ j_{a'}(\xi) = j_{\max a'+1,l-1} \circ j_{a'}(\xi) = \cdots = j_{a'}(\xi),$$

so $j_{a' \cup \{l\}}(\xi) = j_{a'}(\xi) \in I_{a \setminus \{l\}} \cap I_b$.

2. $c \neq \emptyset$: By canceling the c-part of the elementary embeddings from $j_a(\xi) = j_b(\eta)$, we have

$$j_{\max a'+1,l} \circ j_{a'}(\xi) = j_l \circ j_{\max b'+1,l} \circ j_{b'}(\eta).$$

Hence $j_{\max a'+1,l} \circ j_{a'}(\xi) \in \operatorname{ran} j_l = \operatorname{ran}(j_{l-1} \cdot j_{l-1})$. We may view this equality as

$$j_{l-1}(j_{\max a'+1,l-1} \circ j_{a'}(\xi)) \in \operatorname{ran} j_{l-1}(j_{l-1}),$$

so $j_{\max a'+1,l-1} \circ j_{a'}(\xi) \in \operatorname{ran} j_{l-1}$. Repeating the same manipulation several times, we have $j_{a'}(\xi) \in \operatorname{ran} j_{\max a'+1}$. Now, let us prove that

$$j_{a'}(\xi) \in \operatorname{ran} j_{\max a'+1} \implies \xi \in \operatorname{ran} j_{|a'|}. \tag{4.2}$$

Let $m = |a'|$. Then

$$\begin{aligned} j_{a'}(\xi) &= j_{a'(m-2)+1,a'(m-1)}(j_{a' \setminus \{a'(m-1)\}}(\xi)) \\ &\in \operatorname{ran} j_{a'(m-1)+1} = \operatorname{ran}(j_{a'(m-1)-1} \cdot j_{a'(m-1)}). \end{aligned}$$

Thus

$$j_{a'(m-2)+1,a'(m-1)-1}(j_{a'\setminus\{a'(m-1)\}}(\xi)) \in \operatorname{ran} j_{a'(m-1)}.$$

By repeating the previous manipulation, we have

$$j_{a'\setminus\{a'(m-1)\}}(\xi) \in \operatorname{ran} j_{a'(m-2)+2}$$

Thus, the tedious repetition gives (4.2). Now let $m = |a'|$, $n = |c|$, $\xi = j_m(\xi')$, and

$$k = j_{c(n-2)+1,c(n-1)} \circ \cdots \circ j_{l,c(0)}.$$

Then

$$\begin{aligned}
j_a(\xi) &= k \circ j_{a'(m-1)+1,l} \circ j_{a'}(\xi) = k \circ j_{a'(m-1)+1,l} \circ j_{a'}(j_m(\xi')) \\
&= k \circ j_{a'(m-1)+1,l} \circ j_{a'(m-2)+1,a'(m-1)} \circ \cdots j_{0,a'(0)} \circ j_m(\xi') \\
&= k \circ j_{a'(m-1)+1,l} \circ j_{a'(m-2)+1,a'(m-1)} \circ \cdots \\
&\qquad \circ j_{a'(0)-1,a'(1)} \circ j_{a'(0)+m} \circ j_{0,a'(0)}(\xi') \\
&= k \circ j_{a'(m-1)+1,l} \circ j_{a'(m-2)+1,a'(m-1)} \circ \cdots \\
&\qquad \circ j_{a'(1)+(m-1)} \circ j_{a'(0)-1,a'(1)} \circ j_{0,a'(0)}(\xi') \\
&\qquad \vdots \\
&= k \circ j_{a'(m-1)+1,l} \circ j_{a'(m-1)+1} \circ j_{a'(m-2)+1,a'(m-1)} \\
&\qquad \circ \cdots \circ j_{a'(0)-1,a'(1)} \circ j_{0,a'(0)}(\xi') \\
&= k \circ j_l \circ j_{a'(m-1)+1,l} \circ j_{a'(m-2)+1,a'(m-1)} \circ \cdots \circ j_{a'(0)-1,a'(1)} \circ j_{0,a'(0)}(\xi') \\
&= j_{c(n-2)+1,c(n-1)} \circ \cdots \circ j_{l,c(0)} \circ j_l \circ j_{a'(m-1)+1,l} \circ j_{a'}(\xi') \\
&= j_{c(n-2)+1,c(n-1)} \circ \cdots \circ j_{a'(m-1)+1,c(0)} \circ j_{a'}(\xi') = j_{a'\cup c}(\xi').
\end{aligned}$$

Hence $j_a(\xi) = j_{a\setminus\{l\}}(\xi) \in I_{a\setminus\{l\}} \cap I_b$. □

Then the following claim is immediate:

Proposition 4.4.3. *For every $\xi < \lambda$, a finite subset $a \subseteq \omega$ satisfying $\xi \in I_a$ with the least cardinality uniquely exists. Furthermore, if $\xi < \kappa_n$, then the corresponding a is a subset of n.*

Proof. For the uniqueness, if $a \neq b$ satisfies $\xi \in I_a \cap I_b$, then $\xi \in I_{a\cap b}$, and $|a \cap b| < |a|, |b|$. Now let us prove the existence and the last claim. If $\xi < \kappa_n$, then $\xi \in I_n$ (Recall that $j_{\mathsf{Id}_n} = \mathsf{Id}_n$.) That is, we have

$$n \in \{a \in [\omega]^{<\omega} \mid \xi \in I_a\}.$$

Clearly, we can find a member of $\{a \in [\omega]^{<\omega} \mid \xi \in I_a\}$ of the least cardinality. □

Now, let us define the support function as follows:

Definition 4.4.4. For $\xi < \kappa_n$, $\mathrm{supp}_n(\xi)$ is the unique subset $a \subseteq n$ of the least cardinality such that $\xi \in I_a$. Alternatively, $\mathrm{supp}_n(\xi) = \bigcap\{a \subseteq n \mid \xi \in I_a\}$.

Lemma 4.4.5. supp_n *is a natural transformation from Ω^1_{M} to $[\cdot]^{<\omega}$: That is, for every $f : m \to n$ and $\xi \in \Omega^1_{\mathsf{M}}(m) = \kappa_m$,*

$$\mathrm{supp}_n(\Omega^1_{\mathsf{M}}(f)(\xi)) = f^{"}[\mathrm{supp}_m(\xi)].$$

Furthermore, supp *satisfies the support condition: That is, for every $f : m \to n$,*

$$\{\xi < \kappa_n \mid \mathrm{supp}_n(\xi) \subseteq \mathrm{ran}\, f\} \subseteq \mathrm{ran}\, \Omega^1_{\mathsf{M}}(f).$$

Proof. First, let us observe that for two increasing $f : m \to n$, $g : n \to k$,

$$I_{g\circ f} = \{j_{g\circ f}(\xi) \mid \xi < \kappa_m\} = \{j_g(j_f(\xi)) \mid \xi < \kappa_m\} = j_g[I_f].$$

Hence $I_{f[a]} = j_f[I_a]$. Now suppose that $\mathrm{supp}_m(\xi) = a$. Then $\xi \in I_a$ and so

$$j_f(\xi) \in I_{f[a]} \implies \mathrm{supp}_n\, j_f(\xi) \subseteq f[a].$$

If $\mathrm{supp}_n j_f(a) \neq f[a]$, then there is $a' \subsetneq a$ such that $j_f(\xi) \in I_{f[a']}$. It implies $\xi \in I_{a'}$, contradicting with that $\mathrm{supp}_m(\xi) = a$. For the support condition, $\mathrm{supp}_n(\xi) \subseteq \operatorname{ran} f$ implies there is $a \subseteq m$ such that $\mathrm{supp}_n(\xi) \subseteq f[a]$. Hence there is $\eta < \kappa_m$ such that $\xi = j_{f[a]}(\eta) = j_f(j_a(\eta))$, so $\xi \in \operatorname{ran} j_f$. □

4.4.2 The Martin flower for ordinals

So far, we have defined Ω^1_{M} only for natural numbers. We want to define it to other ordinals, and the natural choice should be

$$\Omega^1_{\mathsf{M}}(\alpha) = \kappa_\alpha.$$

However, many parts of the definition of Ω^1_{M} for natural numbers do not work smoothly: For example, for $f : \alpha \to \beta$, the naive definition for $\Omega^1_{\mathsf{M}}(f)$ will introduce infinitely long embedding composition that is unclear to formulate. But we still need to define $\Omega^1_{\mathsf{M}}(f)$. To address this issue, we define the support of an ordinal first, then define $\Omega^1_{\mathsf{M}}(f)$.

Proposition 4.4.6. *Let $\alpha \geq \omega$. Then every element of κ_α has the form*

$$j_{\alpha_{n-2}+1,\alpha_{n-1}} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}(\xi) \tag{4.3}$$

for some $n < \omega$, $\xi < \kappa_n$, $\alpha_0 < \cdots < \alpha_{n-1} < \alpha$.

Proof. We prove the following by induction on α: If α is limit and for every $m < \omega$, every element of $\kappa_{\alpha+m}$ has the form (4.3) for some $n < \omega$, $\xi < \kappa_n$, $\alpha_0 < \cdots < \alpha_{n-1} < \alpha+m$.

The previous claim holds for $\alpha = 0$ trivially. For a general limit $\alpha > 0$, observe that every ordinal in M_α has the form $j_{\beta,\alpha}(\eta)$ for some limit $\beta < \alpha$ (including 0) and $\eta \in M_\beta$. Then

$$j_{\beta,\alpha}(\eta) < \kappa_{\alpha+m} = j_{\beta,\alpha}(\kappa_{\beta+m}) \implies \eta < \kappa_{\beta+m}.$$

Hence by the inductive hypothesis, there is $n < \omega$, $\xi < \kappa_n$ and $\alpha_0 < \cdots < \alpha_{n-1} < \beta+m$ such that $\eta = j_{\alpha_{n-2}+1,\alpha_{n-1}} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}(\xi)$. Now let l be the least natural number such that $\alpha_l \geq \beta$. Then for every $k \geq l$, α_k takes the form $\beta + e_k$ for some $e_k < \omega$. Hence

$$\begin{aligned}
j_{\beta,\alpha}(\eta) &= j_{\beta,\alpha}(j_{\alpha_{n-2}+1,\alpha_{n-1}} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}(\xi)) \\
&= (j_{\beta,\alpha} \circ j_{\beta+e_{k-2}+1,\beta+e_{k-1}} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0})(\xi) \\
&= (j_{\alpha+e_{k-2}+1,\alpha+e_{k-1}} \circ j_{\beta,\alpha} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0})(\xi) \\
&\qquad\qquad \vdots \\
&= (j_{\alpha+e_{k-2}+1,\alpha+e_{k-1}} \circ \cdots \circ j_{\alpha+e_l+1,\alpha+e_{l+1}} \circ j_{\beta,\alpha} \circ j_{\alpha_{l-1},\alpha_l} \circ \cdots \circ j_{0,\alpha_0})(\xi) \\
&= (j_{\alpha+e_{k-2}+1,\alpha+e_{k-1}} \circ \cdots \circ j_{\alpha+e_l+1,\alpha+e_{l+1}} \circ j_{\beta,\alpha} \circ j_{\beta,\beta+e_l} \circ j_{\alpha_{l-1},\beta} \circ \cdots \circ j_{0,\alpha_0})(\xi) \\
&= (j_{\alpha+e_{k-2}+1,\alpha+e_{k-1}} \circ \cdots \circ j_{\alpha+e_l+1,\alpha+e_{l+1}} \circ j_{\alpha,\alpha+e_l} \circ j_{\beta,\alpha} \circ j_{\alpha_{l-1},\beta} \circ \cdots \circ j_{0,\alpha_0})(\xi) \\
&= (j_{\alpha+e_{k-2}+1,\alpha+e_{k-1}} \circ \cdots \circ j_{\alpha+e_l+1,\alpha+e_{l+1}} \circ j_{\alpha_{l-1},\alpha+e_l} \circ \cdots \circ j_{0,\alpha_0})(\xi)
\end{aligned}$$

and $\alpha + e_{k-1} < \alpha + m$. This finishes the proof. □

Now, let us use the notation

$$j_{\{\alpha_0,\cdots,\alpha_{n-1}\}}(\xi) = j_{\alpha_{n-2}+1,\alpha_{n-1}} \circ \cdots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}(\xi).$$

Similar to the finite case, define

$$I_{\{\alpha_0,\cdots,\alpha_{n-1}\}} = \{j_{\{\alpha_0,\cdots,\alpha_{n-1}\}}(\xi) \mid \xi < \kappa_n\}.$$

Then we can see that the proof for Lemma 4.4.2 also works for general I_a, so we can define the support for ordinals in κ_α. We can also prove that if $\xi \in \kappa_\alpha$ has support a, then there is a unique $t_\xi < \kappa_{|a|}$ such that $\xi = j_a(t_\xi)$. Moreover, we have the following:

Lemma 4.4.7. *Let $\xi < \kappa_m$ be such that* $\mathrm{supp}(\xi) = m$. *For $\eta_0 < \cdots < \eta_{m-1} < \alpha$, we have* $\mathrm{supp}(j_{\{\eta_0,\cdots,\eta_{m-1}\}}(\xi)) = \{\eta_0, \cdots, \eta_{m-1}\}$ *and* $t_{j_{\{\eta_0,\cdots,\eta_{m-1}\}}(\xi)} = \xi$.

Proof. By the proof of Lemma 4.4.5 with $f(i) = \eta_i$ and $a = m$, we have $\operatorname{supp}(j_{\{\eta_0,\cdots,\eta_{m-1}\}}(\xi)) = \{\eta_0,\cdots,\eta_{m-1}\}$. The remaining equality follows from the definition of $t_{j_{\{\eta_0,\cdots,\eta_{m-1}\}}(\xi)}$. □

Now let us define $\Omega^1_{\mathsf{M}}(f)$ and show that Ω^1_{M} preserves function composition and satisfies the support condition:

Proposition 4.4.8. *Let $f : \alpha \to \beta$. Define $\Omega^1_{\mathsf{M}}(f)$ by*

$$\Omega^1_{\mathsf{M}}(f)(\xi) = j_{f[\operatorname{supp}\xi]}(t_\xi).$$

Then we have the following:

1. *$\Omega^1_{\mathsf{M}}(f) = j_f$ for $f : m \to n$, $m \leq n < \omega$.*
2. *$\Omega^1_{\mathsf{M}}(g \circ f) = \Omega^1_{\mathsf{M}}(g) \circ \Omega^1_{\mathsf{M}}(f)$.*
3. *$\operatorname{supp}(\Omega^1_{\mathsf{M}}(f)(\xi)) = f[\operatorname{supp}\xi]$.*

Proof.

1. Let $\xi < \kappa_m$ and $g : k \to \operatorname{supp}\xi$ be an increasing enumeration of the support of ξ. Then $\xi = j_g(t_\xi)$, and $\Omega^1_{\mathsf{M}}(f)(\xi) = j_{f\circ g}(t_\xi) = j_f(j_g(t_\xi)) = j_f(\xi)$ by Lemma 4.4.1.
2. Let $f : \alpha \to \beta$ and $g : \beta \to \gamma$. For $\xi < \kappa_\alpha$, we have $\Omega^1_{\mathsf{M}}(g\circ f)(\xi) = j_{(g\circ f)[\operatorname{supp}\xi]}(t_\xi)$ and
$$(\Omega^1_{\mathsf{M}}(g) \circ \Omega^1_{\mathsf{M}}(f))(\xi) = j_{g[\operatorname{supp} j_{f[\operatorname{supp}\xi]}(t_\xi)]}(t_{j_{f[\operatorname{supp}\xi]}(t_\xi)}).$$
Then by Lemma 4.4.7, we have that the right-hand side is equal to $j_{g[f[\operatorname{supp}\xi]]}(t_\xi)$.
3. Follows from Lemma 4.4.7. □

Furthermore, we can see that it is a preflower:

Proposition 4.4.9. *Ω^1_{M} is a preflower. In particular, if j is iterable, then Ω^1_{M} is a flower.*

Proof. Let $\alpha \leq \beta$ and $\iota\colon \alpha \to \beta$ be the insertion map (i.e., $\iota(\xi) = \xi$.) Then $\Omega^1_{\mathsf{M}}(\iota)(\xi) = j_{\mathrm{supp}\,\xi}(t_\xi) = \xi$, so $\Omega^1_{\mathsf{M}}(\iota)\colon \kappa_\alpha \to \kappa_\beta$ is also an insertion map. □

4.4.3 The universality of the Martin flower

In this section, we prove the Martin flower embeds every countable flower, whose proof is motivated by [48, 2.1(a)]. In fact, we have something stronger:

Proposition 4.4.10. *Ω^1_{M} is universal. In fact, if $F \in V_\kappa$ is a flower, then F embeds to Ω^1_{M}.*

Proof. Suppose that F is constant (i.e., every F-term is nullary.) Since $\Omega^1_{\mathsf{M}}(0) = \kappa_0$ and every constant dilator in $V_\kappa = V_{\kappa_0}$ has ordertype $< \kappa_0$, we have the desired embeddability result.

Now suppose that F is nonconstant (i.e., there is a non-nullary term in F). Since $F \in V_\kappa$, $F(\alpha) \in V_\kappa$ for each $\alpha < \kappa$. Moreover, $F(\kappa) = \bigcup_{\alpha<\kappa} F(\alpha)$ and each $F(\alpha)$ is an initial segment of $F(\kappa)$. Hence the ordertype of $F(\kappa)$ is $\leq \kappa$. Since F is not nullary, we have that $F(\xi)$ is a *proper* initial segment of $F(\eta)$ for $\eta > \xi \geq \omega$. This shows $F(\kappa) \cong \kappa$.

Now let us fix $c_0\colon F(\kappa_0) \cong \kappa_0$, and define $c_n = j_{0,n}(c_0)\colon F(\kappa_n) \to \kappa_n$. Then define $\iota_n\colon F(n) \to \Omega^1_{\mathsf{M}}(n)$ by

$$\iota_n(t(e_0, \cdots, e_{l-1})) = c_n(t(\kappa_{e_0}, \cdots, \kappa_{e_{l-1}})),$$

where $t \in \mathrm{field}(F)$ has arity l and $e_0 < \cdots < e_{l-1} < n$. We first claim that $\iota\colon F \restriction \mathbb{N} \to$

$\Omega^1_{\mathsf{M}} \restriction \mathbb{N}$ gives a natural transformation: For an increasing map $f : m \to n$, we have

$$\begin{aligned}
j_f(c_m) = j_f(j_{0,m}(c_0)) &= j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)} \circ j_{0,m}(c_0) \\
&= j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{f(0),f(0)+m} \circ j_{0,f(0)}(c_0) \\
&= j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{f(0)+1,f(0)+m} \circ j_{0,f(0)+1}(c_0) \\
&= j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(1),f(1)+m-1} \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)+1}(c_0) \\
&= j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(1)+1,f(1)+m-1} \circ j_{f(0)+1,f(1)+1} \circ j_{0,f(0)+1}(c_0) \\
&\vdots \\
&= j_{f(m-1),f(m-1)+1} \circ j_{f(m-2)+1,f(m-1)} \circ \cdots \circ j_{f(0)+1,f(1)} \circ j_{0,f(0)+1}(c_0) \\
&= j_{0,f(m-1)+1}(c_0) = c_{f(m-1)+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Omega^1_{\mathsf{M}}(f)(\iota_m(t(e_0, \cdots, e_{l-1}))) &= j_f(c_m(t(\kappa_{e_0}, \cdots, \kappa_{e_{l-1}}))) \\
&= c_{f(m-1)+1}(t(\kappa_{f(e_0)}, \cdots, \kappa_{f(e_{l-1})})). \quad (4.4)
\end{aligned}$$

Here t is fixed by j_f since $F \in V_\kappa$. Also,

$$\iota_n(\Omega^1_{\mathsf{M}}(f)(t(e_0, \cdots, e_{l-1}))) = \iota_n(t(f(e_0), \cdots, f(e_{l-1}))) = c_n(t(\kappa_{f(e_0)}, \cdots, \kappa_{f(e_{l-1})})). \quad (4.5)$$

But observe that for $m < n$ and $x \in F(\kappa_m) \subseteq V_{\kappa_m}$, $c_m(x) = j_{m,n}(c_m(x)) = j_{m,n}(c_m)(j_{m,n}(x)) = c_n(x)$. Hence, the right-hand side of (4.4) and that of (4.5) are the same.

We finalize the proof by showing that the transformation ι also preserves the support transformation. By letting $l = m$ and $e_i = i$ in the previous proof, we have

$$\begin{aligned}
\iota_n(t(f(0), \cdots, f(m-1))) &= c_{f(m-1)+1}(t(\kappa_{f(0)}, \cdots, \kappa_{f(m-1)})) \\
&= j_f(c_m(t(\kappa_0, \cdots, \kappa_{m-1}))).
\end{aligned}$$

By Lemma 4.4.5, it suffices to show that

$$\mathrm{supp}_m(c_m(t(\kappa_0, \cdots, \kappa_{m-1}))) = m.$$

$c_m(t(\kappa_0, \cdots, \kappa_{m-1})) < \kappa_m$ implies $c_m(t(\kappa_0, \cdots, \kappa_{m-1})) \in I_{\{0,1,\cdots,m-1\}}$. Now suppose that $c_m(t(\kappa_0, \cdots, \kappa_{m-1})) \in I_{m\setminus\{e\}}$ for some $e < m$, so there is $x < \kappa_{m-1}$ such that

$$c_m(t(\kappa_0, \cdots, \kappa_{m-1})) = j_e(x).$$

Then we have $t(\kappa_0, \cdots, \kappa_{m-1}) = c_m^{-1}(j_e(x)) = j_e(c_{m-1}^{-1}(x))$, so $t(\kappa_0, \cdots, \kappa_{m-1}) \in \operatorname{ran} j_e$. However, we defined $t(\kappa_0, \cdots, \kappa_{m-1})$ as a pair $(t, \{\kappa_0, \cdots, \kappa_{m-1}\})$, so we have $\kappa_e \in \operatorname{ran} j_e$, a contradiction. □

Remark 4.4.11. In the proof of Proposition 4.4.10, observe that for each $t \in \mathrm{field}(F)$ of arity n, $\iota_n(t(0, 1, \cdots, n-1)) = c_n(t(\kappa_0, \cdots, \kappa_{n-1}))$. $t(\kappa_0, \cdots, \kappa_{n-1})$ over $F(\kappa_n)$ is a limit ordinal since all of $\kappa_0, \cdots, \kappa_{n-1}$ are limit. Hence ι maps every F-term to a limit ordinal, so we have not only an embedding $\iota\colon F \to \Omega^1_{\mathsf{M}}$, but also that $\operatorname{ran}\iota$ is a set of limit ordinals, which we will call a *limit embedding*.

4.5 The measurability of the Martin Flower

The main goal of this section is to construct a measure family for the Martin flower to establish its measurability, which is the heart of this chapter. Martin's proof of determinacy [58] hints at how to construct a measure family, but deciphering the precise construction step is not easy: Martin defined a dependent product of measures, and the product is done along a tree. However, dilators and flowers themselves appear not to have a tree-like structure. Here, we need dendrograms, and we define the measure by the dependent product of measures along a dendrogram tree. We also need to specify how to traverse

a dendrogram to define a product, and here is where we cast trekkable dendrograms (cf. Definition 4.2.9).

Throughout the remaining part of the chapter, 'dendrogram' means a dendrogram for flowers with no nullary terms. In particular, every dendrogram we will see has a unique node of length 0, which is not terminal. Furthermore, every dendrogram we will consider is finite except in subsection 4.5.6, where we also consider countable dendrograms.

4.5.1 Construction of a measure family

We shall define ν^d for each trekkable dendrogram d, which is a measure over the set of embeddings from $\mathrm{Fun}(d)$ to Ω^1_{M}. The construction of a measure family will take the following steps:

1. For a trekkable dendrogram d and an embedding $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$, we will define the subsidiary space $D^{d,\beta}$, which is a set of tuples of elementary embeddings, and a measure $\hat{\nu}^{d,\beta}$ over $D^{d,\beta}$.
2. We define $\nu^{d,\beta}$ by projecting $\hat{\nu}^{d,\beta}$.
3. We will show that $\nu^{d,\beta}$ does not depend on the choice of β. We also show that $\nu^{d,\beta}$ gives the same measure for an isomorphic d.

$\hat{\nu}^{d,\beta}$ will be a dependent product of measures over d. Trees are not linear orders, and this is why we use trekkable dendrograms instead of arbitrary dendrograms: Trekkable dendrograms provide a way to traverse a given dendrogram when we take a product. $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ associates each node in d an Ω^1_{M}-term, so we will think of $\beta(x^\bullet)$ an Ω^1_{M}-term associated with $x \in d$. For that reason, we write $\beta(s^\bullet)$ as $\beta(s)$.

Definition 4.5.1. Let d be a trekkable dendrogram and $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$. For $s \in d$, let us define $\hat{\nu}^{d,\beta}_s$ and the corresponding domain $D^{d,\beta}_s$ as follows:

1. $D^{d,\beta}_0 = \{0\}$ and $\hat{\nu}^{d,\beta}_0$ is the trivial measure.
2. If $\mathrm{lh}\, s = 1$, define
   - $D^{d,\beta}_s = \{\vec{k} \cup \{(s,k')\} \mid \vec{k} \in D^{d,\beta}_{s-1} \wedge k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\}$.
   - $X \in \hat{\nu}^{d,\beta}_\sigma \iff \left\{\vec{k} \in D^{d,\beta}_{s-1} \mid \left\{k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)} \mid \vec{k} \cup \{(s,k')\} \in X\right\} \in \mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\right\} \in \hat{\nu}^{d,\beta}_{s-1}$.
3. Suppose that $\mathrm{lh}(s) > 1$. Then we have some $t \multimap s$ with $a = \mathbf{e}(t)$. Define
   - $D^{d,\beta}_s = \{\vec{k} \cup \{(s,k')\} \mid \vec{k} \in D^{d,\beta}_{s-1} \wedge k' \in \mathsf{Emb}^{j_a(\vec{k}_t)}_{\beta(s)}\}$.
   - $X \in \hat{\nu}^{d,\beta}_s \iff \left\{\vec{k} \in D^{d,\beta}_{s-1} \mid \left\{k' \in \mathsf{Emb}^{j_a(\vec{k}_t)}_{\beta(s)} \mid \vec{k} \cup \{(s,k')\} \in X\right\} \in \mu^{j_a(\vec{k}_t)}_{\beta(s)}\right\} \in \hat{\nu}^{d,\beta}_{s-1}$.

Here $\vec{k}_s = \vec{k}(s)$, so $\vec{k}_s$ is the sth component of $\vec{k}$. Then define $D^{d,\beta} = D^{d,\beta}_{|d|-1}$ and $\hat{\nu}^{d,\beta} = \hat{\nu}^{d,\beta}_{|d|-1}$.

We can easily see that $\hat{\nu}^{d,\beta}$ is a $(\min \beta)$-complete measure over $D^{d,\beta}$. In particular, $\min \beta \geq \kappa_0$ since there is no nullary term in d, so $\hat{\nu}^{d,\beta}$ is κ_0-complete.

To improve the readability of the following proofs, we use the measure quantifier notation introduced in Definition 4.1.15; For example, we can express the definition of $\hat{\nu}^{d,\beta}_s$ for a successor s of $t > 0$ as follows:

$$X \in \hat{\nu}^{d,\beta}_s \iff \forall (\hat{\nu}^{d,\beta}_{s-1})\vec{k} \in D^{d,\beta}_{s-1} \forall (\mu^{j_a(\vec{k}_t)}_{\beta(s)})k' \in \mathsf{Emb}^{j_a(\vec{k}_t)}_{\beta(s)}[\vec{k} \cup \{(s,k')\} \in X].$$

Then we have

$$\forall (\hat{\nu}^{d,\beta})\vec{k}\phi(\vec{k}) \iff \forall (\hat{\mu}^{d,\beta}_1)k^1 \forall (\hat{\mu}^{d,\beta}_2)k^2 \cdots \forall (\hat{\mu}^{d,\beta}_{m-1})k^{m-1}\phi(k^1, \cdots, k^{m-1}), \quad (4.6)$$

where $\hat{\mu}_s^{d,\beta}$ is a unit measure appearing when we define $\hat{\nu}_s^{d,\beta}$, so

$$\hat{\mu}_s^{d,\beta} = \begin{cases} \mu_{\beta(s)}^{j_1 \restriction V_{\kappa_1+\beta(s)}}, & \mathrm{lh}\, s = 1, \\ \mu_{\beta(s)}^{j_a(\vec{k}_t)}, & \mathrm{lh}\, s > 1,\ d \vDash t \multimap s, \text{ and } a = \mathbf{e}^d(t). \end{cases} \tag{4.7}$$

Going back to the construction of the measure family of Martin dilator, elements of Ω_{M}^1 are ordinals and not elementary embeddings, and elements of $D^{d,\beta}$ are tuples indexed by non-zero members of d that can be non-terminal nodes. Hence, $\hat{\nu}^{d,\beta}$ cannot serve as a desired measure. We can define a 'correct' measure by projecting $\hat{\nu}^{d,\beta}$:

Definition 4.5.2. Let us define a measure $\nu^{d,\beta}$ over $(\Omega_{\mathsf{M}}^1)^{\mathrm{Fun}(d)}$ as follows:

$$X \in \nu^{d,\beta} \iff \{\vec{k} \in D^{d,\beta} \mid \{(s, \mathrm{crit}\, \vec{k}_s) \mid s \in \mathrm{term}(d)\} \in X\} \in \hat{\nu}^{d,\beta}.$$

$\nu^{d,\beta}$ is a projection of $\hat{\nu}^{d,\beta}$, so is a κ_0-complete ultrafilter. We will see later that $\nu^{d,\beta}$ does not depend on β, and only depends on the isomorphic type of d. Hence, we can write $\nu^{d,\beta}$ as ν^d and regard it as a measure over the set of embeddings from a finite dilator d to Ω_{M}^1.

4.5.2 The correct concentration of the measure family

In this subsection, we prove that ν^d is a measure over the set $(\Omega_{\mathsf{M}}^1)^d$. The following theorem is a major intermediate step to prove that ν^d concentrates on $(\Omega_{\mathsf{M}}^1)^d$:

Theorem 4.5.3. *$\hat{\nu}^{d,\beta}$ concentrates to the set*

$$\{\vec{k} \in D^{d,\beta} \mid s^\bullet \mapsto \mathrm{crit}\, \vec{k}_s \textit{ is an embedding from } \mathrm{Fun}(d^\bullet) \textit{ to } \Omega_{\mathsf{M}}^1\}.$$

Then let us prove Theorem 4.5.3. First, let us prove the following lemma, which will have a significant role in the latter arguments:

Lemma 4.5.4. *For $\hat{\nu}^{d,\beta}$-almost every $\vec{k} \in D^{d,\beta}$, for $s \in d$ and $n = \mathrm{lh}\, s$, if $d \vDash r \multimap s$ and $a = \mathbf{e}^d(r) < \max(1, n-1)$, x, then*

$$\sup_{\xi < \kappa_{n-1}} j_{a+1}(\xi) < \mathrm{crit}\, \vec{k}_s < \vec{k}_s(\mathrm{crit}\, \vec{k}_s) \leq \kappa_n.$$

If $n \geq 2$, then we additionally have

$$\mathrm{crit}\, \vec{k}_s < \vec{k}_s(\mathrm{crit}\, \vec{k}_s) = j_a(\mathrm{crit}\, \vec{k}_r) < \kappa_n.$$

In particular, for $\hat{\nu}^{d,\beta}$-almost every $\vec{k}$, we have

1. $\kappa_{n-1} < \mathrm{crit}\, \vec{k}_s < \kappa_n$,
2. $j_{a+1}(\mathrm{crit}\, \vec{k}_r) < \mathrm{crit}\, \vec{k}_s$.

Proof. If $n = 1$, then for $\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}$-almost all $\vec{k}$ we have $\vec{k}_s \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}$. Clearly if $k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}$ then $\mathrm{crit}\, k' < \kappa_1$. Moreover, $\kappa_0 = \sup_{\xi<\kappa_0} j_1(\xi)$ and

$$\forall (\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}) k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)} [\mathrm{crit}\, k' > \kappa_0] \iff \kappa_1 = \mathrm{crit}\, j_1 > j_1(\kappa_0) = \kappa_0.$$

Hence for almost all $k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}$ we have $\sup_{\xi<\kappa_0} j_1(\xi) < \mathrm{crit}\, k'$.

If $d \vDash r \multimap s$ and $a = \mathbf{e}^d(r) < n-1$, then $\vec{k}_s \in \mathsf{Emb}^{j_a(\vec{k}_r)}_{\beta(s)}$ for $\hat{\nu}^{d,\beta}_s$-almost all $\vec{k} \in D^{d,\beta}$. Hence for $\hat{\nu}^{d,\beta}_s$-almost all $\vec{k}$,

$$\mathrm{crit}\, \vec{k}_s < \mathrm{crit}\, j_a(\vec{k}_r) = j_a(\mathrm{crit}\, \vec{k}_r) < j_a(\kappa_{n-1}) = \kappa_n.$$

For the lower bound, let us inductively assume that the lower bound inequality holds for

r. Observe that

$$\begin{aligned}
&\forall(\mu^{j_a(\vec{k}_r)}_{\beta(s)})k' \in \mathsf{Emb}^{j_a(\vec{k}_r)}_{\beta(s)} \left[\sup_{\xi<\kappa_{n-1}} j_{a+1}(\xi) < \operatorname{crit} k'\right] \\
\iff & j_a(\vec{k}_r)\left(\sup_{\xi<\kappa_{n-1}} j_{a+1}(\xi)\right) < j_a(\operatorname{crit}\vec{k}_r) \\
\iff & j_a(\vec{k}_r)\left(\sup_{\xi<j_a(\kappa_{n-2})} (j_a\cdot j_a)(\xi)\right) < j_a(\operatorname{crit}\vec{k}_r) \\
\iff & \vec{k}_r\left(\sup_{\xi<\kappa_{n-2}} j_a(\xi)\right) < \operatorname{crit}\vec{k}_r. \qquad (4.8)
\end{aligned}$$

We prove that the last inequality (4.8) holds for $\hat{\nu}^{d,\beta}_{s-1}$-almost all $\vec{k}$; Equivalently,

$$\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \cdots \left[\vec{k}_r\Big(\sup_{\xi<\kappa_{n-2}} j_a(\xi)\Big) < \operatorname{crit}\vec{k}_r\right],$$

where we have measure quantifiers for elementary embeddings indexed by t such that $r<t<s$. However, t-indexed elementary embeddings for $r<t<s$ do not appear in the bracketed formula, so we can remove them by Lemma 4.1.16. Thus what we prove is equivalent to the inequality (4.8) for $\hat{\nu}^{d,\beta}_r$-almost all $\vec{k} \in D^{d,\beta}_r$.

The inductive assumption on the lower bound for r implies $\kappa_{n-1} < \operatorname{crit}\vec{k}_r$ for $\hat{\nu}^{d,\beta}_r$-almost all $\vec{k} \in D^{d,\beta}_r$, so

$$\sup_{\xi<\kappa_{n-2}} j_a(\xi) < j_a(\kappa_{n-2}) = \kappa_{n-1} < \operatorname{crit}\vec{k}_r.$$

Hence we have for $\hat{\nu}^{d,\beta}_r$-almost all $\vec{k} \in D^{d,\beta}_r$,

$$\vec{k}_r\left(\sup_{\xi<\kappa_{n-2}} j_a(\xi)\right) = \sup_{\xi<\kappa_{n-2}} j_a(\xi) < \operatorname{crit}\vec{k}_r. \qquad \square$$

Lemma 4.5.5. *Let $s \in d$ and $\operatorname{lh} s = n \geq 1$. Then for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$,*

1. $\operatorname{crit}\vec{k}_s \notin \operatorname{ran} j_m$ *for* $m<n$.
2. $j_0(\operatorname{crit}\vec{k}_s) > j_1(\operatorname{crit}\vec{k}_s) > \cdots > j_{n-2}(\operatorname{crit}\vec{k}_s) > j_{n-1}(\operatorname{crit}\vec{k}_s) > j_n(\operatorname{crit}\vec{k}_s) = \operatorname{crit}\vec{k}_s$.

In particular, $\mathrm{supp}^{\Omega^1_{\mathsf{M}}}(\mathrm{crit}\,\vec{k}_s) = n$ *and* $(n, \mathrm{crit}\,\vec{k}_s) \in \mathrm{Tr}(\Omega^1_{\mathsf{M}})$.

Proof. 1. The case $\mathrm{lh}\,s = 1$ is easy since $\kappa_0 < \mathrm{crit}\,\vec{k}_s < \kappa_1$ for $\hat{\nu}^{d,\beta}$.a.e $\vec{k}$.

Now suppose that $d \vDash r \multimap s$ for some r and $a = \mathbf{e}^d(r) < n-1$. Note that $\mathrm{crit}\,\vec{k}_s < \kappa_n$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$ by Lemma 4.5.4, so for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$,

$$\mathrm{crit}\,\vec{k}_s \in \mathrm{ran}\,j_m \iff \mathrm{crit}\,\vec{k}_s \in \mathrm{ran}\,j_m \restriction \kappa_{n-1}.$$

Thus we claim $\mathrm{crit}\,\vec{k}_s \notin \mathrm{ran}\,j_m \restriction \kappa_{n-1}$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$, which is equivalent to

$$\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall\Big(\mu^{j_a(\vec{k}_r)}_{\beta(s)}\Big)k' \in \mathsf{Emb}^{j_a(\vec{k}_r)}_{\beta(s)}\Big[k' \notin \mathrm{ran}\,j_m \restriction \kappa_{n-1}\Big].$$

Moreover,

$$\forall\Big(\mu^{j_a(\vec{k}_r)}_{\beta(s)}\Big)k' \in \mathsf{Emb}^{j_a(\vec{k}_r)}_{\beta(s)}\Big[k' \notin \mathrm{ran}\,j_m \restriction \kappa_{n-1}\Big] \iff \mathrm{crit}\,j_a(\vec{k}_r) \notin \mathrm{ran}\Big((j_a(\vec{k}_r))(j_m \restriction \kappa_{n-1})\Big). \quad (4.9)$$

Also, let us observe that Lemma 4.5.4 implies the following for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$ (so also for $\hat{\nu}^{d,\beta}_r$-almost all $\vec{k} \in D^{d,\beta}_r$):

$$\mathrm{crit}\,j_a(\vec{k}_r) = j_a(\mathrm{crit}\,\vec{k}_r) > j_a(\kappa_{n-2}) = \kappa_{n-1} \geq \kappa_m,$$

so $j_a(\vec{k}_r)(\kappa_{n-1}) = \kappa_{n-1}$ and $j_a(\vec{k}_r)(\kappa_m) = \kappa_m$. Hence the right-hand-side of (4.9) is equivalent to

$$\mathrm{crit}\,j_a(\vec{k}_r) \notin \mathrm{ran}\Big((j_a(\vec{k}_r) \cdot j_m) \restriction \kappa_{n-1}\Big).$$

Here observe that if $\xi < \kappa_{n-1}$, then $(j_a(\vec{k}_r))(\xi) = \xi$, so

$$(j_a(\vec{k}_r) \cdot j_m)(\xi) = (j_a(\vec{k}_r) \cdot j_m)((j_a(\vec{k}_r)(\xi)) = (j_a(\vec{k}_r))(j_m(\xi)),$$

so $\mathrm{ran}\Big((j_a(\vec{k}_r) \cdot j_m) \restriction \kappa_{n-1}\Big) \subseteq \mathrm{ran}\,j_a(\vec{k}_r)$. However, for every elementary embedding k', $\mathrm{crit}\,k' \notin \mathrm{ran}\,k'$, so we have the right-hand side of (4.9).

2. Let $m < n-1$. The previous item and Lemma 4.1.12 applied to $\xi = \operatorname{crit} \vec{k}_s$ and $k = j_m \restriction V_{\kappa_n+\omega}$, $\delta = \kappa_n$ gives $j_{m+1}(\operatorname{crit} \vec{k}_s) < j_m(\operatorname{crit} \vec{k}_s)$. The last inequality follows from that $\operatorname{crit} \vec{k}_s < \kappa_n$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$. □

Lemma 4.5.6. *The following holds for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$: Let $s \in d$ and $d \vDash r \multimap s$, $\operatorname{lh} s > 1$, and $a = \mathbf{e}^d(r) < \operatorname{lh} r$. Then*

$$\forall \gamma < \operatorname{crit} \vec{k}_r (j_a(\gamma) < \operatorname{crit} \vec{k}_s).$$

Hence for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $\operatorname{crit} \vec{k}_r$ is the least ordinal γ such that $\operatorname{crit} \vec{k}_s \leq j_a(\gamma)$. Furthermore, we have

$$\sup_{\gamma < \operatorname{crit} \vec{k}_r} j_a(\gamma) < \operatorname{crit} \vec{k}_s.$$

Proof. Observe that

$$\begin{aligned}
&\forall (\hat{\nu}^{d,\beta}) \vec{k} \in D^{d,\beta} [\forall \gamma < \operatorname{crit} \vec{k}_r (j_a(\gamma) < \operatorname{crit} \vec{k}_s)] \\
\iff &\forall (\hat{\nu}^{d,\beta}_r) \vec{k} \in D^{d,\beta}_r \forall \left(\mu^{j_a(\vec{k}_r)}_{\beta(s)} \right) k' \in \mathsf{Emb}^{j_a(\vec{k}_r)}_{\beta(s)} \left[\forall \gamma < \operatorname{crit} \vec{k}_r [j_a(\gamma) < \operatorname{crit} k'] \right] \\
\iff &\forall (\hat{\nu}^{d,\beta}_r) \vec{k} \in D^{d,\beta}_r \left[\forall \gamma < j_a(\vec{k}_r)(\operatorname{crit} \vec{k}_r) [j_a(\vec{k}_r)(j_a \restriction V_{\vec{k}_r(\operatorname{crit} \vec{k}_r)})(\gamma) < \operatorname{crit}(j_a(\vec{k}_r))] \right].
\end{aligned}$$

Also, for $\hat{\nu}^{d,\beta}$-almost all $\vec{k} \in D^{d,\beta}$,

$$\operatorname{crit} j_a(\vec{k}_r) = j_a(\operatorname{crit} \vec{k}_r) > \operatorname{crit} \vec{k}_r$$

since $a \leq \operatorname{lh} r - 1$. Hence $\operatorname{crit} j_a \leq \kappa_{\operatorname{lh} r - 1} \leq \operatorname{crit} \vec{k}_r$ for $\hat{\nu}^{d,\beta}$-almost all almost all $\vec{k} \in D^{d,\beta}$ by Lemma 4.5.4. This implies

$$j_a(\vec{k}_r)(\operatorname{crit} \vec{k}_r) = \operatorname{crit} \vec{k}_r.$$

Hence

$$\begin{aligned}
&\forall \gamma < j_a(\vec{k}_r)(\operatorname{crit} \vec{k}_r)\Big[j_a(\vec{k}_r)(j_a \restriction V_{\vec{k}_r(\operatorname{crit} \vec{k}_r)})(\gamma) < \operatorname{crit}(j_a(\vec{k}_r))\Big] \\
\iff &\forall \gamma < \operatorname{crit} \vec{k}_r\Big[j_a(\vec{k}_r)(j_a \restriction V_{\vec{k}_r(\operatorname{crit} \vec{k}_r)})(\gamma) < \operatorname{crit}(j_a(\vec{k}_r))\Big] \\
\iff &\forall \gamma < \operatorname{crit} \vec{k}_r\Big[j_a(\vec{k}_r)(j_a(\gamma)) < \operatorname{crit}(j_a(\vec{k}_r))\Big] \\
\iff &\forall \gamma < \operatorname{crit} \vec{k}_r\Big[\vec{k}_r(\gamma) < \operatorname{crit}(\vec{k}_r)\Big].
\end{aligned}$$

The last condition clearly holds, which finishes the proof. The last inequality follows from that $\operatorname{crit} \vec{k}_r < \operatorname{crit} \vec{k}_s$ are inaccessible cardinals.[1] □

Now we prove that $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ is a dilator embedding from $\mathrm{Fun}(d^\bullet)$ to Ω^1_{M} for almost all $\vec{k}$:

Theorem 4.5.7. *Let $s, t \in d$, $s, t \neq 0$ and Д be an arity diagram between $s^\bullet$ and $t^\bullet$ in* $\mathrm{Fun}(d^\bullet)$. *Then for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$,*

$$\mathrm{Fun}(d^\bullet) \vDash s^\bullet <_{\text{Д}} t^\bullet \implies \Omega^1_{\mathsf{M}} \vDash \operatorname{crit} \vec{k}_s <_{\text{Д}} \operatorname{crit} \vec{k}_t.$$

By the Elementary comparison decomposition theorem Theorem 4.2.11, it suffices to show Theorem 4.5.7 for elementary comparison relations $<_{\text{Д}}$. The following implies for almost all $\vec{k}$, $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ preserves elementary comparison relations of type (A):

Proposition 4.5.8. *For $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ preserves elementary comparison relations of type (A).*

Proof. Suppose that $s, t \in d$, $d \vDash t \multimap s$, $\operatorname{lh} t = n > 0$ and $e = \mathbf{e}^d(t)$. There is only one elementary comparison relation Д between $s^\bullet$ and $t^\bullet$ of type (A), namely,

$$\mathrm{Fun}(d^\bullet) \vDash s^\bullet <_{\text{Д}} t^\bullet \iff \mathrm{Fun}(d^\bullet)(n+1) \vDash s^\bullet((n+1) \setminus \{e\}) < t^\bullet(n+1).$$

[1] Its proof does not require the axiom of choice since the critical point κ of an elementary embedding is inaccessible in the sense that V_κ is a model of second-order ZF, so it is closed under a limit of increasing ordinals below κ of length $< \kappa$.

Also, $\Omega^1_{\mathsf{M}} \vDash \operatorname{crit} \vec{k}_s <_{\text{Д}} \operatorname{crit} \vec{k}_t$ is equivalent to

$$j_{\{0,1,\cdots,n\}\setminus\{e\}}(\operatorname{crit} \vec{k}_s) < j_{\{0,1,\cdots,n\}}(\operatorname{crit} \vec{k}_t).$$

Hence it suffices to show the above inequality for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$: Observe that $j_{\{0,1,\cdots,n\}}$ is the identity and $j_{\{0,1,\cdots,n\}\setminus\{e\}} = j_e$. However, Lemma 4.5.4 implies $\operatorname{crit} \vec{k}_s < j_e(\operatorname{crit} \vec{k}_t)$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, as desired. □

Then let us turn to the elementary comparison relations of type (B).

Proposition 4.5.9. *For $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ preserves elementary comparison relations of type (B).*

Proof. Let $s, t, r \in d$ be such that $d \vDash r \multimap s, t$ and $d \vDash s \lessdot t$. The type (B) comparison relation is $<_{\text{Д}}$ for the trivial Д, so it suffices to show the following: For $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $\operatorname{crit} \vec{k}_s < \operatorname{crit} \vec{k}_t$. This follows from the following computation:

$$\begin{aligned}
&\forall (\hat{\nu}^{d,\beta})\vec{k} \in D^{d,\beta}[\operatorname{crit} \vec{k}_s < \operatorname{crit} \vec{k}_t] \\
\iff &\forall (\hat{\nu}^{d,\beta}_t)\vec{k} \in D^{d,\beta}_t[\operatorname{crit} \vec{k}_s < \operatorname{crit} \vec{k}_t] \\
\iff &\forall (\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall (\mu^{k'}_{\beta(s)}) k^0 \in \mathsf{Emb}^{k'}_{\beta(s)} \forall (\mu^{k'}_{\beta(t)}) k^1 \in \mathsf{Emb}^{k'}_{\beta(t)}[\operatorname{crit} k^0 < \operatorname{crit} k^1]. \\
\iff &\forall (\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall (\mu^{k'}_{\beta(s)}) k^0 \in \mathsf{Emb}^{k'}_{\beta(s)}[(k' \restriction V_{\operatorname{crit} k' + \beta(t)})(\operatorname{crit}(k^0)) < \operatorname{crit} k']. \\
\iff &\forall (\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r[(k'(k' \restriction V_{\operatorname{crit} k' + \beta(t)}))(\operatorname{crit} k') < k'(\operatorname{crit} k')]. \qquad (4.10)
\end{aligned}$$

Here

$$k' = \begin{cases} j_{\mathbf{e}^d(r)}(\vec{k}_r), & \text{if } \operatorname{lh} r \geq 1 \\ j_1 \restriction V_{\kappa_1 + \beta(t)} & \text{if } \operatorname{lh} r = 0 \end{cases}$$

We can see that (4.10) follows from Lemma 4.1.12 and $\operatorname{crit} k' \notin \operatorname{ran} k'$ for *every* elementary embedding k'. □

Proposition 4.5.10. *For $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ preserves elementary comparison relations of type (C).*

Proof. Let $s, t \in d$ be two members with a common immediate predecessor r, and $a = \mathbf{e}^d(r)$, $m = \mathrm{lh}\, s = \mathrm{lh}\, t$. The corresponding comparison relation is

$$d^\bullet(\omega) \vDash s^\bullet((m+1) \setminus \{a+1\}) < t^\bullet((m+1) \setminus \{a\}).$$

Hence we will prove the following for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$:

$$j_{(m+1)\setminus\{a+1\}}(\mathrm{crit}\, \vec{k}_s) < j_{(m+1)\setminus\{a\}}(\mathrm{crit}\, \vec{k}_t),$$

which is equivalent to

$$j_{a+1}(\mathrm{crit}\, \vec{k}_s) < j_a(\mathrm{crit}\, \vec{k}_t).$$

If $\mathrm{lh}\, r = 0$, then $a = 0$. By applying j_0 to $\forall \xi < \kappa_0(j_0(\xi) < \mathrm{crit}\, \vec{k}_t)$, we have

$$\forall \xi < \kappa_1[j_1(\xi) < j_0(\mathrm{crit}\, \vec{k}_t)],$$

and the desired result follows from $\mathrm{crit}\, \vec{k}_s < \kappa_1$. If $\mathrm{lh}\, r \geq 1$, then applying j_a to the inequality in Lemma 4.5.6 gives

$$\forall \gamma < j_a(\mathrm{crit}\, \vec{k}_r)[j_{a+1}(\gamma) < j_a(\mathrm{crit}\, \vec{k}_t)],$$

and the conclusion follows from $\mathrm{crit}\, \vec{k}_s < j_a(\mathrm{crit}\, \vec{k}_r)$. □

Proposition 4.5.11. *For $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$, $s^\bullet \mapsto \mathrm{crit}\, \vec{k}_s$ preserves elementary comparison relations of type (D).*

Proof. Let $r, s, t', t \in d$ be such that $d \vDash r \multimap s$, $d \vDash r \multimap t' \multimap t$ and $d \vDash s \lessdot t'$. We also write $a = \mathbf{e}^d(t')$, and $m = \mathrm{lh}\, s$ (so $m + 1 = \mathrm{lh}\, t$). Then the corresponding comparison relation is

$$d^\bullet(\omega) \vDash s^\bullet((m+1) \setminus \{a\}) < t^\bullet(m+1),$$

and we claim that $j_a(\operatorname{crit}\vec{k}_s) < \operatorname{crit}\vec{k}_t$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$. We can see that

$$\begin{aligned}
&\forall(\hat{\nu}^{d,\beta})\vec{k} \in D^{d,\beta}[j_a(\operatorname{crit}\vec{k}_s) < \operatorname{crit}\vec{k}_t] \\
\iff &\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall(\hat{\mu}^{d,\beta}_s)k^0 \forall(\hat{\mu}^{d,\beta}_{t'})k^1 \forall(\mu^{j_a(k^1)}_{\beta(t')})k^2[j_a(\operatorname{crit}k^0) < \operatorname{crit}k^2] \\
\iff &\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall(\hat{\mu}^{d,\beta}_s)k^0 \forall(\hat{\mu}^{d,\beta}_{t'})k^1[j_a(k^1)(j_a(\operatorname{crit}k^0)) < \operatorname{crit}j_a(k^1)] \\
\iff &\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall(\hat{\mu}^{d,\beta}_s)k^0 \forall(\hat{\mu}^{d,\beta}_{t'})k^1[j_a(\operatorname{crit}k^0) < j_a(\operatorname{crit}k^1)] \\
\iff &\forall(\hat{\nu}^{d,\beta}_r)\vec{k} \in D^{d,\beta}_r \forall(\hat{\mu}^{d,\beta}_s)k^0 \forall(\hat{\mu}^{d,\beta}_{t'})k^1[\operatorname{crit}k^0 < \operatorname{crit}k^1]
\end{aligned}$$

and the latter holds by $d^\bullet \vDash s \lessdot t'$ and Proposition 4.5.9. □

4.5.3 Independence of $\hat{\nu}$ from a trekkable order

We defined $\hat{\nu}^{d,\beta}$ for a specific dendrogram d, and we want to guarantee the final measure only depends on the isomorphism type of d. In this subsection, we prove that $\hat{\nu}^{d,\beta}$ and $\hat{\nu}^{d',\beta'}$ are the same if d and d' are isomorphic, and if $h\colon d \to d'$ is the isomorphism then $\beta' \circ h = \beta$.

We first discuss how to transform a given dendrogram into another isomorphic dendrogram. The following lemma says we can turn a dendrogram into another isomorphic one by successively exchanging s and $s+1$ in the dendrogram. We include its proof for completeness.

Lemma 4.5.12. *Say two trekkable dendrograms d and d' are* adjacent witnessed by m *if $m+1 < |d| = |d'|$ and the map $h\colon d \to d'$ switching m and $m+1$ and fixing the others is a dendrogam isomorphism. For two isomorphic trekkable dendrograms d and d', we have a sequence of trekkable dendrograms*

$$d = d_0 \cong d_1 \cong \cdots \cong d_m \cong d'$$

such that for each i, d_i and d_{i+1} are adjacent.

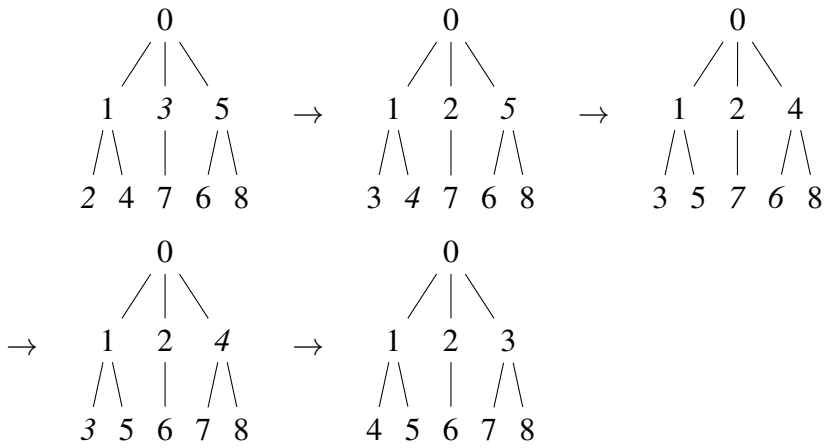


Figure 4.3: Sorting a trekkable dendrogram, with switched numbers italic. We assume $\mathbf{e}(s) = 0$ for every s in the example.

Proof. We claim that we can re-enumerate every trekkable dendrogram under the *level-then-value order*: For a dendrogram d and $s, t \in d$, we say $s <_{\mathsf{LV}} t$ if one of the following holds:

1. $\mathrm{lh}\, s < \mathrm{lh}\, t$, or
2. $\mathrm{lh}\, s = \mathrm{lh}\, t = m$ and $\mathrm{Fun}(d^\bullet)(\omega) \vDash s(m) < t(m)$.

We can see that $<_{\mathsf{LV}}$ is a linear order over d. Furthermore, we can see that if $d \vDash s \multimap t$ or $d \vDash s \lessdot t$, then $d \vDash s <_{\mathsf{LV}} t$. We say a trekkable dendrogram d is *aligned under the level-then-value order* if for $s, t \in d$, $s <_{\mathbb{N}} t$ if and only if $s <_{\mathsf{LV}} t$. Now let us consider the following algorithm: For a given trekkable dendrogram d, let us find the least m such that $m >_{\mathsf{LV}} m+1$. If there is such, let us obtain a new dendrogram d' by swapping m and $(m+1)$; That is, we make d' from d in a way that the map $f : d \to d'$ such that $f(m) = m+1$, $f(m+1) = m$, $f(t) = t$ for $t \neq m, m+1$ is a dendrogram isomorphism. We repeat this process until we get a trekkable dengrogram aligned under the level-then-value order. Figure 4.3 illustrates how it works.

We first claim that if d is a trekkable dendrogram, so is d'. Suppose that we have

$d' \vDash s \multimap t$. We have $d \vDash s \multimap t$ $s, t \notin \{m, m+1\}$, so $s <_{\mathbb{N}} t$. Now suppose that one of s or t is in $\{m, m+1\}$. Note that both s and t cannot be in $\{m, m+1\}$: Otherwise we have either $d \vDash m \multimap m+1$ or $d \vDash m+1 \multimap m$. The first possibility implies $m <_{\mathsf{LV}} m+1$, which contradicts the choice of m. The second possibility is impossible since d is trekkable.

Hence, we have four possible cases: $d' \vDash m \multimap t$, $d' \vDash (m+1) \multimap t$, $d' \vDash s \multimap m$, $d' \vDash s \multimap (m+1)$ with $s, t \notin \{m, m+1\}$. Each cases with the trekkability of d imply $m+1 <_{\mathbb{N}} t$, $m <_{\mathbb{N}} t$, $s <_{\mathbb{N}} m+1$, $s <_{\mathbb{N}} m$ respectively. The first and the fourth imply $m <_{\mathbb{N}} t$ and $s <_{\mathbb{N}} m+1$ respectively. Since $s, t \notin \{m, m+1\}$, the second and the third also imply $m+1 <_{\mathbb{N}} t$, $s <_{\mathbb{N}} m$ respectively. This shows half of the trekkability of d'. By a similar argument, one can show that $d' \vDash s \lessdot t$ implies $s <_{\mathbb{N}} t$, so d' is trekkable.

We finish the proof by showing that our algorithm terminates: For a trekkable dendrogram d, let

$$B(d) = \{(s,t) \in d^2 \mid s <_{\mathbb{N}} t \wedge s >_{\mathsf{LV}} t\}.$$

We claim that $|B(d)| = 1 + |B(d')|$: Observe that there are six types of elements in $B(d)$, namely, $(m, m+1)$, or (s, m) or $(s, m+1)$ for $s <_{\mathbb{N}} m$, (m, t) or $(m+1, t)$ for $t >_{\mathbb{N}} m+1$, or (s, t) for $s <_{\mathbb{N}} t$ and $s, t \notin \{m, m+1\}$. In all cases, the first component is $<_{\mathsf{LV}}$-larger than the second component. The isomorphism from d to d' preserves $<_{\mathsf{LV}}$, and the isomorphism maps each tuples into $(m+1, m)$, $(s, m+1)$, (s, m), $(m+1, t)$, (m, t), or (s, t) respectively. We can see that the second component is $<_{\mathbb{N}}$-larger for all types of tuples except the first. However, the first component is $<_{\mathsf{LV}}$-larger in all types of tuples. This shows our algorithm terminates. □

Note that if d and d' are adjacent dendrograms witnessed by l, then we must have $l \geq 2$. The previous algorithm turns a trekkable dendrogram d into another isomorphic trekkable dendrogram d' by turning d into the trekkable dendrogram d'' that is aligned

under the breadth-first search order, then turning d'' into d'. The following proposition follows from examining the proof of Lemma 4.5.12, which we record for a later purpose.

Proposition 4.5.13. *Let d be a trekkable dendrogram. Then we can find a sequence of trekkable dendrograms*

$$d \cong d_0 \cong d_1 \cong \cdots \cong d_m$$

such that for each $k < m$, d_k and d_{k+1} are adjacent, and d_m is aligned under the level-then-value order. Furthermore, if d_k and d_{k+1} are adjacent witnessed by l, then l is the least number such that $d \vDash l >_{\mathsf{LV}} (l+1)$. □

The next lemma says that the level-then-value order respects the size of the critical points:

Lemma 4.5.14. *Let d be a trekkable dendrogram and $s, t \in d$. If $s <_{\mathsf{LV}} t$, then* $\operatorname{crit} \vec{k}_s < \operatorname{crit} \vec{k}_t$ *for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$.*

Proof. If $\operatorname{lh} s < \operatorname{lh} t$, then $\operatorname{crit} \vec{k}_s < \kappa_{\operatorname{lh} s} \leq \kappa_{\operatorname{lh} t - 1} \leq \operatorname{crit} \vec{k}_t$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$ by Lemma 4.5.4. Now suppose that $\operatorname{lh} s = \operatorname{lh} t = m$. Then $\operatorname{Fun}(d^\bullet) \vDash s^\bullet(m) < t^\bullet(m)$, so $\operatorname{crit} \vec{k}_s < \operatorname{crit} \vec{k}_t$ for $\hat{\nu}^{d,\beta}$-almost all $\vec{k}$ by Theorem 4.5.7. □

The following theorem says for two adjacent dendrograms d and d', $\hat{\nu}^{d,\beta}$ and $\hat{\nu}^{d',\beta}$ are the same modulo permuting components. For a technical reason in the proof, *we will assume in the rest of the chapter that β is a limit embedding*, that is, $\beta(\sigma)$ is always a limit ordinal for every $s \in d$. See Remark 4.4.11 for a limit embedding.

Theorem 4.5.15. *Let d and d' be adjacent dendrograms witnessed by l, and suppose that l is the least number such that $d \vDash l >_{\mathsf{LV}} (l+1)$. We also fix a limit embedding $\beta \colon \operatorname{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ and the isomorphism $h_l \colon d \to d'$ switching l and $(l+1)$.*[2] *For*

[2] In particular, we have that $h_{l+1} \circ h_l$ is the identity.

$X \subseteq D^{d,\beta}$, we have

$$X \in \hat{\nu}^{d,\beta} \iff h_l^*[X] \in \hat{\nu}^{d',\beta\circ h_l},$$

where $h_l^*[X] = \{p \circ h_l \mid p \in X\}$.

Proof. Let $S = |d|$. Then

$$\forall(\hat{\nu}^{d,\beta})\vec{k}\phi(\vec{k}) \iff \forall(\hat{\mu}_0^{d,\beta})k^0\forall(\hat{\mu}_1^{d,\beta})k^1 \cdots \forall(\hat{\mu}_{S-1}^{d,\beta})k^{S-1}\phi(k^0,\cdots,k^{S-1}), \quad (4.11)$$

where $\hat{\mu}_s^{d,\beta}$ is as given in (4.7). For $m < S$, let us define $X \mid^d s \subseteq D_s^{d,\beta}$ by

$$\vec{k} \in X \mid^d s \iff \forall(\hat{\mu}_{s+1}^{d,\beta})\hat{k}^{s+1} \cdots \forall(\hat{\mu}_{S-1}^{d,\beta})\hat{k}^{S-1}\Big[\vec{k} \cup \{(t,\hat{k}^t) \mid s < t < S\} \in X\Big] \quad (4.12)$$

Note that $\hat{\mu}_t^{d,\beta}$ may depend on some of $k^0,\cdots,k^s$. We have

$$X \in \hat{\nu}^{d,\beta} \iff \forall(\hat{\mu}_0^{d,\beta})k^0 \cdots \forall(\hat{\mu}_s^{d,\beta})k^s\Big[\{(t,k^t) \mid t \le s\} \in X \mid^d s\Big].$$

We assume that l is the least number such that $d \vDash l >_{\mathsf{LV}} l+1$. Note that $\mathrm{lh}^d(l) > 1$, otherwise, both l and $(l+1)$ have 0 as a common immediate predecessor in d, so l cannot witness d and d' are adjacent. Moreover, for $s = 1,2,\cdots,l-1,l+1$, $\mathrm{lh}^d(s) \le \mathrm{lh}^d(l)$.

Now let $l' = l \restriction (\mathrm{lh}^d(l) - 1)$ and $a = \mathbf{e}^d(l')$. By the assumption on l, we have $\mathrm{lh}^d(s) \le \mathrm{lh}^d(l)$ for every $s <_{\mathbb{N}} l$. Then we are tempted to argue

$$\begin{aligned} X \in \hat{\nu}^{d,\beta} &\iff \forall(\hat{\mu}_0^{d,\beta})k^0 \cdots \forall(\hat{\mu}_{l-1}^{d,\beta})k^{l-1}\forall(\hat{\mu}_l^{d,\beta})k^l\forall(\hat{\mu}_{l+1}^{d,\beta})k^{l+1} \\ &\qquad \Big[\{(s,k^s) \mid s \le l+1\} \in X \mid^d (l+1)\Big] \\ &\iff \forall(\hat{\mu}_0^{d,\beta})k^0 \cdots \forall(\hat{\mu}_{l-1}^{d,\beta})k^{l-1}\forall\Big(j_a(k^{l'})(\hat{\mu}_{l+1}^{d,\beta})\Big)k^{l+1} \in j_a(k^{l'})(\mathrm{dom}_{l+1}^{d,\beta}) \\ &\qquad \Big[\Big\{(s,j_a(k^{l'})(k^s)) \mid s < l\Big\} \cup \Big\{(l,j_a(k^{l'})),(l+1,k^{l+1})\Big\} \\ &\qquad\qquad \in j_a(k^{l'})(X \mid^d (l+1))\Big], \end{aligned} \quad (4.13)$$

where

$$\mathrm{dom}_s^{d,\beta} = \mathrm{dom}_s^{d,\beta}(k^0, \cdots, k^{l-1})$$
$$= \begin{cases} \mathsf{Emb}_{\beta(s)}^{j_1 \restriction V_{\kappa_1+\beta(s)}} & \mathrm{lh}^d(s) = 1, \\ \mathsf{Emb}_{\beta(s)}^{j_a(k^{s'})} & \mathrm{lh}^d(s) > 1,\ d \vDash s' \multimap s,\ \mathbf{e}^d(s') = a. \end{cases}$$

However, we need to check

$$\forall(\hat{\mu}_0^{d,\beta})k^0 \cdots \forall(\hat{\mu}_{l-1}^{d,\beta})k^{l-1}\forall(\hat{\mu}_{l+1}^{d,\beta})k^{l+1}$$
$$\left[\mathrm{dom}_{l+1}^{d,\beta}, \hat{\mu}_{l+1}^{d,\beta}, k^0, \cdots, k^{l-1}, X \mid^d (l+1) \in \mathrm{dom}\, j_a(k^{l'})\right]$$

to ensure the equivalence (4.13) works, otherwise, we do not know if we can apply $j_a(k^{l'})$ to the sets above. From now on, let us omit the expression 'almost all,' which should be clear from context. Also, we fix the immediate predecessor l' of l, and $a = \mathbf{e}^d(l')$.

Lemma 4.5.16. *We have* $\mathrm{crit}\, k^m < \mathrm{crit}\, k^l$ *for* $m = 0, 1, \cdots, l-1, l+1$.

Proof. By the assumption on l, we have $d \vDash 0, 1, \cdots, l-1, l+1 <_{\mathsf{LV}} l$. Hence, we have a desired result by Lemma 4.5.14. □

Let us recall that

$$\mathrm{dom}\, j_a(k^{l'}) = V_{j_a(\mathrm{crit}\, k^{l'})+j_a(\beta(l'))}.$$

In addition, for $s = 1, \cdots, N-1, N+1$ and almost all k^s, $k^s \in V_{k^s(\mathrm{crit}\, k^s)+\beta(s)+99}$ and

$$k^s(\mathrm{crit}\, k^s) = \begin{cases} \kappa_1 & \text{If } \mathrm{lh}\, s = 1, \\ j_b(\mathrm{crit}\, k^{s'}) & \text{If } \mathrm{lh}^d(s) > 1,\ d \vDash s' \multimap s,\ \mathbf{e}^d(s') = b. \end{cases}$$

Hence to see $k^s \in \mathrm{dom}\, j_a(k^{l'})$, it suffices to show:

Lemma 4.5.17. *For* $s = 1, \cdots, l-1, l+1$,

$$k^s(\mathrm{crit}\, k^s) + \beta(s) + 99 < j_a(\mathrm{crit}\, k^{l'}) + \beta(l). \tag{4.14}$$

Furthermore, if $\mathrm{lh}^d(s) < \mathrm{lh}^d(l)$*, then we have*

$$k^s(\mathrm{crit}\, k^s) + \beta(s) + 99 < j_a(\mathrm{crit}\, k^{l'}). \tag{4.15}$$

Proof. We write the immediate predecessor of s in d by s', and $b = \mathbf{e}^d(s')$. If $\mathrm{lh}^d(s) = 1$, then $b = 0$ and $\beta(s) < \kappa_1 < \mathrm{crit}\, k^l < j_a(\mathrm{crit}\, k^{l'})$. Then we have (4.15) since $j_a(\mathrm{crit}\, k^{l'})$ is inaccessible.

Now suppose that $\mathrm{lh}^d(s) > 1$. By the assumption on l, we have $d \vDash s <_{\mathsf{LV}} l$. If $\mathrm{lh}^d(s) < \mathrm{lh}^d(l)$, then we have

$$j_b(\mathrm{crit}\, k^{s'}),\ \beta(s) < \kappa_{\mathrm{lh}^d(s)} \leq \kappa_{\mathrm{lh}^d(l)-1} \leq j_a(\mathrm{crit}\, k^{l'}).$$

Then the inaccessibility of $j_a(\mathrm{crit}\, k^{l'})$ and the equality $j_b(\mathrm{crit}\, k^{s'}) = k^s(\mathrm{crit}\, k^s)$ implies (4.15). Otherwise, we have $\mathrm{lh}^d(s) = \mathrm{lh}^d(l)$ since $\mathrm{lh}^d(s) \leq \mathrm{lh}^d(l)$ holds. Following Lemma 4.2.12, we have two possible cases:

1. Suppose that s and l have the same immediate predecessor in d, i.e., $s' = l'$. Since $\beta(s) < \beta(l)$ and $\beta(l)$ is a limit ordinal, we have (4.14). (Note that $s = l + 1$ is impossible in this case; Otherwise, l and $(l + 1)$ have the common immediate predecessor in d.)

2. Otherwise, Lemma 4.2.12 implies $\mathrm{Fun}(d^\bullet) \vDash s'(\mathrm{lh}^d(s) \setminus \{b\}) < l(\mathrm{lh}^d(l))$. Thus for almost all $\vec{k}$, we have $j_b(\mathrm{crit}\, \vec{k}_{s'}) < \mathrm{crit}\, \vec{k}_l$. This shows the following holds for almost all $\vec{k}$:

   $$\vec{k}_s(\mathrm{crit}\, \vec{k}_s) = j_b(\mathrm{crit}\, \vec{k}_{s'}) < \mathrm{crit}\, \vec{k}_l < j_a(\mathrm{crit}\, \vec{k}_{l'}).$$

   Also, $d \vDash s <_{\mathsf{LV}} l$ implies $\beta(s) < \beta(l)$. Since $\beta(l)$ is limit, we have $\beta(s) + 99 < \beta(l)$. Combining all of this, we have (4.14) for almost all $\vec{k}$. (Note that by Lemma 4.5.21, we can also derive (4.15) in this case. However, we do not need this strengthened inequality in our purpose.) □

Lemma 4.5.18. *The rank of* $\mathrm{dom}^{d,\beta}_{l+1}$ *and* $\hat{\mu}^{d,\mu}_{l+1}$ *are less than* $j_a(\mathrm{crit}\, k^{l'})$*, so* $\mathrm{dom}^{d,\beta}_{l+1}, \hat{\mu}^{d,\beta}_{l+1} \in \mathrm{dom}\, j_a(k^{l'})$ *and they are fixed by* $j_a(k^{l'})$.

Proof. Let us divide the case:

1. Case $\mathrm{lh}^d(l+1) = 1$: Then the rank of $\mathrm{dom}^{d,\beta}_{l+1}$ and $\hat{\mu}^{d,\beta}_{l+1}$ are no more than $\kappa_1 + \beta(l+1) + 99$. Observe that $\mathrm{lh}^d(l) > \mathrm{lh}^d(l+1) \geq 1$, so $\mathrm{lh}^d(l) \geq 2$.

   If $\mathrm{lh}^d(l) > 2$, then $j_a(\mathrm{crit}\, k^{l'}) > \mathrm{crit}\, k^l \geq \kappa_2$, so we have $\beta(l+1) < \kappa_2 < j_a(\mathrm{crit}\, k^{l'})$, which implies $\kappa_1 + \beta(l+1) + 99 < j_a(\mathrm{crit}\, k^{l'})$.

   If $\mathrm{lh}^d(l) = 2$, then $a = 0$ and $j_a(\mathrm{crit}\, k^{l'}) > j_0(\kappa_0) = \kappa_1 \geq \beta(l+1)$, so again $\kappa_1 + \beta(l+1) + 99 < j_a(\mathrm{crit}\, k^{l'})$.

2. Case $\mathrm{lh}^d(l+1) > 1$: Suppose that $d \vDash (l+1)' \multimap (l+1)$ and $b = \mathbf{e}^d((l+1)')$. The rank of $\mathrm{dom}^{d,\beta}_{l+1}$ and $\hat{\mu}^{d,\beta}_{l+1}$ are no more than $j_b(\mathrm{crit}\, k^{(l+1)'}) + \beta(l+1) + 99$. Observe that $\mathrm{lh}^d((l+1)') < \mathrm{lh}^d(l')$, and Lemma 4.5.4 implies

   $$\mathrm{crit}\, k^{(l+1)'} < \kappa_{\mathrm{lh}^d(l+1)'} \quad \text{and} \quad \kappa_{\mathrm{lh}^d(l')} < \mathrm{crit}\, k^{l'}.$$

   Also, note that $b < \mathrm{lh}^d(l+1)'$ and $a < \mathrm{lh}^d(l')$, so we have

   $$j_b(\mathrm{crit}\, k^{(l+1)'}) < \kappa_{\mathrm{lh}^d(l+1)'} \leq \kappa_{\mathrm{lh}^d(l')} = j_a(\kappa_{\mathrm{lh}^d(l')-1}) < j_a(\mathrm{crit}\, k^{l'}).$$

   By Lemma 4.5.21, we have

   $$\beta(l+1) < \mathrm{crit}\, k^{l+1} < \kappa_{\mathrm{lh}^d(l+1)'} \leq \kappa_{\mathrm{lh}^d(l')} < \mathrm{crit}\, k^l < j_a(\mathrm{crit}\, k^{l'}).$$

   Then by the inaccessibility of $j_a(\mathrm{crit}\, k^{l'})$, we have

   $$j_b(\mathrm{crit}\, k^{(l+1)'}) + \beta(l+1) + 99 < j_a(\mathrm{crit}\, k^{l'}).$$

□

Lemma 4.5.19. $X \mid^d (l+1) \in \mathrm{dom}\, j_a(k^{l'})$.

Proof. It suffices to show that $D^{d,\beta}_{l+1}$ has rank less than $j_a(\operatorname{crit} k^{l'}) + \beta(l)$. $D^{d,\beta}_{l+1}$ is a set of tuples of elementary embeddings, and we proved in Lemma 4.5.17 that each component of a tuple has rank less than $j_a(\operatorname{crit} k^{l'}) + \beta(l)$. Hence the tuple also has rank less than $j_a(\operatorname{crit} k^{l'}) + \beta(l)$, so the rank of $D^{d,\beta}_{l+1}$ is also less than $j_a(\operatorname{crit} k^{l'}) + \beta(l)$. □

Hence (4.13) holds, and is equivalent to

$$\begin{aligned}
&\forall (\hat{\mu}^{d,\beta}_0) k^0 \dots \forall (\hat{\mu}^{d,\beta}_{l-1}) k^{l-1} \forall (j_a(k^{l'})(\hat{\mu}^{d,\beta}_{l+1})) k^{l+1} \in j_a(k^{l'})(\mathrm{dom}^{d,\beta}_{l+1}) \\
&\quad \Big[\Big\{ (s, j_a(k^{l'})(k^s)) \mid s < l \Big\} \cup \Big\{ (l, j_a(k^{l'})), (l+1, k^{l+1}) \Big\} \in j_a(k^{l'})(X \mid^d (l+1)) \Big] \\
\iff &\forall (\hat{\mu}^{d,\beta}_0) k^0 \dots \forall (\hat{\mu}^{d,\beta}_{l-1}) k^{l-1} \forall (\hat{\mu}^{d,\beta}_{l+1}) k^{l+1} \forall (\hat{\mu}^{d,\beta}_{l}) k^{l} \\
&\quad \Big[\{ (s, k^s) \mid s \leq l+1 \} \in X \mid^d (l+1) \Big] \\
\iff &\forall (\hat{\mu}^{d,\beta}_0) k^0 \dots \forall (\hat{\mu}^{d,\beta}_{l-1}) k^{l-1} \forall (\hat{\mu}^{d',\beta}_{l}) k^{l+1} \forall (\hat{\mu}^{d',\beta}_{l+1}) k^{l} \\
&\quad \Big[\{ (s, k^s) \mid s \leq l-1 \} \cup \{ (l, k^{l+1}), (l+1, k^l) \} \in h^*_l[X \mid^d (l+1)] \Big].
\end{aligned}$$

That is, we can switch the order between k^{l+1} and k^l. □

4.5.4 Independence of $\nu^{d,\beta}$ from β

In this subsection, we prove that $\nu^{d,\beta}$ does not depend on the choice of β.

Lemma 4.5.20. *For a finite flower d with no nullary terms and embeddings $\beta, \gamma \colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$, suppose that $\beta(s) \leq \gamma(s)$ for every $s \in d$.*

If we define $\pi^d_{\beta,\gamma} \colon D^{d,\gamma} \to D^{d,\beta}$ by

$$\pi^d_{\beta,\gamma}(\vec{k})(s) = \vec{k}_s \restriction V_{\operatorname{crit} \vec{k}_s + \beta(s)},$$

for $s \in \mathrm{Fun}(d^\bullet)$, then

$$\forall Y \subseteq D^{d,\gamma} \Big[Y \in \hat{\nu}^{d,\beta} \iff (\pi^d_{\beta,\gamma})^{-1}[Y] \in \hat{\nu}^{d,\gamma} \Big].$$

Proof. We prove it by induction on $s \in d$ as follows:

For each $s \in d$ define $\pi^{d}_{\beta,\gamma,s} \colon D^{d,\gamma}_{s} \to D^{d,\beta}_{s}$ by

$$\pi^{d}_{\beta,\gamma,s}(\vec{k}) = \{(t, \vec{k}_t \restriction V_{\mathrm{crit}\,\vec{k}_t+\beta(t)}) \mid t \in d,\ t \leq_{\mathbb{N}} s\}.$$

Then we prove the following:

$$\forall Y \subseteq D^{d,\gamma}_{s}[Y \in \hat{\nu}^{d,\beta}_{s} \iff (\pi^{d}_{\beta,\gamma,s})^{-1}[Y] \in \hat{\nu}^{d,\gamma}_{s}],$$

which is equivalent to

$$\forall Y \subseteq D^{d,\gamma}_{s}\left[\forall(\hat{\nu}^{d,\beta}_{s})\vec{k} \in D^{d,\beta}_{s}[\vec{k} \in Y] \iff \forall(\hat{\nu}^{d,\gamma}_{s})\vec{k} \in D^{d,\gamma}_{\sigma}[\pi^{d}_{\beta,\gamma,s}(\vec{k}) \in Y]\right].$$

If $\mathrm{lh}\, s = 1$, then the inductive hypothesis and Lemma 4.1.11 implies

$$\begin{aligned}
Y \in \hat{\nu}^{d,\beta}_{s} &\iff \forall\left(\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\right) k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)} \\
&\qquad \left[\forall(\hat{\nu}^{d,\beta}_{s-1})\vec{k} \in D^{d,\beta}_{s-1}[\vec{k} \cup \{(s,k')\} \in Y]\right] \\
(\text{by } 4.1.11) &\iff \forall\left(\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\right) k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\gamma(\sigma)}}_{\gamma(\sigma)} \\
&\qquad \left[\forall(\hat{\nu}^{d,\beta}_{s-1})\vec{k} \in D^{d,\beta}_{s-1}[\vec{k} \cup \{(s,k' \restriction V_{\kappa_1+\beta(s)})\} \in Y]\right] \\
(\text{Ind.}) &\iff \forall\left(\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\right) k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\gamma(\sigma)}}_{\gamma(\sigma)} \\
&\qquad \left[\forall(\hat{\nu}^{d,\gamma}_{s-1})\vec{k} \in D^{d,\gamma}_{s-1}[\pi^{d}_{\beta,\gamma,s-1}(\vec{k}) \cup \{(s,k' \restriction V_{\kappa_1+\beta(s)})\} \in Y]\right] \\
&\iff \forall\left(\mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}\right) k' \in \mathsf{Emb}^{j_1 \restriction V_{\kappa_1+\gamma(s)}}_{\gamma(s)} \\
&\qquad \left[\forall(\hat{\nu}^{d,\gamma}_{s-1})\vec{k} \in D^{d,\gamma}_{s-1}\left[\pi^{d}_{\beta,\gamma,s}(\vec{k} \cup \{(s,k')\}) \in Y\right]\right] \\
&\iff \forall(\hat{\nu}^{d,\gamma}_{s})\vec{k} \in D^{d,\gamma}_{s}[\pi^{d}_{\beta,\gamma,s}(\vec{k}) \in Y] \iff (\pi^{d}_{\beta,\gamma,s})^{-1}[Y] \in \hat{\nu}^{d,\gamma}_{s}.
\end{aligned}$$

The remaining case is similar, so we omit it. □

The following lemma is necessary to prove the next proposition, which roughly says the measure $\hat{\nu}^{d,\beta}$ sees an 'initial segment of certain conditions' as small.

Lemma 4.5.21. *Suppose that we are given a finite dendrogram d representing a flower without nullary terms and $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$. If we are given a sequence of ordinals $\alpha_s < \kappa_{\mathrm{lh}\, s}$ for each $s \in d$ satisfying the followng: For every $s \in d$, if $d \vDash s' \multimap s$ and $a = \mathbf{e}^d(s')$, then $\alpha_s \leq j_a(\alpha_{s'})$. Then*

$$\textstyle\prod_{s\in d}\{k^s \mid \mathrm{crit}\, k^s > \alpha_s\} \in \hat{\nu}^{d,\beta}.$$

Proof. We prove it by induction on $(d, <_{\mathbb{N}})$: That is, we prove for each $s \in d$,

$$\textstyle\prod_{t\leq_{\mathbb{N}} s}\{k^t \mid \mathrm{crit}\, k^t > \alpha_t\} \in \hat{\nu}^{d,\beta}_s. \tag{4.16}$$

Suppose that (4.16) holds for $t <_{\mathbb{N}} s$. Then

$$\begin{aligned}
&\textstyle\prod_{t\leq_{\mathbb{N}} s}\{k^t \mid \mathrm{crit}\, k^t > \alpha_t\} \in \hat{\nu}^{d,\beta}_s \\
\iff &\Big\{\vec{k} \in D^{d,\beta}_{s-1} \;\Big|\; \{k^s \in \mathsf{Emb}^{k'}_{\beta(s)} \mid \\
&\qquad\qquad \forall t \leq_{\mathbb{N}} s[\mathrm{crit}\, \vec{k}_t > \alpha_t] \wedge \mathrm{crit}\, k^s > \alpha_s\} \in \mu^{k'}_{\beta(s)}\Big\} \in \hat{\nu}^{d,\beta}_{s-1} \\
\iff &\Big\{\vec{k} \in D^{d,\beta}_{s-1} \;\Big|\; \forall t \leq_{\mathbb{N}} s[\mathrm{crit}\, \vec{k}_t > \alpha_t] \wedge \\
&\qquad\qquad \{k^s \in \mathsf{Emb}^{k'}_{\beta(s)} \mid \mathrm{crit}\, k^s > \alpha_s\} \in \mu^{k'}_{\beta(s)}\Big\} \in \hat{\nu}^{d,\beta}_{s-1}
\end{aligned} \tag{4.17}$$

Where $d \vDash s' \multimap s$, $a = \mathbf{e}^d(s')$, and

$$k' = \begin{cases} j_a(\vec{k}_{s'}), & \text{if } \mathrm{lh}^d\, s' \geq 2, \\ j_1 \restriction V_{\kappa_1+\beta(s)} & \text{if } \mathrm{lh}^d\, s' = 1. \end{cases}$$

If $\mathrm{lh}^d(s') = 1$, then $\{k^s \in \mathsf{Emb}^{j_1\restriction V_{\kappa_1+\beta(s)}}_{\beta(s)} \mid \mathrm{crit}\, k^s > \alpha_s\} \in \mu^{j_1\restriction V_{\kappa_1+\beta(s)}}_{\beta(s)}$ holds since $\alpha_s < \kappa_1$. If $\mathrm{lh}^d(s') \geq 2$, then

$$\{k^s \in \mathsf{Emb}^{j_a(\vec{k}_{s'})}_{\beta(s)} \mid \mathrm{crit}\, k^s > \alpha_s\} \in \mu^{j_a(\vec{k}_{s'})}_{\beta(s)} \iff j_a(\mathrm{crit}\, \vec{k}_{s'}) > j_a(\vec{k}_{s'})(\alpha_s),$$

and the latter inequality holds for $\hat{\nu}^{d,\beta}_{s-1}$-almost all $\vec{k}$ since we inductively assumed that $\forall t \leq_{\mathbb{N}} s-1[\mathrm{crit}\, \vec{k}_t > \alpha_\tau]$ holds for $\hat{\nu}^{d,\beta}_{s-1}$-almost all $\vec{k}$, and

$$j_a(\mathrm{crit}\, \vec{k}_{s'}) > j_a(\alpha_{s'}) = j_a(\vec{k}_{s'}(\alpha_{s'})) = j_a(\vec{k}_{s'})(j_a(\alpha_{s'})) \geq j_a(\vec{k}_{s'})(\alpha_s),$$

where the first equality holds since $\alpha_{s'} < \operatorname{crit} \vec{k}_{s'}$. □

Proposition 4.5.22. *$\nu^{d,\beta}$ does not depend on the choice of β.*

Proof. First, we claim that if $\beta(s) \leq \gamma(s)$ for every $s \in d$, then $X \in \nu^{d,\beta} \iff X \in \nu^{d,\gamma}$. By Lemma 4.5.20,

$$\begin{aligned} X \in \nu^{d,\beta} &\iff \left\{\vec{k} \in D^{d,\beta} \;\middle|\; \{(s, \operatorname{crit} \vec{k}_s) \mid s \in \operatorname{term}(d)\} \in X\right\} \in \hat{\nu}^{d,\beta} \\ &\iff (\pi^d_{\beta,\gamma})^{-1}\left\{\vec{k} \in D^{d,\beta} \;\middle|\; \{(s, \operatorname{crit} \vec{k}_s) \mid s \in \operatorname{term}(d)\} \in X\right\} \in \hat{\nu}^{d,\gamma} \\ &\iff \left\{\vec{k} \in D^{d,\gamma} \;\middle|\; \{(s, \operatorname{crit}(\pi^d_{\beta,\gamma}(\vec{k}))_s) \mid s \in \operatorname{term}(d)\} \in X\right\} \in \hat{\nu}^{d,\gamma}. \end{aligned}$$

Since $\operatorname{crit}(\pi^d_{\beta,\gamma}(\vec{k}))_s = \operatorname{crit} \vec{k}_s$, we have that the last formula is equivalent to $X \in \hat{\nu}^{d,\gamma}$. For a general case, let $p \colon \operatorname{Ord} \times \operatorname{Ord} \to \operatorname{Ord}$ be the order isomorphism, where $\operatorname{Ord} \times \operatorname{Ord}$ follows the lexicographic order. If we let $\alpha_s = p(\beta(s), \gamma(s))$, then it satisfies the condition of Lemma 4.5.21. Hence, the combination of Lemma 4.5.21 and Theorem 4.5.3 implies there is an embedding $\delta \colon \operatorname{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ such that $\delta(s) \geq \alpha_s$ for every $s \in d$. It is clear that $\alpha_s \geq \beta(s), \gamma(s)$, so we have an embedding δ pointwise dominating β, γ. □

Hence the choice of β is irrelevant of $\nu^{d,\beta}$, so we drop β and write ν^d instead of $\nu^{d,\beta}$. Theorem 4.5.3 immediately implies

Corollary 4.5.23. *ν^d concentrates on $(\Omega^1_{\mathsf{M}})^{\operatorname{Fun}(d)}$.*

Proof. Let $\beta \colon \operatorname{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ be an embedding. By Theorem 4.5.3, we have

$$\{\vec{k} \in D^{d,\beta} \mid s \mapsto \operatorname{crit} \vec{k}_s \text{ is an embedding from } \operatorname{Fun}(d^\bullet) \text{ to } \Omega^1_{\mathsf{M}}\} \in \hat{\nu}^{d,\beta}.$$

Hence by the definition of $\nu^{d,\beta}$, we also have

$$\{\gamma \in (\Omega^1_{\mathsf{M}})^{\operatorname{Fun}(d)} \mid \gamma \text{ is a dilator embedding}\} \in \nu^{d,\beta} = \nu^d.$$ □

For a finite flower F with no nullary terms, we can define ν^F with the help of ν^d:

Definition 4.5.24. Let F be a finite flower with no nullary terms. We define ν^F over the set of embeddings from F to Ω^1_{M} as follows:

$$X \in \nu^F \iff \{\gamma \in (\Omega^1_{\mathsf{M}})^{\mathrm{Fun}(d)} \mid \gamma \circ h \in X\} \in \nu^d,$$

where d is a trekkable dendrogram with the isomorphism $h \colon \mathrm{Fun}(d) \to F$.

4.5.5 The coherence of the measure family

From the remaining part of the chapter, we show that ν^d witnesses the measurability of Ω^1_{M}. We first verify the coherence of the measure family.

Proposition 4.5.25. *Let d, d' be a finite trekkable dendrogram with no nullary terms such that $|d'| - |d| = 1$, and there is $s < |d'|$ and a trekkable dendrogram morphism $h \colon d \to d'$ satisfying*

$$h(t) = \begin{cases} t & \text{if } t <_{\mathbb{N}} s, \\ t+1 & \text{if } t \geq_{\mathbb{N}} s. \end{cases}$$

For $\beta \colon \mathrm{Fun}((d')^\bullet) \to \Omega^1_{\mathsf{M}}$ and $X \subseteq D^{d,\beta}$, we have

$$X \in \hat{\nu}^{d,\beta\circ h} \iff (h^*)^{-1}[X] \in \hat{\nu}^{d',\beta},$$

where h^* is a map defined over the set of embeddings $\mathrm{Fun}((d')^\bullet) \to \Omega^1_{\mathsf{M}}$ by $h^*(\beta) = \beta \circ h$.

Proof. Let $|d| = m$. Following the notation in the proof of Theorem 4.5.15, (4.11), we have

$$\begin{aligned} &\forall(\hat{\nu}^{d,\beta\circ h})\vec{k}[\vec{k} \in X] \\ &\qquad \iff \forall(\hat{\mu}_1^{d,\beta\circ h})k^1 \forall(\hat{\mu}_2^{d,\beta\circ h})k^2 \cdots \forall(\hat{\mu}_{m-1}^{d,\beta\circ h})k^{m-1}[\{(t, k^t) \mid t < m\} \in X]. \end{aligned}$$

Observe that in d', s is a terminal node. This means no other measure components $\hat{\mu}_t^{d,\beta}$ in defining $\hat{\nu}^{d',\beta}$ depends on the sth component. Hence for $t \geq_{\mathbb{N}} s$, $\hat{\mu}_t^{d,\beta\circ h} = \hat{\mu}_{t+1}^{d',\beta}$. We also have $\hat{\mu}_t^{d,\beta\circ h} = \hat{\mu}_t^{d',\beta}$ for $t <_{\mathbb{N}} s$, so Lemma 4.1.16 implies

$$\begin{aligned}\forall(\hat{\nu}^{d,\beta\circ h})\vec{k}[\vec{k} \in X] &\iff \forall(\hat{\mu}_1^{d',\beta})k^1\forall(\hat{\mu}_2^{d',\beta})k^2 \cdots \forall(\hat{\mu}_m^{d',\beta})k^m \\ &\qquad [\{(h^{-1}(t), k^t) \mid t \leq m, t \neq s\} \in X]. \\ &\iff \forall(\hat{\nu}^{d',\beta})\vec{k}[\vec{k} \in (h^*)^{-1}[X]].\end{aligned}$$

□

We can derive the coherence by applying Proposition 4.5.25 several times.

Lemma 4.5.26. *Let d and d' be finite dendrograms, $|d| < |d'|$, and $f\colon d \to d'$ a trekkable dendrogram morphism. Then we can find a sequence of trekkable dendrograms $d_0, d_1, \cdots, d_m$ and $f_l\colon d_l \to d_{l+1}$ $(l < m)$ such that each f_l is trekkable, $d_0 = d$, $d_m = d'$, $f = f_{m-1} \circ \cdots \circ f_0$, and $|d_{l+1}| - |d_l| = 1$ for every $l < m$.*

Proof. Let $d'_0 \subseteq d'$ be the range of f and let $\{s_0, \cdots, s_{m-1}\}$ be the $<_{\mathbb{N}}$-increasing enumeration of $d' \setminus d'_0$. By the trekkability of d', for each $l < m$, the set $d'_l = d'_0 \cup \{s_0, \cdots, s_{l-1}\} \subseteq d'$ is closed under immediate predecessors. Then let us find a trekkable dendrogram d_l, a trekkable dendrogram morphism $g_l\colon d'_l \to d_l$, and f_l making the following diagram commutes:

$$\begin{array}{ccccccc} d'_0 & \xrightarrow{\subseteq} & d'_1 & \xrightarrow{\subseteq} & \cdots & \xrightarrow{\subseteq} & d'_m \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & & & \downarrow{\scriptstyle h_m} \\ d_0 & \xrightarrow[f_0]{} & d_1 & \xrightarrow[f_1]{} & \cdots & \xrightarrow[f_{m-1}]{} & d_m \end{array}$$

We can see that d_l and f_l satisfy the desired properties. □

Proposition 4.5.27. *Let d and d' be finite dendrograms and $f\colon d \to d'$ a trekkable dendrogram morphism. For $\beta\colon \mathrm{Fun}((d')^\bullet) \to \Omega^1_{\mathsf{M}}$ and $X \subseteq D^{d,\beta}$, we have*

$$X \in \hat{\nu}^{d,\beta\circ f} \iff (f^*)^{-1}[X] \in \hat{\nu}^{d',\beta}.$$

Proof. We only consider the case $|d| < |d'|$. Let $d_0, \cdots, d_m$ and $f_l \colon d_l \to d_{l+1}$ $(l < m)$ be the sequence of trekkable dendrograms and morphisms given by Lemma 4.5.26. Then we have

$$X \in \hat{\nu}^{d,\beta\circ f} \iff (f_0^*)^{-1}[X] \in \hat{\nu}^{d_1,\beta\circ f_{l-1}\circ\cdots\circ f_1} \iff \cdots \iff (f^*)^{-1}[X] \in \hat{\nu}^{d',\beta}. \quad \square$$

Hence we have

Theorem 4.5.28 (Coherence). *Let F, F' be finite flowers with no nullary terms and $f \colon F \to F'$ be an embedding. For $X \subseteq (\Omega^1_{\mathsf{M}})^F$, we have*

$$X \in \nu^F \iff (f^*)^{-1}[X] \in \nu^{F'}.$$

Proof. By replacing F and f if necessary, we may assume that $F \subseteq F'$ and f is the inclusion map. Let d' be a trekkable dendrogram with an isomorphism $h' \colon F' \cong \mathrm{Fun}(d')$, and consider a subdendrogam $\hat{d} \subseteq d'$ such that $h'[F] = \mathrm{Fun}(\hat{d})$. Then we can find a trekkable dendrogram d with an isomorphism $g \colon d \to \hat{d}$ that is also $<_{\mathbb{N}}$-increasing. It is easy to see that $g \colon d \to d'$ is a trekkable dendrogram morphism, and we can find an isomorphism $h \colon F \to \mathrm{Fun}(d)$ making the following diagram commute:

$$\begin{array}{ccc} F' & \xrightarrow{h'} & \mathrm{Fun}(d') \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \mathrm{Fun}(g)} \\ F & \xrightarrow[h]{} & \mathrm{Fun}(d) \end{array}$$

Then by Proposition 4.5.27,

$$\begin{aligned} X \in \nu^F &\iff \{\gamma \in (\Omega^1_{\mathsf{M}})^{\mathrm{Fun}(d)} \mid \gamma \circ h \in X\} \in \nu^d \\ &\iff \{\gamma \in (\Omega^1_{\mathsf{M}})^{\mathrm{Fun}(d')} \mid (\gamma \circ \mathrm{Fun}(g)) \circ h \in X\} \in \nu^{d'} \\ &\iff \{\gamma \in (\Omega^1_{\mathsf{M}})^{\mathrm{Fun}(d')} \mid (\gamma \circ h') \circ f \in X\} \in \nu^{d'} \\ &\iff \{\gamma \in (\Omega^1_{\mathsf{M}})^{\mathrm{Fun}(d')} \mid \gamma \circ h' \in (f^*)^{-1}[X]\} \in \nu^{d'} \\ &\iff (f^*)^{-1}[X] \in \nu^{F'}. \end{aligned}$$ $\square$

4.5.6 The ω_1-completeness of the measure family

We finish this section by proving that the measure family we have constructed is ω_1-complete. The main idea of the proof is somewhat similar to that of Lemma 4.3.9, but the argument is more complicated since we iterate measures along a tree and use a dependent product. As we did in the previous subsection, we handle the trekkable dendrogram version of the ω_1-completeness first and transfer it into the flower version. *We use the axiom of dependent choice in this subsection.*

Let F be a countable flower with no nullary terms. Then its cell decomposition $\mathrm{Cell}(F)$ is a dendrogram. In particular, $\mathrm{Cell}(F)$ is locally well-founded and $\mathrm{Fun}(\mathrm{Cell}(F)^\bullet)(n)$ is well-ordered for each n. This means the level-then-value order $<_{\mathsf{LV}}$ over $\mathrm{Cell}(F)$ is a well-order. Hence, we can re-label elements of $\mathrm{Cell}(F)$ into ordinals and form a trekkable dendrogram C isomorphic to $\mathrm{Cell}(F)$. By Proposition 4.2.8, the cell decomposition of F (and also C) is a tree with the top node 0.

Let D be a countable trekkable dendrogram with no nullary terms, and $f_i \colon d_i \to D$ be an increasing dendrogram morphism for each $i < \omega$. Let us also fix $\beta \colon \mathrm{Fun}(D^\bullet) \to \Omega^1_{\mathsf{M}}$, which will be a uniform bound for $\hat{\nu}^{d_i, \beta \circ f_i}$. We want to find a sequence of elementary embeddings $\langle \tilde{k}_s \mid s \in D \rangle$ such that for each $i < \omega$, $\{(s, \tilde{k}_{f_i(s)}) \mid s \in d_i\} \in X_i$. We will find the desired sequence 'cell-by-cell': More precisely, from a given $\tilde{k}_s$, we will find $\tilde{k}_t$ for every immediate successor t of s. We need subsidiary notions for the proof: First, we need a 'section' of a measure for a given sequence of elementary embeddings $\vec{k}$:

Definition 4.5.29. Let d be a trekkable dendrogram and $\beta \colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$. A sequence $\vec{k}$ of elementary embeddings is (d, β)-*coherent* if

1. $\mathrm{dom}\, \vec{k} \subseteq d$ and is closed under immediate predecessors.
2. $\mathrm{dom}\, \vec{k}_s = V_{\mathrm{crit}\, \vec{k}_s + \beta(s)}$ for every $s \in \mathrm{dom}\, \vec{k}$.

Definition 4.5.30. Let $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ and $\vec{k}$ be a (d,β)-coherent sequence of elementary embeddings. We define $D^{d,\beta}_s[\vec{k}]$ and $\hat{\nu}^{d,\beta}_s[\vec{k}]$ similar to Definition 4.5.1, but with 'skipping' the embeddings occurring in $\vec{k}$. More precisely, we define them as follows:

1. $D^{d,\beta}_0[\vec{k}] = \emptyset$ and $\hat{\nu}^{d,\beta}_0[\vec{k}]$ is the trivial measure.
2. If $s \in \mathrm{dom}\,\vec{k}$, $D^{d,\beta}_s[\vec{k}] = D^{d,\beta}_{s-1}[\vec{k}]$ and $\hat{\nu}^{d,\beta}_s[\vec{k}] = \hat{\nu}^{d,\beta}_{s-1}[\vec{k}]$.
3. If $\mathrm{lh}\, s = 1$ and $s \notin \mathrm{dom}\,\vec{k}$, define
   - $D^{d,\beta}_s[\vec{k}] = \{\vec{k}' \cup \{(s,k'')\} \mid \vec{k}' \in D^{d,\beta}_{s-1}[\vec{k}] \wedge k'' \in \mathsf{Emb}^{j_1{\restriction}V_{\kappa_1+\beta(s)}}_{\beta(s)}\}$.
   - $X \in \hat{\nu}^{d,\beta}_s[\vec{k}] \iff \{\vec{k}' \in D^{d,\beta}_{s-1}[\vec{k}] \mid \{k'' \in \mathsf{Emb}^{j_1{\restriction}V_{\kappa_1+\beta(s)}}_{\beta(s)} \mid \vec{k}' \cup \{(s,k'')\} \in X\} \in \mu^{j_1{\restriction}V_{\kappa_1+\beta(s)}}_{\beta(s)}\} \in \hat{\nu}^{d,\beta}_{s-1}[\vec{k}]$.
4. If $s \notin \mathrm{dom}\,\vec{k}$, $d \vDash t \multimap s$, $\mathbf{e}(t) = a$. Define
   - $D^{d,\beta}_s[\vec{k}] = \{\vec{k}' \cup \{(s,k'')\} \mid \vec{k}' \in D^{d,\beta}_{s-1}[\vec{k}] \wedge k'' \in \mathsf{Emb}^{j_a((\vec{k}\cup\vec{k}')_t)}_{\beta(s)}\}$.
   - $X \in \hat{\nu}^{d,\beta}_s[\vec{k}] \iff \{\vec{k}' \in D^{d,\beta}_{s-1}[\vec{k}] \mid \{k'' \in \mathsf{Emb}^{j_a((\vec{k}\cup\vec{k}')_t)}_{\beta(s)} \mid \vec{k}' \cup \{(s,k'')\} \in X\} \in \mu^{j_a((\vec{k}\cup\vec{k}')_t)}_{\beta(s)}\} \in \hat{\nu}^{d,\beta}_{s-1}[\vec{k}]$.

$D^{d,\beta}[\vec{k}]$, $\hat{\nu}^{d,\beta}[\vec{k}]$ are $D^{d,\beta}_s[\vec{k}]$ and $\hat{\nu}^{d,\beta}_s[\vec{k}]$ for the final element s of d respectively.

For $X \subseteq D^{d,\beta}$ and a sequence $\vec{k}$, we define $X[\vec{k}] = \{\vec{k}' \in D^{d,\beta}[\vec{k}] \mid \vec{k} \cup \vec{k}' \in X\}$.

Every measure we used in the definition of $\hat{\nu}^{d,\beta}_s[\vec{k}]$ is countably complete, so $\hat{\nu}^{d,\beta}_s[\vec{k}]$ is also countably complete. We also define a generalization of (4.12):

Definition 4.5.31. Let d be a finite trekkable dendrogram and $\beta\colon \mathrm{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ an embedding, $\vec{k}$ a (d,β)-coherent sequence, and $d' \subseteq d$ is closed under immediate predecessors. Suppose that $\mathrm{dom}\,\vec{k} \subseteq d'$ and both $\mathrm{dom}\,\vec{k}$ and d' are closed under nodes with the same immediate predecessor; That is, if $t \in d'$ and $d \vDash s \multimap t, t'$, then $t' \in d'$.

For $X \in \hat{\nu}^{d,\beta}[\vec{k}]$, let us define $X \restriction^{d,\beta,\vec{k}} d'$ by

$$\vec{k}' \in X \restriction^{d,\beta,\vec{k}} d' \iff \forall(\mu^{d,\beta}_{s_0})k^{s_0} \dots \forall(\mu^{d,\beta}_{s_{m-1}})k^{s_{m-1}}(\vec{k}' \cup \{(s_i, k^i) \mid i < m\} \in X),$$

where $\{s_i \mid i < m\}$ is the increasing enumeration of $d \setminus d'$ and $\mu^{d,\beta}_s$ is a unit measure occurring in the definition of $\hat{\nu}^{d,\beta}[\vec{k}]$:

$$\mu^{d,\beta}_s = \mu^{d,\beta}_s(\vec{k} \cup \vec{k}', \langle k^{s_i} \mid s_i < s\rangle)$$
$$= \begin{cases} \mu^{j_1 \restriction V_{\kappa_1+\beta(s)}}_{\beta(s)} & \operatorname{lh} s = 1, \\ \mu^{j_a(k^{s'})}_{\beta(s)} & d \vDash s' \multimap s,\ \mathbf{e}^d(s') = a,\ s' \notin \operatorname{dom}\vec{k} \cup \vec{k}', \\ \mu^{j_a((\vec{k}\cup\vec{k}')_{s'})}_{\beta(s)} & d \vDash s' \multimap s,\ \mathbf{e}^d(s') = a,\ s' \in \operatorname{dom}\vec{k} \cup \vec{k}'. \end{cases}$$

We also define a measure $\hat{\nu}^{d,\beta}[\vec{k}] \restriction d'$ over $D^{d,\beta}[\vec{k}] \restriction^{d,\beta,\vec{k}} d'$ by

$$X \in \hat{\nu}^{d,\beta}[\vec{k}] \restriction d' \iff \forall(\mu^{d,\beta}_{t_0})k^{t_0} \dots \forall(\mu^{d,\beta}_{t_{p-1}})k^{t_{p-1}}(\{(t_i, k^{t_i}) \mid i < p\} \in X),$$

where $\{t_i \mid i < p\}$ is the increasing enumeration of $d' \setminus \operatorname{dom}\vec{k}$.

Lemma 4.5.32. *Let d be a finite trekkable dendrogram, $d' \subseteq d$ being closed under immediate predecessors, $\beta\colon \operatorname{Fun}(d^\bullet) \to \Omega^1_{\mathsf{M}}$ an embedding, and $\vec{k}$ a (d,β)-coherent sequence of elementary embeddings such that $\operatorname{dom}\vec{k} \subseteq d'$ and both d' and $\operatorname{dom}\vec{k}$ are closed under nodes with the same immediate predecessor. Then*

1. *For $X \subseteq D^{d,\beta}[\vec{k}]$, $X \in \hat{\nu}^{d,\beta}[\vec{k}] \iff X \restriction^{d,\beta,\vec{k}} d' \in \hat{\nu}^{d,\beta}[\vec{k}] \restriction d'$.*
2. *For $X \subseteq D^{d,\beta}[\vec{k}]$, if $\vec{k}'$ is a (d,β)-coherent sequence such that $\vec{k} \cup \vec{k}' \in D^{d,\beta}$, then $\vec{k}' \in X \restriction^{d,\beta,\vec{k}} \operatorname{dom}\vec{k}' \iff X[\vec{k}'] \in \hat{\nu}^{d,\beta}[\vec{k} \cup \vec{k}']$.*

Proof. The main idea of the proof is that we can switch the order of measure quantifiers in the definition of $\hat{\nu}^{d,\beta}[\vec{k}]$ and $\hat{\nu}^{d,\beta}[\vec{k}] \restriction d'$ as long as the measure order is trekkable. More precisely, suppose that d and $\hat{d}$ are trekkable dendrograms and $h\colon d \to \hat{d}$ is an isomorphism. Then we have

- For each $X \subseteq D^{d,\beta}[\vec{k}]$,

$$X \in \hat{\nu}^{d,\beta}[\vec{k}] \iff \forall(\mu^{\hat{d},\beta\circ h}_{h(s_0)})k^{s_0} \cdots \forall(\mu^{\hat{d},\beta\circ h}_{h(s_{m-1})})k^{s_{m-1}}\Big[\{(s_i, k^{s_i}) \mid i < m\} \in X\Big],$$

  where $\langle s_i \mid i < m\rangle$ is the enumeration of $d \setminus \operatorname{dom}\vec{k}$ such that $\langle h(s_i) \mid i < m\rangle$ is increasing.

- For $X \subseteq D^{d,\beta}[\vec{k}]$, if $\langle t_i \mid i < p\rangle$ is an enumeration of $d \setminus d'$ such that $\langle h(t_i) \mid i < p\rangle$ is increasing, we have

$$\vec{k}' \in X \restriction^{d,\beta,\vec{k}} d' \iff \forall(\mu^{\hat{d},\beta\circ h}_{h(t_0)})k^{t_0} \cdots \forall(\mu^{\hat{d},\beta\circ h}_{h(t_{p-1})})k^{t_{p-1}}\Big[\vec{k}' \cup \{(t_i, k^{t_i}) \mid i < p\} \in X\Big],$$

- For $X \subseteq D^{d,\beta}[\vec{k}] \restriction^{d,\beta,\vec{k}} d'$, if $\langle t_i \mid i < p\rangle$ is an enumeration of $d' \setminus \operatorname{dom}\vec{k}$ such that $\langle h(t_i) \mid i < p\rangle$ is increasing, we have

$$X \in \hat{\nu}^{d,\beta}[\vec{k}] \restriction d' \iff \forall(\mu^{\hat{d},\beta\circ h}_{h(t_0)})k^{t_0} \cdots \forall(\mu^{\hat{d},\beta\circ h}_{h(t_{p-1})})k^{t_{p-1}}\Big[\{(t_i, k^{t_i}) \mid i < p\} \in X\Big],$$

Its proof follows from the proof of Theorem 4.5.15, so we omit its details. Let us apply the previous observation to prove the lemma:

1. Let $\hat{d}$ be a dendrogram isomorphic to d, whose field is a natural number, but enumerates elements of d' first, then enumerates those of $d \setminus d'$; That is, if $h \colon d \to \hat{d}$ is an isomorphism, $s \in d'$, $t \in d \setminus d'$, then $h(s) <_{\mathbb{N}} h(t)$. Such $\hat{d}$ exists and is trekkable by the assumption that d' is closed under nodes with the same immediate predecessor. Suppose that $\langle h(s_i) \mid i < m\rangle$ increasingly enumerates $d \setminus \operatorname{dom}\vec{k}$,

and $\langle h(s_i) \mid i < p\rangle$ increasingly enumerates $d' \setminus \operatorname{dom}\vec{k}$. Then for $X \subseteq D^{d,\beta}[\vec{k}]$,

$$\begin{aligned}
&X \in \hat{\nu}^{d,\beta}[\vec{k}] \\
\iff &\forall(\mu^{\hat{d},\beta\circ h}_{h(s_0)})k^{s_0} \dots \forall(\mu^{\hat{d},\beta\circ h}_{h(s_{m-1})})k^{s_{m-1}}\left[\left\{(s_i, k^{s_i}) \;\middle|\; i < m\right\} \in X\right] \\
\iff &\forall(\mu^{\hat{d},\beta\circ h}_{h(s_0)})k^{s_0} \dots \forall(\mu^{\hat{d},\beta\circ h}_{h(s_{p-1})})k^{s_{p-1}}\left[\left\{(s_i, k^{s_i}) \;\middle|\; i < p\right\} \in X \restriction^{d,\beta,\vec{k}} d'\right] \\
\iff &X \restriction^{d,\beta,\vec{k}} d' \in \hat{\nu}^{d,\beta}[\vec{k}] \restriction d'.
\end{aligned}$$

2. Let $d' = \operatorname{dom}\vec{k}'$, and $\hat{d}$ be a dendrogram isomorphic to d whose field is a natural number but enumerates elements of d' first, then enumerates elements of $d \setminus d'$. If $\langle h(s_i) \mid i < m\rangle$ increasingly enumerates $d \setminus \operatorname{dom}\vec{k}$, and $\langle h(s_i) \mid i < p\rangle$ increasingly enumerates $d' \setminus \operatorname{dom}\vec{k}$, then

$$\begin{aligned}
&\vec{k}' \in X \restriction^{d,\beta,\vec{k}} d' \\
\iff &\forall(\mu^{\hat{d},\beta\circ h}_{h(s_p)})k^{s_p} \dots \forall(\mu^{\hat{d},\beta\circ h}_{h(s_{m-1})})k^{s_{m-1}}\left[\vec{k}' \cup \left\{(s_i, k^{s_i}) \;\middle|\; p \le i < m\right\} \in X\right] \\
\iff &\forall(\mu^{\hat{d},\beta\circ h}_{h(s_p)})k^{s_p} \dots \forall(\mu^{\hat{d},\beta\circ h}_{h(s_{m-1})})k^{s_{m-1}}\left[\left\{(s_i, k^{s_i}) \;\middle|\; p \le i < m\right\} \in X[\vec{k}']\right] \\
\iff &X[\vec{k}'] \in \hat{\nu}^{d,\beta}[\vec{k} \cup \vec{k}'].
\end{aligned}$$

□

The following theorem will immediately imply the ω_1-completeness of the measure family. Before stating the proof, let me remark that the following proof is a variant of the standard proof of "Every measurable cardinal is Ramsey."

Theorem 4.5.33. *Let D be a countable trekkable dendrogram with no nullary terms, $\beta\colon \operatorname{Fun}(D^\bullet) \to \Omega^1_{\mathsf{M}}$ a limit embedding, $\{d_i \mid i < \omega\}$ a sequence of finite trekkable dendrograms, and $f_i\colon d_i \to D$ an increasing dendrogram morphism such that $D = \bigcup_{i<\omega} \operatorname{ran} f_i$.*

If $X_i \in \hat{\nu}^{d_i,\beta\circ f_i}$ for each $i < \omega$, then we can find a family of elementary embeddings $\langle \tilde{k}_s \mid s \in D\rangle$ such that $\{(s, \tilde{k}_{f_i(s)}) \mid s \in d_i\} \in X_i$ for each $i < \omega$ and $s^\bullet \mapsto \operatorname{crit}\tilde{k}_s$ for $s \in D$ is an embedding from $\operatorname{Fun}(D^\bullet)$ to Ω^1_{M}.

Proof. By Theorem 4.5.3, we may assume that for every $\vec{k} \in X_i$, the map $s^\bullet \mapsto \operatorname{crit} \vec{k}_s$ is an embedding from $\mathrm{Fun}(d_i^\bullet)$ to Ω^1_{M}. Now define $D_0 = \{0\}$, $D_{\alpha+1} = D_\alpha \cup \{\xi \in D \mid \alpha \multimap \xi\}$, and $D_\alpha = \bigcup_{\xi<\alpha} D_\xi$ for a limit α. Then each $D_\alpha \subseteq D$ is closed under immediate predecessors and $\alpha \in D_\alpha$ for every $\alpha \in D$. Clearly, each D_α is closed under nodes with the same immediate predecessor. We first find $\tilde{k}_s$ for $s \in D_1$ satisfying the following: For each $i < \omega$,

$$\{(s, \tilde{k}_{f_i(s)}) \mid f_i(s) \in D_1\} \in X_i \restriction^{d_i, \beta \circ f_i, \{\}} f_i^{-1}[D_1].$$

Observe that

$$X_i \restriction^{d_i, \beta \circ f_i, \{\}} f_i^{-1}[D_1] \in \hat{\nu}^{d_i, \beta \circ f_i} \restriction f_i^{-1}[D_1]$$

and $\hat{\nu}^{d_i, \beta \circ f_i} \restriction f_i^{-1}[D_i]$ is the product of measures of the form $\mu^{j_1 \restriction V_{\kappa_1 + \beta \circ f_i(s)}}_{\beta \circ f_i(s)}$ for $s \in d_i$ such that $f_i(s) \in D_1$. If we take $\gamma(0) = \sup\{\beta(s) \mid D \vDash 0 \multimap s\} < \kappa_1$, then Proposition 4.1.18 and Countable Choice imply for each $i < \omega$ we can find $Y_i^1 \in \mu^{j_1 \restriction V_{\kappa_1 + \gamma(0)}}_{\gamma(0)}$ such that

$$\Delta^{j_1 \restriction V_{\kappa_1 + \gamma(0)}}_{N_i^1} \cap \prod_{D \vDash 0 \multimap f_i(s)} \pi^{j_1 \restriction V_{\kappa_1 + \gamma(0)}}_{\beta \circ f_i(s), \gamma(0)}[Y_i^1] \subseteq X_i \restriction^{d_i, \beta \circ f_i, \{\}} f_i^{-1}[D_1], \tag{4.18}$$

where $N_i^0 = |\{s \in d_i \mid D \vDash 0 \multimap f_i(s)\}|$. We have $Y^1 := \bigcap_i Y_i^1 \in \mu^{j_1 \restriction V_{\kappa_1 + \gamma(1)}}_{\gamma(1)}$, so by Proposition 4.1.19, we can choose $\overline{k}_s \in Y^1$ for each $s \in D_1 \setminus D_0$ such that $D \vDash s < s'$ implies

$$\overline{k}_s \restriction V_{\operatorname{crit} \overline{k}_s + \beta(s)} \in \operatorname{ran}\left(\overline{k}_s \restriction V_{\operatorname{crit} \overline{k}_{s'} + \beta(s')}\right).$$

Then take $\tilde{k}_s = \overline{k}_s \restriction V_{\operatorname{crit} \overline{k}_s + \beta(s)}$. Combining with (4.18), we have

$$\left\{(s, \tilde{k}_{f_i(s)}) \;\middle|\; f_i(s) \in D_1\right\} \in X_i \restriction^{d_i, \beta \circ f_i, \{\}} f_i^{-1}[D_1] \tag{4.19}$$

for each $i < \omega$. Note that if we write $\vec{\tilde{k}}^{\alpha, i} = \{(s, \tilde{k}_{f_i(s)}) \mid f_i(s) \in D_\alpha\}$, then (4.19) becomes $\vec{\tilde{k}}^{1,i} \in X_i \restriction^{d_i, \beta \circ f_i, \{\}} \operatorname{dom} \vec{\tilde{k}}^{1,i}$, which is equivalent to $X_i[\vec{\tilde{k}}^{1,i}] \in \hat{\nu}^{d_i, \beta \circ f_i}[\vec{\tilde{k}}^{1,i}]$.

Now, let us inductively assume that we have found $\tilde{k}_s$ for $s \in D_\alpha$ such that for every $i < \omega$, $X_i[\vec{\tilde{k}}^{\alpha,i}] \in \hat{\nu}^{d_i,\beta\circ f_i}[\vec{\tilde{k}}^{\alpha,i}]$. We will find $\tilde{k}_s$ for $s \in D$ such that $D \vDash \alpha \multimap s$ such that

$$\{(s, \tilde{k}_{f_i(s)}) \mid s \in d_i \wedge D \vDash \alpha \multimap f_i(s)\} \in X_i[\vec{\tilde{k}}^{\alpha,i}] \restriction^{d_i,\beta\circ f_i,\vec{\tilde{k}}^{\alpha,i}} (f_i^{-1}[D_{\alpha+1}])[\vec{\tilde{k}}^{\alpha,i}].$$

Note that $\alpha \in D_\alpha$, so $\tilde{k}_\alpha$ is defined. From the inductive assumption, we have

$$X_i[\vec{\tilde{k}}^{\alpha,i}] \restriction^{d_i,\beta\circ f_i,\vec{\tilde{k}}^{\alpha,i}} (f_i^{-1}[D_\alpha])[\vec{\tilde{k}}^{\alpha,i}] \in \hat{\nu}^{d_i,\beta\circ f_i}[\vec{\tilde{k}}^{\alpha,i}] \restriction^{d_i,\beta\circ f_i,\vec{\tilde{k}}^{\alpha,i}} (f_i^{-1}[D_\alpha])[\vec{\tilde{k}}^{\alpha,i}], \tag{4.20}$$

The measure in (4.20) is a product of measures of the form $\mu^{j_a(\tilde{k}_\alpha)}_{\beta\circ f_i(s)}$ for $s \in d_i$ with $f_i(s) \in D_\alpha$, where $a = \mathbf{e}^D(\alpha)$. Now take $\gamma(\alpha) = \sup\{\beta(s) \mid D \vDash \alpha \multimap s\}$, then Proposition 4.1.18 and Countable Choice imply for each $i < \omega$, we can find $Y_i^\alpha \in \mu^{j_a(\tilde{k}_\alpha)}_{\gamma(\alpha)}$ such that

$$\Delta^{j_a(\tilde{k}_\alpha)}_{N_i^{\alpha+1}} \cap \prod_{D\vDash\alpha\multimap f_i(s)} \pi^{j_a(\tilde{k}_\alpha)}_{\beta\circ f_i(s),\gamma(\alpha)}[Y_i^{\alpha+1}] \subseteq X_i[\vec{\tilde{k}}^{\alpha,i}] \restriction^{d_i,\beta\circ f_i,\vec{\tilde{k}}^{\alpha,i}} (f_i^{-1}[D_\alpha])[\vec{\tilde{k}}^{\alpha,i}], \tag{4.21}$$

where $N_i^{\alpha+1} = \left|\{s \in d_i \mid D \vDash \alpha \multimap f_i(s)\}\right|$. We have $Y^{\alpha+1} := \bigcap_i Y_i^{\alpha+1} \in \mu^{j_a(\tilde{k}_\alpha)}_{\gamma(\alpha)}$, so by Proposition 4.1.19, we can choose $\overline{k}_s \in Y^{\alpha+1}$ for each $s \in D_{\alpha+1} \setminus D_\alpha$ such that $D \vDash s < s'$ and $\alpha \multimap s, s'$ imply

$$\overline{k}_s \restriction V_{\operatorname{crit}\overline{k}_s+\beta(s)} \in \operatorname{ran}\left(\overline{k}_s \restriction V_{\operatorname{crit}\overline{k}_{s'}+\beta(s')}\right).$$

Now take $\tilde{k}_s = \overline{k}_s \restriction V_{\operatorname{crit}\overline{k}_s+\beta(s)}$ for $s \in D_{\alpha+1} \setminus D_\alpha$ as before. Combining with (4.21), we have

$$\{(s, \tilde{k}_{f_i(s)}) \mid f_i(s) \in D_{\alpha+1} \setminus D_\alpha\} \in X_i[\vec{\tilde{k}}^{\alpha,i}] \restriction^{d_i,\beta\circ f_i,\vec{\tilde{k}}^{\alpha,i}} (f_i^{-1}[D_\alpha])[\vec{\tilde{k}}^{\alpha,i}] \tag{4.22}$$

Note that if we let $\vec{k}'$ be the left-hand-side of (4.22), then (4.22) is equivalent to $X_i[\vec{\tilde{k}}^{\alpha,i}][\vec{k}'] \in \hat{\nu}^{d_i,\beta\circ f_i}[\vec{\tilde{k}}^{\alpha,i} \cup \vec{k}']$, and $X_i[\vec{\tilde{k}}^{\alpha,i}][\vec{k}'] = X_i[\vec{\tilde{k}}^{\alpha,i} \cup \vec{k}']$. It shows the inductive hypothesis for $\alpha+1$. For limit α, observe that for each $i < \omega$ we can find $\xi < \alpha$ such that $X_i[\vec{\tilde{k}}^{\alpha,i}] = X_i[\vec{\tilde{k}}^{\xi,i}]$ and $\hat{\nu}^{d_i,\beta\circ f_i}[\vec{\tilde{k}}^{\alpha,i}] = \hat{\nu}^{d_i,\beta\circ f_i}[\vec{\tilde{k}}^{\xi,i}]$ since each d_i is finite.

We finalize the proof by showing that $s^\bullet \mapsto \operatorname{crit} \tilde{k}_s$ is an embedding from $\operatorname{Fun}(D^\bullet)$ to Ω^1_{M}. Let α be the least ordinal such that $D = D_\alpha$. For $s, t \in D$, we can find $i < \omega$ such that $s, t \in \operatorname{ran} f_i$. For an arity diagram Д, we have

$$D \vDash s <_{\text{Д}} t \implies d_i \vDash f_i^{-1}(s) <_{\text{Д}} f_i^{-1}(t) \implies \Omega^1_{\mathsf{M}} \vDash \operatorname{crit} \tilde{k}_s <_{\text{Д}} \operatorname{crit} \tilde{k}_t. \qquad \square$$

Theorem 4.5.34 (ω_1-completeness). *Let D be a countable flower with no nullary terms and $\{d_i \mid i < \omega\}$ be a countable family of finite subflowers of D. If $X_i \in \nu^d$ for each $i < \omega$, then we can find an embedding $f \colon D \to \Omega^1_{\mathsf{M}}$ such that for each $i < \omega$, $f \restriction d_i \in X_i$.*

Proof. By replacing d_i with a larger dilator and X_i with its pullback if necessary, we may assume that $D = \bigcup_{i<\omega} d_i$. Let $\hat{d}_i$ and $\hat{D}$ be trekkable dendrograms such that $\operatorname{Fun}(\hat{d}_i) \cong d_i$ and $\operatorname{Fun}(\hat{D}) \cong D$. Let $h \colon \operatorname{Fun}(\hat{D}) \to D$ be an isomorphism, and let $f_i \colon \hat{d}_i \to \hat{D}$ be an embedding such that $h \circ \operatorname{Fun}(f_i)$ is an isomorphism from $\operatorname{Fun}(\hat{d}_i)$ to d_i. Let us also fix a limit embedding $\beta \colon \operatorname{Fun}(\hat{D}^\bullet) \to \Omega^1_{\mathsf{M}}$, then

$$\hat{X}_i := \{\vec{k} \in D^{\hat{d}_i,\beta} \mid \{(h \circ \operatorname{Fun}(f_i)(s), \operatorname{crit} \vec{k}_s) \mid s \in \operatorname{term}(\hat{d}_i)\} \in X_i\} \in \hat{\nu}^{d_i,\beta}.$$

Hence by Theorem 4.5.33, we can find $\langle \tilde{k}_s \mid s \in \hat{D} \rangle$ such that

1. $\{(s, \tilde{k}_{f_i(s)}) \mid s \in \hat{d}_i\} \in \hat{X}_i$ for each $i < \omega$, and
2. The map $s^\bullet \mapsto \operatorname{crit} \tilde{k}_s$ is an embedding from $\operatorname{Fun}(D^\bullet)$ to Ω^1_{M}, and

Hence, the function $s^\bullet \mapsto \operatorname{crit} \tilde{k}_s$ restricted to (the isomorphic copy of) $\operatorname{term}(\hat{D})$ is a function witnessing the ω_1-completeness. $\square$

4.6 Final remarks

Let me finish this chapter with my viewpoint and future research directions about the connection between large ptykes (i.e., ptykes with large cardinal properties), determinacy,

and homogeneous Suslin representation of projective sets.

I have worked with an iterable cardinal in this chapter due to its simplicity over a Woodin cardinal and a measurable cardinal above. However, I expect every proof of $\mathbf{\Pi}^1_n$-determinacy from a large cardinal can be decomposed into a construction of a measurable $(n-1)$-ptyx and a proof of $\mathbf{\Pi}^1_n$-determinacy from a measurable $(n-1)$-ptyx. Hence, I conjecture we can also construct a measurable dilator from a Woodin cardinal and a measurable cardinal above, and a measurable n-ptyx from n many Woodin cardinals and a measurable above. However, Martin's measurable dilator from an iterable cardinal is expected to have a stronger property than an expected measurable dilator from a Woodin cardinal and a measurable above — On the one hand, every measure associated with Martin's measurable dilator is κ-complete for an associated iterable cardinal κ. On the other hand, the following will appear in the work in preparation [7]: If κ is a measurable cardinal with a Woodin cardinal $\delta < \kappa$, then for each $\alpha < \delta$, the universal dilator induced from a normal measure over κ has a family of α-complete measures.

It is well-known that the existence of a measurable cardinal is strictly stronger than $\mathbf{\Pi}^1_1$-determinacy. $\mathbf{\Pi}^1_1$-determinacy is equivalent to the assertion that every real has a sharp. Similarly, it is reasonable to guess that the existence of a measurable dilator is strictly stronger than $\mathbf{\Pi}^1_2$-determinacy. It is known by [68, Corollary 2.2] that $\mathbf{\Pi}^1_{n+1}$-determinacy is equivalent to "For every real R, $M^\sharp_n(R)$ exists and ω_1-iterable." I guess that if we know how to construct a measurable n-ptykes from n Woodin cardinals and a measurable above, then we should also be able to construct a half-measurable n-ptyx from the assertion "For every real R, $M^\sharp_n(R)$ exists and ω_1-iterable." Kechris [49] stated without proof that Projective Determinacy is equivalent to 'For every n, there is a half-measurable n-ptyx,' and he noted that the level-by-level equivalence should hold, but 'it has not been proved yet.' I also conjecture that the existence of half-measurable n-ptyx

is equivalent to $\mathbf{\Pi}^1_{n+1}$-determinacy.

We finish this chapter with a possible connection with the homogeneous Suslin representation of projective sets: As stated before, a homogeneous Suslin representation of a $\mathbf{\Pi}^1_1$-set into an 'effective part' corresponding to a predilator D and a measurable cardinal κ. More precisely, we can decompose a homogeneous Suslin representation of a $\Pi^1_1[R]$-set into an 'effective part' corresponding to an R-recursive predilator D and a measurable cardinal κ. I expect that a homogeneous Suslin representation of a $\Pi^1_n[R]$ can be decomposed into an 'effective part' given by an $(n+1)$-preptyx P and a measurable n-ptyx Ω, and so $P(\Omega)$ forms a homogeneous Suslin representation. It is interesting to ask if every homogeneous tree representation of a given definable set is decomposed into an 'effective object,' which should be a generalization of ptykes and a large-cardinal-like object.

CHAPTER 5

CASE Π^1_3, Σ^1_4, AND BEYOND

Ever drifting down the stream—
Lingering in the golden gleam—
Life, what is it but a dream?

Lewis Carroll, *Through the Looking-Glass* Ch. XII

In this chapter, we examine the Π^1_3 and Σ^1_4 pointclasses, and isolate Walsh-like characterization of the Π^1_3- and Σ^1_4-consequence comparisons of theories extending $\mathbf{\Delta}^1_2$-Determinacy.

5.1 Preliminaries

5.1.1 Second-order arithmetic and Determinacy

In this chapter, we work with a fragment of arithmetic bolstered by a fragment of projective determinacy. The following proposition says that adding a fragment of determinacy bolsters the underlying theory:

Proposition 5.1.1 (ACA_0)**.** $\mathbf{\Pi}^1_n$-Det *proves* Σ^1_{n+1}-DC_0. *Similarly,* $\mathbf{\Delta}^1_n$-Det *proves* Σ^1_n-DC_0.

Proof. We only prove the former assertion; The latter follows from essentially the same proof. Suppose that we have $\forall^1 X \exists^1 Y \phi(X, Y)$ for a Σ^1_{n+1}-formula ϕ possibly with parameters. By pulling out the foremost existential quantifier in P and grouping it with

$\exists^1 Y$, we may assume that ϕ is Π^1_n. Now consider the game $\mathcal{G}_\phi$ defined as follows: Player I and II play two reals X and Y, and Player I wins if and only if $\neg\phi(X, Y)$ holds. By the assumption, Player I does not have a winning strategy for $\mathcal{G}_\phi$, which is a $\mathbf{\Sigma}^1_n$-game. Hence, Player II has a winning strategy σ. Now let A_0 be a given real. Let us recursively define $A_{n+1} = \sigma * A_n$, then we can see that $\langle A_n \mid n \in \mathbb{N}\rangle$ witness dependent choice for $\phi(X, Y)$. □

It is well-known that for $n \geq 1$, $\mathbf{\Delta}^1_{2n}$-Determinacy implies $\mathbf{\Pi}^1_{2n}$-Determinacy, under the name *determinacy transfer*. The currently known proof heavily uses set-theoretic notions, so it is unclear whether the same argument carries over to a fragment of second-order arithmetic. We leave the following as a conjecture:

Conjecture 5.1.2. ACA_0 proves $\mathbf{\Delta}^1_{2n}$-Det implies $\mathbf{\Pi}^1_{2n}$-Det.

As a partial result, Schlutzenberg [87] proved that full second-order arithmetic Z_2 proves $\mathbf{\Delta}^1_2$-Det implies $\mathbf{\Pi}^1_2$-Det. A careful examination of his argument shows that it also works over ACA_0. As a corollary, $\mathbf{\Delta}^1_2$-Det implies Σ^1_3-DC_0. However, we pretend as if we do not know this implication in this chapter. The main reason is that many results in this chapter are generalizable to Π^1_{2n+1} and Σ^1_{2n+2} for $n \geq 2$, and we do not know yet if $\mathbf{\Delta}^1_{2n}$-Det implies Σ^1_{2n+1}-DC_0 for $n \geq 2$.

5.1.2 Pseododilators

Let us recall the notions in Section 1.3: A pseudodilator is a predilator that is not a dilator. For a pseudodilator d, the least ordinal α making $d(\alpha)$ ill-founded is a *climax* and is denoted by $\mathsf{Clim}(d)$. For two pseudodilators d and d', the statement $\mathsf{Clim}(d) \leq \mathsf{Clim}(d')$

is equivalent to a Π^1_2-assertion

$$\mathsf{Grow}_{\Pi^1_2}(d, d') \equiv \mathsf{Grow}_{\Pi^1_2}(d, d') \equiv \forall^1\alpha\big[\mathsf{WO}(\alpha) \to [\mathsf{WO}(d(\alpha)) \to \mathsf{WO}(d'(\alpha))]\big].$$

The next lemma says that $\mathsf{Clim}(d) \leq \mathsf{Clim}(d')$ isequivalent to a Σ^1_2-assertion when d is a pseudodilator:

Lemma 5.1.3 (Σ^1_2-DC_0). *Suppose that d is a pseudodilator and d' is a predilator. Then $\mathsf{Grow}_{\Pi^1_2}(d, d')$ holds if and only if the following statement holds:*

$$\mathsf{Grow}_{\Sigma^1_2}(d, d') \equiv \exists^1\alpha\big[\mathsf{WO}(\alpha) \wedge \neg\mathsf{WO}(d(\alpha)) \wedge \forall^1\beta[\beta < \alpha \to \mathsf{WO}(d'(\beta))]\big].$$

Proof. Suppose that $\mathsf{Grow}_{\Pi^1_2}(d, d')$ holds. Since d is a pseudodilator, there is a well-order α such that $d(\alpha)$ is ill-founded. Now we claim that there is a least such α, that is, α satisfying

$$\forall^1\beta\big[\beta < \alpha \to \mathsf{WO}(d'(\beta))\big]. \tag{5.1}$$

Suppose not, and assume that the following holds:

$$\forall^1\alpha\exists^1\beta\Big[\big[\mathsf{WO}(\alpha) \wedge \neg\mathsf{WO}(d(\alpha))\big] \to \big[\beta < \alpha \wedge \neg\mathsf{WO}(d'(\beta))\big]\Big].$$

By $\mathsf{Grow}_{\Pi^1_2}(d, d')$, we get

$$\forall^1\alpha\exists^1\beta\Big[\big[\mathsf{WO}(\alpha) \wedge \neg\mathsf{WO}(d(\alpha))\big] \to \big[\beta < \alpha \wedge \neg\mathsf{WO}(d(\beta))\big]\Big].$$

We can see that the bracketed formula is Σ^1_2. Hence by Σ^1_2-DC_0, we can find a sequence $\langle\alpha_n \mid n \in \mathbb{N}\rangle$ and a sequence of embeddings $\langle f_n \mid n \in \mathbb{N}\rangle$ such that for each n,

1. $\mathsf{WO}(\alpha_n)$ and $\neg\mathsf{WO}(d(\alpha_n))$.
2. $f_n \colon \alpha_{n+1} \to \alpha_n$ witnesses $\alpha_{n+1} < \alpha_n$.

However, it leads to a contradiction. Now let α be a well-order satisfying (5.1). By $\mathsf{Grow}_{\Pi^1_2}(d, d')$, we get $d'(\beta)$. In sum, we proved

$$\neg\mathsf{WO}(d(\alpha)) \wedge \forall^1\beta(\beta < \alpha \to \mathsf{WO}(d'(\beta))),$$

so we have $\mathsf{Grow}_{\Sigma^1_2}(d, d')$.

Conversely, assume that we have $\mathsf{Grow}_{\Sigma^1_2}(d, d')$, and let α be a well-order witnessing $\mathsf{Grow}_{\Sigma^1_2}(d, d')$. Let γ be any well-order, and suppose that $d(\gamma)$ is well-founded. We claim that $d'(\gamma)$ is well-founded: If $\gamma \geq \alpha$, then $d(\alpha)$ is also well-founded, contradicting the assumption. Hence $\gamma < \alpha$, and it clearly implies $\mathsf{WO}(d'(\gamma))$. □

We can also see that $\mathsf{Grow}_{\Pi^1_2}(d, d')$ and $\mathsf{Grow}_{\Sigma^1_2}(d, d')$ are equivalent also when d' is a pseudodilator and there is no requirement on d:

Corollary 5.1.4 (Σ^1_2-DC_0)**.** *If d' is a pseudodilator and d is a predilator, then* $\mathsf{Grow}_{\Pi^1_2}(d, d')$ *iff* $\mathsf{Grow}_{\Sigma^1_2}(d, d')$.

Proof. If $\mathsf{Grow}_{\Pi^1_2}(d, d')$ and d' is a pseudodilator, then so is d. Hence we get $\mathsf{Grow}_{\Sigma^1_2}(d, d')$. Conversely, if $\mathsf{Grow}_{\Sigma^1_2}(d, d')$, then d is a pseudodilator so we have $\mathsf{Grow}_{\Pi^1_2}(d, d')$. □

Hence $\mathsf{Grow}_{\Pi^1_2}(d, d')$ is Δ^1_2-expressible over Σ^1_2-DC_0, which is a subtheory of $\mathsf{ACA}_0 + \mathbf{\Pi}^1_2$-Det. The next result says that Σ^1_2-AC_0 proves $\mathsf{Grow}_{\Pi^1_2}$ is a prewellorder over the class of pseudodilators.

Proposition 5.1.5. *1.* (ACA_0) $\mathsf{Grow}_{\Pi^1_2}$ *is transitive: That is,* $\mathsf{Grow}_{\Pi^1_2}(d_0, d_1)$ *and* $\mathsf{Grow}_{\Pi^1_2}(d_1, d_2)$ *imply* $\mathsf{Grow}_{\Pi^1_2}(d_0, d_2)$.

2. (ACA_0) *If e_0, e_1 are dilators, then* $\mathsf{Grow}_{\Pi^1_2}(d_0, d_1)$ *is equivalent to* $\mathsf{Grow}_{\Pi^1_2}(d_0 + e_0, d_1 + e_1)$.

3. ($\mathsf{ACA}_0 + \Pi^1_1$-TI_0) *If d is a pseudodilator, then the least well-order making a predilator ill-founded exists. That is, for a pseudodilator d, a well-order α satisfying the following exists:*

$$\neg\mathsf{WO}(d(\alpha)) \wedge \forall^0 i \in \mathrm{field}(\alpha)\mathsf{WO}(d(\alpha \restriction i)). \tag{5.2}$$

4. ($\mathsf{ATR}_0 + \Pi^1_1$-TI_0) *For a pseudodilator d, a well-order α satisfying (5.2) is unique up to isomorphism.*

5. ($\mathsf{ATR}_0 + \Pi^1_1$-TI_0) *For two pseudodilators d and d', either $\mathsf{Grow}_{\Pi^1_2}(d, d')$ or $\mathsf{Grow}_{\Pi^1_2}(d', d)$ holds.*

6. (Σ^1_2-AC_0) *There is no sequence of pseudodilators $\langle d_n \mid n \in \mathbb{N}\rangle$ such that $\neg\mathsf{Grow}_{\Pi^1_2}(d_n, d_{n+1})$ for every $n \in \mathbb{N}$.*

Proof. The first two are immediate by the definition of $\mathsf{Grow}_{\Pi^1_2}$.

3. Suppose that d is a predilator and α is a well-order such that $d(\alpha)$ is ill-founded. We are done if α satisfies (5.2). Suppose not, we claim that there is $i \in \mathrm{field}(\alpha)$ such that

$$\neg\mathsf{WO}(d(\alpha \restriction i)) \wedge \forall j <_\alpha i\ \mathsf{WO}(d(\alpha \restriction j)).$$

Suppose not, then we have for every $i \in \mathrm{field}(\alpha)$,

$$\forall j <_\alpha i\ \mathsf{WO}(d(\alpha \restriction j)) \to \neg\mathsf{WO}(d(\alpha \restriction i)).$$

Hence by Π^1_1-TI_0, $\mathsf{WO}(d(\alpha \restriction i))$ for all $i \in \mathrm{field}(\alpha)$, so α satisfies (5.2), a contradiction.

4. Suppose that α and β be least well-orders making a pseudodilator d ill-founded. If α and β are not isomorphic, then one is isomorphic to an initial segment of the other, which is a theorem of ATR_0. However, it implies a contradiction.

5. Let α and α' be the least well-orders making d and d' ill-founded respectively. We claim that $\alpha \leq \alpha'$ implies $\mathsf{Grow}_{\Pi^1_2}(d, d')$: Let β be a well-order such that $d(\beta)$ is well-founded. If $\beta \geq \alpha$, then $d(\beta)$ is ill-founded, so $\beta < \alpha$. It implies $\beta < \alpha'$, so $d'(\beta)$ is also well-founded.

6. Suppose that we have a sequence $\langle d_n \mid n \in \mathbb{N} \rangle$ of pseudodilators such that $\neg\mathsf{Grow}_{\Pi^1_2}(d_n, d_{n+1})$ for every $n \in \mathbb{N}$. By $\neg\mathsf{Grow}_{\Pi^1_2}(d_n, d_{n+1})$, we have

$$\forall^0 n \exists^1 \alpha [\mathsf{WO}(\alpha) \wedge \mathsf{WO}(d_n(\alpha)) \wedge \neg\mathsf{WO}(d_{n+1}(\alpha))].$$

Thus by Σ^1_2-AC_0, we can choose $\langle \alpha_n \mid n \in \mathbb{N} \rangle$ such that

$$\forall^0 n [\mathsf{WO}(\alpha_n) \wedge \mathsf{WO}(d_n(\alpha_n)) \wedge \neg\mathsf{WO}(d_{n+1}(\alpha_n))].$$

We also have $\mathsf{Grow}_{\Pi^1_2}(d_{n+1}, d_n)$ by the fifth item, so we have

$$\forall^0 n [\mathsf{WO}(d_{n+1}(\alpha_n)) \to \mathsf{WO}(d_n(\alpha_n))].$$

If $\alpha_n \leq \alpha_{n+1}$, then $d_{n+1}(\alpha_n) \leq d_n(\alpha_n)$, so $d_{n+1}(\alpha_n)$ is well-founded, a contradiction. Hence $\alpha_{n+1} < \alpha_n$, so $\langle \alpha_n \mid n \in \mathbb{N} \rangle$ is an infinite descending sequence of well-orders, a contradiction. □

5.1.3 Additional facts about 2-preptykes

In this subsection, we introduce facts about 2-preptykes that we need to develop the contents in this chapter. First, we observe that the definitions in Section E.7 and arguments in Appendix D and Section E.6 generalize to n-preptykes so that we can cell-decompose n-preptykes into n-predendrograms. We will not repeat the definition of n-predendrograms in this dissertation.

The following proposition is a version of Lemma 1.1.5.

Proposition 5.1.6 (ACA_0)**.** *Let P_0 and P_1 be two 2-preptykes of field subsets of $\mathbb{N}$. Then we can construct a new 2-preptyx $P = P_0 \vee P_1$ such that the following holds:*

1. *P is a 2-ptyx iff either P_0 or P_1 is a 2-ptyx.*
2. *Every thread predendrogram of P takes the form $1 + D_0 + D_1$ for some thread predendrogram D_0 of P_0, and a thread predendrogram D_1 of P_1.*
3. *If both of P_0 and P_1 are recursive, then so is P.*
4. *Suppose that P_0 is a 2-pseudoptyx, P_1 is a 2-ptyx, and b_0 is a thread of P_0 whose associated predendrogram d_0 is a dilator. For each thread b_1 of P_1 whose associated predendrogram is d_1, we can find a thread b of P whose associated predendrogram is $1 + d_0 + d_1$ in a (b_0, b_1, P_0, P_1)-recursive way.*

Proof. We assume P_0 and P_1 are 2-predendrograms. We construct a 2-predendrogram P in which levels of P_0 and P_1 occur alternatingly. The field of P will be a set of finite sequences, and we define the immediate successor relation of P by sequence extension by length 1. We first define each level and local comparison relation of P, then define the priority predendrogram relation.

1. The level 0 of P consists of $\langle x \rangle$ for $x \in P_0$ with P_0-length 0. We compare $\langle x \rangle$ and $\langle y \rangle$ by comparing the unique component under $<^{P_0}$.
2. The level 1 of P consists of $\langle x_0, x_1 \rangle$ for $x_0 \in P_0$ and $x_1 \in P_1$. The local comparison relation compares the last component under $<^{P_1}$.
3. The level $2n$ of P for $n \geq 1$ consists of

$$\langle x_0, x_1, \cdots, x_{2n-1}, x_{2n} \rangle$$

such that $P_0 \vDash x_0 \multimap x_2 \multimap \cdots \multimap x_{2n}$ and $P_1 \vDash x_1 \multimap x_3 \multimap \cdots \multimap x_{2n-1}$. The local comparison relation of P at this level compares the last component under $\lessdot^{P_0}$.

4. We similarly define the level $(2n+1)$ of P for $n \geq 1$. The detail is left to the reader.

Now for $\sigma = \langle x_0, x_1, \cdots, x_n \rangle \in P$, we define $\pi^P(\sigma)$ as follows:

1. If $\sigma = \langle x_0 \rangle$, $\pi^P(\sigma) = \emptyset$.
2. If $\sigma = \langle x_0, x_1 \rangle$, $\pi^P(\sigma)$ is the singleton 1-predendrogram $\{0\}$.
3. Now let $\sigma = \langle x_0, x_1, \cdots, x_{2n-1}, x_{2n} \rangle$ and $x_{2n-1} \in P_1$, $x_{2n} \in P_0$. Then $\pi^P(\sigma)$ is the 1-predendrogram of the form

$$\{0\} + \{2i+1 \mid i \in \pi^{P_0}(x_{2n})\} + \{2i+2 \mid i \in \pi^{P_1}(x_{2n-1})\},$$

   where $+$ indicates how we impose the local comparison relation at the root level, and in each component we follow the immediate successor relations, comparison rules and priority orders from $\pi^{P_0}(x_{2n})$ and $\pi^{P_1}(x_{2n-1})$ respectively.

4. We similarly define $\pi^P(\sigma)$ when $\sigma = \langle \langle x_0, x_1, \cdots, x_{2n}, x_{2n+1} \rangle$ in the form

$$\{0\} + \{2i+1 \mid i \in \pi^{P_0}(x_{2n})\} + \{2i+2 \mid i \in \pi^{P_1}(x_{2n+1})\}.$$

It is clear that P is locally well-founded iff both P_0 and P_1 are locally well-founded. Also, $b = \langle x_0, x_1, x_2, x_3, \cdots \rangle$ is an infinite chain of P iff $P_0 \vDash x_0 \multimap x_2 \multimap x_4 \multimap \cdots$ and $P_1 \vDash x_1 \multimap x_3 \multimap x_5 \multimap \cdots$. By definition of π^P, the associated thread predendrogram of b in P takes the form $1 + d_0 + d_1$, where d_i is the associated predendrogram of $\langle x_i, x_{2+i}, x_{4+i}, \cdots \rangle$ in P_i $(i = 0, 1)$. If both P_0 and P_1 are recursive, then their cell decompositions are also recursive, so P is recursive as well. The last condition is immediate by the definition. $\square$

A *2-preflower* is a 2-preptyx P such that

1. An arity of every P-term is a finite (pre)flower, and
2. If D, E are predilators such that D is an initial segment of E and if $\int^1$ is the flowerfication operator, then $P(\int^1 D)$ is an initial segment of $P(\int^1 E)$.

For a 2-predendrogram P, we can define its 2-preflowerfication $Q = \int^2 P$ as follows:

Definition 5.1.7 (ACA_0)**.** Let P be a 2-predendrogram. We define a new 2-predendrogram $\int^2 P$ as follows: We first introduce a new element $*$ not in the field of P, and we make $*$ the unique root of $\int^2 P$. $*$ has every root of P as immediate successors over $\int^2 P$. The local comparison relation of $\int^2 P$ inherits that of P.

Now let us define the priority predendrogram relation over $\int^2 P$. $\pi^{\int^2 P}(*)$ is the empty 1-predendrogram. For the other nodes, we will define $\pi^{\int^2 P}(x)$ by the flowerfication $\int^1 \pi^P(x)$, but $\int^1 \pi^P(x)$ is not a priority dendrogram since its members are not natural numbers. However, nodes of $\int^1 \pi^P(x)$ are either a unique root or a natural number coming from $\pi^P(x)$. We get $\pi^{\int^2 P}(x)$ from $\int^1 \pi^P(x)$ by replacing each m to $(m+1)$, and the root to 0.

Proposition 5.1.8 (ACA_0)**.** *If P is a 2-ptyx, then so is $\int^2 P$. In fact, for a 2-preptyx P and an infinite branch b of (a cell decomposition of) P with the associated thread pseudodilator D, the associated branch in $\int^2 P$ has the thread pseudodilator isomorphic to $\int^1 D$.*

Proof. The latter claim shows the former claim, since we know that if D is a pseudodilator, then so is $\int^1 D$. Hence, we only prove the latter claim.

Let $b\colon x_0 \multimap x_1 \multimap x_2 \multimap \cdots$ be an infinite branch of (the cell decomposition of) P. The associated infinite branch of $\int^2 P$ is $b'\colon * \multimap x_0 \multimap x_1 \multimap \cdots$. For the thread predi-

lator $D = \bigcup_{i\in\mathbb{N}} \pi^P(x_i)$ associated with b, the associated predilator of b' is isomorphic to

$$\bigcup_{i\in\mathbb{N}} \int^2(\pi^P(x_i)) = \int^2(\bigcup_{i\in\mathbb{N}} \pi^P(x_i)) = \int^2 D. \qquad \square$$

5.2 Π^1_3-prewellorder for 2-ptykes

In this section, we introduce how to compare two 2-ptykes. We first define the 2-ptyx comparison without any determinacy hypothesis, and prove that it gives a prewellorder. Using a Π^1_3-prewellorder over the collection of countable (or recursive) 2-ptykes may not be the only way to compare the Π^1_3-consequences of a theory, but using 2-ptykes gives a concrete notions to compare Π^1_3-consequences.

There are at least two ways to compare two 2-ptykes, although both rely on Moschovakis' argument for the prewellordering property of $\mathbf{\Pi}^1_3$ sets under $\mathbf{\Delta}^1_2$-determinacy. One starts from the definition of 2-ptykes and presents a way to compare a pair (P, d) of a 2-preptyx and a predilator in a Δ^1_2-way, then turns it into a comparison of two 2-ptykes. The other uses the cell decomposition of a 2-preptyx and compares threads of two 2-ptykes. I do not know if the two comparisons are equivalent, and I think the thread-based comparison is more natural than the application-based comparison.

5.2.1 Comparing 2-ptykes via applying dilators

For the first comparison, let us introduce the following collection:

$$\mathcal{P}^2 = \{(P, d) \in \mathsf{PrePtyx}^2 \times \mathsf{PreDil} \mid \neg\mathsf{Dil}(d) \vee \mathsf{WO}(P(d))\}.$$

Note that P is a 2-ptyx iff $\forall^1 D[(P,D) \notin \mathcal{P}^2]$. Now we compare two elements (P,d) and (Q,d') in $\mathcal{P}^2$. The basic idea is to compare $P(d)$ and $Q(d')$, but they may not be comparable when one of them is not well-ordered. In the case when $P(d)$ is not well-ordered, d is a pseudodilator, so we compare $\mathsf{Clim}(d)$ instead. That is, we compare $\min(\mathrm{otp}(P(d)), \mathsf{Clim}(d))$ and $\min(\mathrm{otp}(Q(d')), \mathsf{Clim}(d'))$ instead when we take $\mathsf{Clim}(d) = \infty$ when d is a dilator, and $\mathrm{otp}(A) = \infty$ when A is ill-founded. We first find a pseudodilator whose climax is $\min(\mathrm{otp}(P(d)), \mathsf{Clim}d)$:

Proposition 5.2.1 (ACA_0)**.** *Let $(P,d) \in \mathcal{P}^2$. If we take*

$$c_{P,d} = (P(d) \to \omega^*) + d$$

Here $(\alpha \to \beta)$ is an implicational predilator defined in Lemma 1.1.4. Then $c_{P,d}$ is a pseudodilator, and

$$\mathsf{Clim}(c_{P,d}) = \begin{cases} \mathrm{otp}(P(d)), & \text{if } \mathsf{WO}(P(d)), \\ \mathsf{Clim}(d), & \text{if } \neg\mathsf{Dil}(d). \end{cases} \quad \square$$

Hence $\mathcal{P}^2$ has a Σ^1_2-prewellorder given by

$$(P,d) \leq_{\mathcal{P}^2} (Q,d') \iff \mathsf{Grow}_{\Sigma^1_2}(c_{P,d}, c_{Q,d'}).$$

Now we define a 2-ptyx comparison assuming $\mathbf{\Delta}^1_2$-Determinacy by considering the following game:

Definition 5.2.2 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-Det)**.** A game $\mathcal{G}_{\mathsf{Appl}^2}(P,Q)$ for 2-preptykes P and Q is defined as follows: Player I and II play predilators over $\mathbb{N}$, which we will call d_{I} and d_{II} respectively. In each turn, each player plays finitely (possibly zero) many terms for predilators with the comparison rules for the new terms. Player I wins $\mathcal{G}_{\mathsf{Appl}^2}(P,Q)$ if we have $(Q,d_{\mathrm{II}}) <_{\mathcal{P}^2} (P,d_{\mathrm{I}})$.

Now take

$$P \trianglelefteq_{\mathsf{Appl}^2} Q \iff \text{II has a winning strategy in } \mathcal{G}_{\mathsf{Appl}^2}(P, Q).$$

The following result comes from the proof of the First Periodicity Theorem (cf. [67, 6B]):

Proposition 5.2.3 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *Let P, Q, and R be ptykes.*

1. *If we define $P \triangleleft_{\mathsf{Appl}^2} Q$ by*

$$P \triangleleft_{\mathsf{Appl}^2} Q \iff P \trianglelefteq_{\mathsf{Appl}^2} Q \wedge Q \not\trianglelefteq_{\mathsf{Appl}^2} P$$

   then $P \triangleleft_{\mathsf{Appl}^2} Q$ iff Player I has a winning strategy for $\mathcal{G}_{\mathsf{Appl}^2}(Q, P)$.

2. *Either $P \trianglelefteq_{\mathsf{Appl}^2} Q$ or $Q \trianglelefteq_{\mathsf{Appl}^2} P$ holds.*

3. *$P \trianglelefteq_{\mathsf{Appl}^2} Q$ and $Q \trianglelefteq_{\mathsf{Appl}^2} R$ implies $P \trianglelefteq_{\mathsf{Appl}^2} R$.*

4. *(Σ^1_3-AC_0) $\trianglelefteq_{\mathsf{Appl}^2}$ is well-founded.*

Proof. The first part is followed by modifying the proof of [67, 6B.1, Lemma 4]. The second and the third follow immediately. For the last one, suppose that we have a sequence

$$P_0 \rhd_{\mathsf{Appl}^2} P_1 \rhd_{\mathsf{Appl}^2} P_2 \rhd_{\mathsf{Appl}^2} \cdots.$$

Then for each n, Player I has a winning strategy for $\mathcal{G}_{\mathsf{Appl}^2}(P_n, P_{n+1})$. Since the statement '$\sigma$ is a winning strategy for I in $\mathcal{G}_{\mathsf{Appl}^2}(P_n, P_{n+1})$' is Π^1_2, we can choose a sequence $\langle \sigma_n \mid n \in \mathbb{N} \rangle$ of winning strategies by Σ^1_3-AC_0 such that σ_n is a winning strategy for Player I in $\mathcal{G}_{\mathsf{Appl}^2}(P_n, P_{n+1})$. From σ_n, we can find predilators d_n such that $(P_n, d_n) >_{\mathcal{P}^2} (P_{n+1}, d_{n+1})$ holds for each n, contradicting the last statement of Proposition 5.1.5. □

We know that if a linear order embeds into a well-order, then the linear order is well-ordered. The following lemma is a 2-preptyx analogue of the previous statement:

Lemma 5.2.4 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *Suppose that P is a 2-preptyx, Q is a 2-ptyx, and $P \trianglelefteq_{\mathsf{Appl}^2} Q$ holds. Then P is also a 2-ptyx.*

Proof. Let us fix a winning strategy for Player II in the game $\mathcal{G}_{\mathsf{Appl}^2}(P, Q)$. Now suppose that d is any countable dilator, and Player I plays d appropriately in the game $\mathcal{G}_{\mathsf{Appl}^2}(P, Q)$. Player II responds to a predilator d' by playing according to the fixed winning strategy, so we have $\mathsf{Grow}_{\Pi^1_2}(c_{P,d}, c_{Q,d'})$. Since Q is a 2-ptyx, $c_{Q,d'}$ is always a pseudodilator no matter what d' is. By the definition of $\mathsf{Grow}_{\Pi^1_2}$, a predilator $c_{P,d} = (P(d) \to \omega^*) + d$ is also a pseudodilator, but d being a dilator implies $(P(d) \to \omega^*)$ must be a pseudodilator. By Lemma 1.1.4, $P(d)$ is a well-order. This shows P is a 2-ptyx. □

5.2.2 Comparing 2-ptykes via their threads

In this subsection, we define an alternative way to compare two 2-ptykes, which is based on the cell-decomposition of 2-preptykes.

Observe that a 2-preptyx P is a 2-ptyx iff its cell-decomposition $\mathrm{Cell}(P)$ is locally well-founded and every thread predilator of $\mathrm{Cell}(P)$ is not a dilator. We may understand a 2-ptyx as a tree whose infinite branch encodes a pseudodilator. We are tempted to gauge the complexity of a 2-ptyx by considering the supremum of $\mathsf{Clim}(D)$ for a thread pseudodilator D of its cell-decomposition $\mathrm{Cell}(P)$, but it turns out that this is an incorrect way to assign the ordinal complexity. We discuss the reason in subsection 5.2.3 in depth, but roughly speaking, we also have to consider 'hidden threads' of a given 2-ptyx that

does not exist in the current universe V but may exist in an extension of V by a genericity iteration. Hence, we only define how to compare two 2-ptykes under $\mathbf{\Delta}^1_2$-Determinacy.

Definition 5.2.5 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-Det)**.** Let P and Q be two 2-preptykes whose fields are natural numbers. Consider the game $\mathcal{G}_{\vartheta^2}(P, Q)$ defined as follows: Consider the cell-decomposition $\mathrm{Cell}(P)$ and $\mathrm{Cell}(Q)$ of P and Q respectively. We can think of elements of $\mathrm{Cell}(P)$ and $\mathrm{Cell}(Q)$ as natural numbers.

In Stage 0, both players play a node of length 0 from $\mathrm{Cell}(P)$ and $\mathrm{Cell}(Q)$ respectively. In Stage $m \geq 1$, Player I plays a node of length m extending what they played in Stage $(m-1)$. Then Player II plays a node of length m extending what she played in Stage $(m-1)$. One of the players loses the game if they or she is the first player who cannot extend the node further. Otherwise, let us consider the associated thread predilators d_{I} and d_{II} from the infinite branches both players played. Player I wins if and only if $\mathsf{Clim}(d^*_{\mathrm{I}}) > \mathsf{Clim}(d^*_{\mathrm{II}})$.

Note that we can recursively compute d_{I} and d_{II} from the play of each player. Moreover, $\mathsf{Clim}(d^*_{\mathrm{I}}) > \mathsf{Clim}(d^*_{\mathrm{II}})$ is equivalent to a Σ^1_2-assertion $\neg\mathsf{Grow}_{\Pi^1_2}(d^*_{\mathrm{I}}, d^*_{\mathrm{II}})$, which is also Π^1_2 by Lemma 5.1.3. Hence $\mathcal{G}_{\vartheta^2}(P, Q)$ is a Δ^1_2-game with parameter P and Q. If we assume $\mathbf{\Delta}^1_2$-determinacy, then one of the players has the winning strategy. Hence, we can define

$$P \trianglelefteq_{\vartheta^2} Q \iff \text{Player II has a winning strategy in } \mathcal{G}_{\vartheta^2}(P, Q).$$

The proof of the following proposition is more or less similar to that of Proposition 5.2.3, so the proof is left to the reader.

Proposition 5.2.6 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-Det)**.** *Let P, Q, and R be 2-ptykes.*

1. *If we define* $P \lhd_{\vartheta^2} Q$ *by*

$$P \lhd_{\vartheta^2} Q \iff P \trianglelefteq_{\vartheta^2} Q \wedge Q \not\trianglelefteq_{\vartheta^2} P$$

   then $P \lhd_{\vartheta^2} Q$ *iff Player I has a winning strategy for* $\mathcal{G}(Q, P)$.

2. *Either* $P \trianglelefteq_{\vartheta^2} Q$ *or* $Q \trianglelefteq_{\vartheta^2} P$ *holds.*

3. $P \trianglelefteq_{\vartheta^2} Q$ *and* $Q \trianglelefteq_{\vartheta^2} R$ *implies* $P \trianglelefteq_{\vartheta^2} R$.

4. $(\Sigma^1_3\text{-}\mathsf{AC}_0) \trianglelefteq_{\vartheta^2}$ *is well-founded.* □

The following lemma is a thread comparison version of Lemma 5.2.4. Unlike Lemma 5.2.4, we require P to be locally well-founded since the thread game only certifies thread predilators to be pseudodilators. Such a restriction is not problematic since most 2-preptykes we will see are locally well-founded.

Lemma 5.2.7 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2\text{-}\mathsf{Det}$)**.** *Suppose that* P *is a locally well-founded 2-preptyx and* Q *is a 2-ptyx, and* $P \trianglelefteq_{\vartheta^2} Q$ *holds. Then* P *is also a 2-ptyx.*

Proof. From $P \trianglelefteq_{\vartheta^2} Q$, Player II has a winning strategy in $\mathcal{G}_{\vartheta^2}(P, Q)$. Now suppose that d_{I} is a thread predilator of P. By playing d_{I} in $\mathcal{G}_{\vartheta^2}(P, Q)$ by Player I and counterplaying under Player II's winning strategy, we can find a thread predilator d_{II} of Q such that $\mathsf{Grow}_{\Pi^1_2}(d^*_{\mathrm{I}}, d^*_{\mathrm{II}})$. Since Q is a 2-ptyx, d_{II} must be a pseudodilator, and if $d^*_{\mathrm{II}}(\alpha)$ is ill-founded for a well-order α, then $d^*_{\mathrm{I}}(\alpha)$ is also ill-founded. This shows every thread of P is not a dilator, so P is a 2-ptyx. □

The following propositions indicate a 2-flowerfication operator is a 'successor operator' for 2-ptykes under $\trianglelefteq_{\vartheta^2}$.

Proposition 5.2.8 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2\text{-}\mathsf{Det}$)**.** *Let* P *be a 2-ptyx. Then* $P \lhd_{\vartheta^2} \int^2 P$.

Proof. It suffices to show that Player I has a winning strategy in $\mathcal{G}_{\vartheta^2}(\int^2 P, P)$, which is given as follows: Player I first plays the root of the cell decomposition of $\int^2 P$. After this stage, Player I exactly plays what Player II played. If Player II plays the associated thread pseudodilator d, then Player I plays the associated thread pseudodilator $\int^1 d$, and we have $\mathsf{Clim}(d^*) < \mathsf{Clim}(\int^1 d^*)$. □

Proposition 5.2.9 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *Let P and Q be 2-ptykes. If $P \lhd_{\vartheta^2} Q$, then $\int^2 P \trianglelefteq_{\vartheta^2} Q$.*

Proof. Suppose that $P \lhd_{\vartheta^2} Q$ holds, so Player I has a winning strategy in the game $\mathcal{G}_{\vartheta^2}(Q, P)$. Now we claim that Player II has a winning strategy in $\mathcal{G}_{\vartheta^2}(\int^2 P, Q)$. To distinguish the two games, we call $\mathcal{G}_{\vartheta^2}(Q, P)$ the prior game, and $\mathcal{G}_{\vartheta^2}(\int^2 P, Q)$ the latter game.

In the latter game $\mathcal{G}_{\vartheta^2}(\int^2 P, Q)$, Player I must start with the unique top node in the cell decomposition of $\int^2 P$. Now we move the plays from the latter game to that of the prior game by omitting the first play by Player I and switching the players; That is,

- Player I in the prior game precisely follows Player II in the latter game.
- Player II in the prior game precisely follows Player I in the latter game, except for the first play, which is the unique top node in $\mathsf{Cell}(\int^2 P)$.

Player I's strategy in the prior game $\mathcal{G}_{\vartheta^2}(Q, P)$ determines Player II's strategy in the latter game. Now we claim that it is a winning strategy: Suppose that in the latter game, each player played threads $\theta_{\int^2 P}$ and θ_Q respectively. We can find a thread θ_P of P such that $\theta_{\int^2 P} = \int^1 \theta_P$ and $\mathsf{Clim}(\theta_Q) > \mathsf{Clim}(\theta_P)$. We have $\mathsf{Clim}(\theta^*_Q) \geq \mathsf{Clim}(\theta^*_{\int^2 P}) = \mathsf{Clim}(\theta^*_P) + 1$, so Player II wins in the latter game. □

5.2.3 What is compared by 2-ptyx comparisons?

We defined a Π^1_3-prewellorder over the class of countable 2-ptykes under $\mathbf{\Delta}^1_2$-determinacy. Unlike the Π^1_1 and Σ^1_2 cases, however, we have not defined an ordinal characteristic for the countable 2-ptyx we compare; In the Π^1_1 case, we compare two well-orders by comparing their ordertypes. In the Σ^1_2 case, we compare two pseudodilators by comparing their climaxes. In the Π^1_3 case, we do not know which ordinals are compared. I do not (and cannot) answer which ordinals are compared under 2-ptyx comparisons. Despite that, let me present my observations and opinions on this problem with the hope that they shed light on the problem.

Kechris [49] stated that we can compare two countable 2-ptykes by applying a universal dilator Ω^1 obtained from sharps; To be precise, Ω^1 is a universal flower and not a universal dilator, but we can derive an associated universal dilator by considering its derivative $\partial\Omega^1$ instead. If we assume every real has a sharp, then $\partial\Omega^1$ embeds every countable dilator. Hence a countable 2-preptyx P is a 2-ptyx if $P(\partial\Omega^1)$ is well-ordered. $\partial\Omega^1$ has ω_1 many connected components, so if $P(\partial\Omega^1)$ is ill-founded, then there is a countable initial segment D of $\partial\Omega^1$ such that $P(D)$ is ill-founded. That means,

$$P \text{ is a 2-ptyx } \iff \mathsf{WO}(P(\partial\Omega^1)).$$

Hence, we may think $P(\partial\Omega^1)$ as a degree of 2-ptyxness of P. We can compare $P(\partial\Omega^1)$ and $Q(\partial\Omega^1)$ for two countable 2-ptykes P and Q, and Kechris stated without proof that the relation $P(\partial\Omega^1) \leq Q(\partial\Omega^1)$ is Π^1_3 assuming $\mathbf{\Delta}^1_2$-determinacy. I do not know how to prove Kechris' assertion, which may be too optimistic to be true. However, I guess his assertion may hold if we work in the AD-world:

Question 5.2.10. Working over $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, does the relation $P(\partial\Omega^1) \leq Q(\partial\Omega^1)$ provide a Π^1_3-prewellorder over the set of countable 2-ptykes?

Kechris' analysis of homogeneous trees and scales [48] might be relevant to the previous question. Even if Kechris' assertion may be incorrect, its orientation — comparing two 2-ptykes by applying a reasonably large dilator — may still be correct. Before addressing what a reasonably large dilator should be, let me talk about how inner model theorists compare two Π^1_3-statements. For technical convenience, let us only consider how to compare two lightface Π^1_3-statements (or equivalently, recursive 2-ptykes.)

Steel [91, Theorem 4.12] proved that working over ZFC plus $M_1^\sharp$ exists, every Π^1_3-statement ϕ is equivalent to a Σ_1-statement ϕ^* over $M_1|\delta^1_3$. We can use the least $\alpha < \delta^1_3$ satisfying $M_1|\alpha \vDash \phi^*$ as an 'ordinal complexity' of ϕ. Let me note that Aguilera and Pakhomov [5] rely on Steel's method to assign an ordinal complexity of a Π^1_3-assertion, so the way [5] assigns an ordinal to a Π^1_3-assertion is not different from that of Steel. Aguilera-Pakhomov also proved [5, Theorem 36] that the set of ordinal complexity of a Π^1_3-sentence is cofinal in δ^1_3.

This way of assigning an ordinal complexity of a Π^1_3-statement is not random, but closely related to the prewellordering property of Π^1_3: Steel stated that his theorem is an elaboration of Woodin's proof of the prewellordering property of Π^1_3, and Schlutzenberg [86, Definition 2.11] also uses an initial segment of $M_1^\sharp$ to define a prewellorder over a Π^1_3 subset of $\mathbb{N}$. Due to the promised connection between the linear comparison of consequences and the prewellordering property, using an initial segment of $M_1|\delta^1_3$ looks like a correct way to compare Π^1_3-consequences.

While the inner model theorists' method is correct, the connection between their comparison and 2-ptykes is less clear. In the Π^1_1 case, we extract a well-order (or prewellorder) from a given Π^1_1-assertion, and then we can directly compare the well-orders. In the inner model theorists' method, they use genericity iteration to capture the foremost universal quantifier in a Π^1_3-assertion; For example, Steel proved that if $\psi(X)$ is a

Σ^1_2-formula,

$$\forall^1 X\psi(X) \iff \exists\alpha < \delta^1_3[M_1|\alpha \vDash \text{There is a Woodin cardinal } \delta \text{ and } \Vdash_{\mathbb{B}_\delta} \psi(\dot{X})],$$

where $\mathbb{B}_\delta$ is Woodin's extender algebra at δ, and $\dot{X}$ is the generic real name of $\mathbb{B}_\delta$. Then we use the least α with the reasonable condition[1] as a complexity of a Π^1_3-formula $\forall^1 X\psi(X)$. Woodin's extender algebra is impossible to describe without a Woodin cardinal, which bars characterizing the ordinal complexity of a Π^1_3-sentence in the language of dilators and ptykes by naively mimicking Steel's proof.

Perhaps describing the ordinal complexity of a 2-ptyx without any inner model theoretic jargon may not be feasible. Inner model theorists use $M_1^\sharp$ to capture the ordinal complexity of a Π^1_3-sentence. The dilator counterpart of $M_1^\sharp$ should follow by applying the ongoing construction of a measurable dilator from a Woodin cardinal and a measurable above [7] to $M_1^\sharp$. I expect the resulting dilator to satisfy the following:

Conjecture 5.2.11. Suppose that $M_1^\sharp$ exists. Then we have a flower Ω, an inner model M with an M-cardinal δ satisfying the following: For every $\alpha < \delta$, we can find a family of measures

$$\langle \mu_d \mid d \text{ is a finite flower with no constant terms} \rangle$$

such that

1. Each μ_d is α-M-*additive* measure on Ω^d in the sense that for every $\beta < \alpha$ and $\langle X_\xi \mid \xi < \beta \rangle \in M$ such that $X_\xi \in \mu^d$ for every $\xi < \beta$, we have $\bigcap_{\xi<\beta} X_\xi \in \mu_d$.
2. The measure family is coherent and ω_1-complete in the usual sense. In particular, $\mathrm{Ult}(M, D) = \varinjlim_{d\subseteq D} \mathrm{Ult}(M^d, \mu^d)$ is well-founded for a countable flower D with no constant terms even if D is not in M.

[1] More precisely, we demand $M_1|\alpha$ projects to ω.

If we start from $M_1^\sharp$, Ω should be a flower induced by iterating the top measure of $M_1^\sharp$, M should be M_1, and δ should be the unique Woodin cardinal in M_1. I suspect that if a flower Ω admits an inner model M and an M-cardinal δ satisfying the above two conditions, then M thinks δ is a Woodin cardinal. If it is, there might be no reason to refuse a Woodin cardinal to calculate the ordinal complexity of a Π^1_3-statement.

However, there are examples of getting the ordinal characteristic of dilators or ptykes by applying a large object. Chapter 3 is an example. Also, Girard-Ressayre [31, 5.B] proved that if we let π^1_n to be the supremum of the ordertype of Π^1_n-prewellorders over a Π^1_n-definable subset of $\mathbb{N}$ and Ξ_n to be the ordered sum of every recursive n-ptyx, then $\Xi_n(\Xi_{n-1}) = \pi^1_n$ for every $n \geq 2$. Hence, let me raise the following question, which asks if we can find a 'master dilator' Ω providing the ordinal characteristic of a 2-ptyx by application:

Question 5.2.12 (ZFC + $\mathbf{\Delta}^1_2$-Det)**.** Can we find a dilator Ω satisfying the following?

1. Ω embeds every countable dilator.
2. For each real X, $\{P(\Omega) \mid P \text{ is an } X\text{-recursive 2-ptyx}\} = \delta^1_3(X)$, where $\delta^1_3(X)$ is the supremum of the ordertype of $\Delta^1_3[X]$-well-orders.
3. For two 2-ptykes P and Q, $P(\Omega) \leq Q(\Omega)$ is equivalent to a Π^1_3-assertion. (It is plausible if $P(\Omega) \leq Q(\Omega)$ is equivalent to $P \trianglelefteq_{\vartheta^2} Q$.)

Also, can Ω be a differentiation of the direct limit of flowers induced from $M_1^\sharp(A)$ for all real A?

If we just accept an inner model with a single Woodin cardinal and extender algebra, it might be possible to characterize which ordinal is compared under $\trianglelefteq_{\vartheta^2}$:

Conjecture 5.2.13 (ZFC + $\forall r \in \mathbb{R} \exists M_1^\sharp(r)$)**.** For a countable 2-ptyx P, let us define its *2-climax* $\mathsf{Clim}_2(P)$ as the least ordinal η satisfying the following: If δ is the unique

Woodin cardinal in $M_1(P)$, $\mathbb{B}_\delta$ is Woodin's extender algebra computed in $M_1(P)$, and $\dot{X}$ is a generic real name for $\mathbb{B}_\delta$, then

$$M_1(P) \vDash \eta \geq \delta \wedge \Vdash_{\mathbb{B}_\delta} \text{“If } \dot{X} \text{ is a thread predilator of } \check{P}\text{, then } \mathsf{Clim}(\dot{X}) \leq \eta\text{.”}$$

Then $P \trianglelefteq_{\vartheta^2} Q$ iff $\mathsf{Clim}_2(P) \leq \mathsf{Clim}_2(Q)$.

5.3 Case Π^1_3

In this section, let us generalize the results in [103] and [39] to Π^1_3 pointclasses. Throughout this section, we assume $\mathbf{\Delta}^1_2$-Determinacy in the metatheory. For two 2-ptykes P and Q, let us write $P \equiv_{\vartheta^2} Q$ when $P \trianglelefteq_{\vartheta^2} Q$ and $Q \trianglelefteq_{\vartheta^2} P$. We first establish basic properties of $|T|_{\Pi^1_3}$ for a Π^1_3-sound recursive extension T of $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$. We can see that every T-provably recursive 2-ptyx is $\lhd_{\vartheta^2}$-less than the proof-theoretic 2-ptyx of T:

Lemma 5.3.1 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *Let T be a Π^1_3-sound extension of $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$ and Q be a recursive 2-ptyx. If $T \vdash \mathsf{Ptyx}^2(Q)$, then $Q \lhd_{\vartheta^2} |T|_{\Pi^1_3}$.*

Proof. Q embeds to $|T|_{\Pi^1_3}$ so Player II can win the game $\mathcal{G}_{\vartheta^2}(Q, |T|_{\Pi^1_2})$ by playing the same thread Player I plays. It shows $Q \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}$. By Proposition 5.1.8 and Proposition 5.2.8, we have $T \vdash \mathsf{Ptyx}^2(\int^2 Q)$ and $Q \lhd_{\vartheta^2} \int^2 Q \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}$. □

It is well-known that adding a true Σ^1_1-statement to the theory does not change its proof-theoretic ordinal. The following lemma says adding a true Σ^1_3-statement does not change the proof-theoretic 2-ptyx under $\equiv_{\vartheta^2}$:

Lemma 5.3.2 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *Suppose that T is a Π^1_3-sound extension of $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$ and R is a recursive 2-pseudoptyx. Then $|T|_{\Pi^1_3} \equiv_{\vartheta^2} |T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}$.*

Proof. We first argue $|T|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}$: Clearly, $T + \neg\mathsf{Ptyx}^2(R)$ proves every theorem of T, so there is an embedding from $|T|_{\Pi^1_3}$ to $|T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}$. This means, Player II can win the game $\mathcal{G}_{\vartheta^2}(|T|_{\Pi^1_3}, |T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3})$ by exactly playing Player I's play.

Now let us prove $|T+\neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}$. That is, we want Player II's winning strategy of $\mathcal{G}_{\vartheta^2}(|T+\neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}, |T|_{\Pi^1_3})$. For (codes for) two recursive 2-preptykes P and Q, we can effectively compute (the code for) $P \vee Q$. Now let f be a recursive function taking (a code for) a 2-preptyx Q, and returning (a code for) $R \vee Q$. Then consider the following strategy for Player II in $\mathcal{G}_{\vartheta^2}(|T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}, |T|_{\Pi^1_3})$: Since R is not a 2-ptyx, we can fix its thread b whose associated predilator d_b is a dilator. We observe that $|T + \neg\mathsf{Ptyx}^2(R)|_{\Pi^1_3}$ is an ordered sum of recursive 2-preptykes, and by looking at the index of the sum, Player II can see where the thread Player I will play comes from, say Q. So in Stage 0, Player II sees this Q, and computes $f(Q) = R \vee Q$. In further stages, Player II plays in a way that her thread predilator from $R \vee Q$ embeds Player I's thread predilator from Q. After the play, if Player I plays a thread whose associated predilator is d, Player II will play a thread predilator of the form $1 + d_b + d$. Since $\mathsf{Clim}\big((1 + d_b + d)^*\big) \geq \mathsf{Clim}(d^*)$, Player II wins the game. □

Lastly, we can see that $|T|_{\Pi^1_3}$ is the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx bounding every T-provable recursive 2-ptyx. Its proof requires Σ^1_3-AC_0. The previous statements are also provable with the application-based comparison $\trianglelefteq_{\mathsf{Appl}^2}$, but the following statement works only for the thread comparison. The main reason for this is that in application-based comparison, the sum of 2-ptykes literally gives the sum of compared values. In a thread-based comparison, the sum of 2-ptykes gives the supremum of the climaxes of the thread predilators we compare.

Lemma 5.3.3 ($\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det} + \Sigma^1_3$-$\mathsf{AC}_0$)**.** *Let T be a Π^1_3-sound extension of* $\mathsf{ACA}_0 +$

$\mathbf{\Delta}^1_2$-Det. *Suppose that $P \trianglelefteq_{\vartheta^2} Q$ for every recursive T-provably 2-ptyx P and a 2-preptyx Q. Then $|T|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} Q$.*

Proof. For each T-provably recursive 2-ptyx P, let us choose a winning strategy σ_P for Player II in $\mathcal{G}_{\vartheta^2}(P, Q)$ by applying Σ^1_3-AC$_0$. Then consider the following strategy in $\mathcal{G}_{\vartheta^2}(|T|_{\Pi^1_3}, Q)$ for Player II: In Stage 0, Player II can see where the thread will come from, say P. In further stages, Player II conforms σ_P. □

The contraposition of Lemma 5.3.3 gives the following:

Corollary 5.3.4 (ACA$_0$ + $\mathbf{\Delta}^1_2$-Det + Σ^1_3-AC$_0$)**.** *Suppose that T and Q are given as in Lemma 5.3.3. If $Q \triangleleft_{\vartheta^2} |T|_{\Pi^1_3}$, then there is a recursive T-provably 2-ptyx P such that $Q \triangleleft_{\vartheta^2} P$.* □

Let me remark that we have not used the definability condition on T when proving the previous theorems. That is, T can be any Π^1_3-sound theory extending ACA$_0$ + $\mathbf{\Delta}^1_2$-Det. In many cases, we apply the previous results over a sufficiently strong metatheory like ZFC + $\mathbf{\Delta}^1_2$-Det.

The following theorem says comparing the Π^1_3-consequences of theories modulo true Σ^1_3-sentences is equivalent to comparing proof-theoretic 2-ptykes under $\trianglelefteq_{\vartheta^2}$:

Theorem 5.3.5. *For Π^1_3-sound extensions S and T of* ACA$_0$ + $\mathbf{\Delta}^1_2$-Det*, we have*

$$S \subseteq^{\Sigma^1_3}_{\Pi^1_3} T \iff |S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}.$$

Proof. Let us prove the left-to-right direction first. Suppose that P is a 2-ptyx such that $P \triangleleft_{\vartheta^2} |S|_{\Pi^1_3}$. By Lemma 5.3.3, we may assume that P is a recursive 2-ptyx such that $T \vdash \mathrm{Ptyx}^2(P)$. By the assumption, there is a recursive 2-pseudoptyx F such that

$T + \neg\mathsf{Ptyx}^2(F) \vdash \mathsf{Ptyx}^2(P)$. Hence, by Lemma 5.3.2,

$$P \lhd_{\vartheta^2} |T + \neg\mathsf{Ptyx}^2(F)|_{\Pi^1_3} \equiv_{\vartheta^2} |T|_{\Pi^1_3}.$$

This shows $|S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}$.

Conversely, suppose that $|S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3}$ holds, and let $S \vdash^{\Sigma^1_3} \mathsf{Ptyx}^2(P)$. We want to show that $T \vdash^{\Sigma^1_3} \mathsf{Ptyx}(P)$. Let F be a recursive pseudoptyx such that $S + \neg\mathsf{Ptyx}^2(F) \vdash \mathsf{Ptyx}^2(P)$. This shows

$$P \lhd_{\vartheta^2} |S + \neg\mathsf{Ptyx}^2(F)|_{\Pi^1_3} \equiv_{\vartheta^2} |S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3},$$

so $P \lhd_{\vartheta^2} |T|_{\Pi^1_3}$. Hence, by Corollary 5.3.4, we can find a recursive 2-ptyx R such that $T \vdash \mathsf{Ptyx}^2(R)$ and $P \lhd_{\vartheta^2} R$. Note that $P \lhd_{\vartheta^2} R$ is a true Σ^1_3-sentence, and

$$\sigma \equiv \text{“}P \text{ is locally well-founded”}$$

is a true Π^1_1-sentence. Hence by internalizing Lemma 5.2.7, we have

$$T + (P \lhd_{\vartheta^2} R) + \sigma \vdash \mathsf{Ptyx}^2(P).$$

This shows $T \vdash^{\Sigma^1_3} \mathsf{Ptyx}^2(P)$, as desired. □

Note that the previous proof uses Lemma 5.3.3, but only externally in S and T. Hence, we do not need to fortify S and T to include Σ^1_3-AC_0. Now let us prove the second equivalence, connecting the proof-theoretic 2-ptyx comparison and Π^1_3-reflection comparison. Here, $\mathsf{Bool}(\Sigma^1_2)$ is the pointclass of Boolean combinations of lightface Σ^1_2-sets. Equivalently, $\mathsf{Bool}(\Sigma^1_2)$ is a collection of sets defined by formulas of the form $\phi_0 \to \phi_1$ for Σ^1_2-formulas ϕ_0 and ϕ_1.

Theorem 5.3.6. *For* $\mathsf{Bool}(\Sigma^1_2)$*-definable* Π^1_3*-sound extensions* S *and* T *of* $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$*, we have*

$$|S|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |T|_{\Pi^1_3} \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_3} \Pi^1_3\text{-}\mathsf{RFN}(T) \to \Pi^1_3\text{-}\mathsf{RFN}(S). \tag{5.3}$$

Furthermore, we have

$$|S|_{\Pi^1_3} \lhd_{\vartheta^2} |T|_{\Pi^1_3} \iff T \vdash^{\Sigma^1_3} \Pi^1_3\text{-}\mathsf{RFN}(S). \tag{5.4}$$

Proof. Let us prove the left-to-right direction of (5.3). Suppose that we are given $S \subseteq^{\Sigma^1_3}_{\Pi^1_3} T$. Then the following is true:

$$\theta \equiv \forall \phi \in \Pi^1_3(\mathsf{Prv}_S(\phi) \to \mathsf{Prv}^{\Sigma^1_3}_T(\phi)).$$

θ itself is not Σ^1_3 but takes the form $\forall^0\Sigma^1_3$. We can find a Σ^1_3-formula θ' satisfying the following:

1. $\mathsf{ACA}_0 \vdash \theta' \to \theta$,
2. $\Sigma^1_3\text{-}\mathsf{AC}_0 \vdash \theta \to \theta'$.

Hence θ' is also true, and the following is clear:

$$\mathsf{ACA}_0 + \theta' \vdash \Pi^1_3\text{-}\mathsf{RFN}(T) \to \Pi^1_3\text{-}\mathsf{RFN}(S).$$

Conversely, suppose that $|S|_{\Pi^1_3} \rhd_{\vartheta^2} |T|_{\Pi^1_3}$ holds. We claim that $S \vdash^{\Sigma^1_3} \Pi^1_3\text{-}\mathsf{RFN}(T)$. Given this claim, we can easily see that $\mathsf{ACA}_0 \nvdash^{\Sigma^1_3} \Pi^1_3\text{-}\mathsf{RFN}(T) \to \Pi^1_3\text{-}\mathsf{RFN}(S)$. By Corollary 5.3.4, we can find an S-provably recursive 2-ptyx P such that $|T|_{\Pi^1_3} \lhd_{\vartheta^2} P$. Hence, the following statement is true:

$$\theta :\equiv \forall^0 Q \in \mathsf{RecPrePtyx}^2\big[\mathsf{Prv}_T(\mathsf{Ptyx}^2(Q)) \to Q \lhd_{\vartheta^2} P\big].$$

Again, we can find a true Σ^1_3-sentence θ' such that ACA_0 proves $\theta' \to \theta$. By the Π^1_3-completeness of 2-ptykes and the internalized Lemma 5.2.7, we get $S + \theta \vdash \Pi^1_3\text{-}\mathsf{RFN}(T)$.

Now we prove (5.4). We proved one direction of the equivalence. To see the unproven direction, suppose that $T \vdash^{\Sigma^1_3} \Pi^1_3\text{-}\mathsf{RFN}(S)$ but $|T|_{\Pi^1_3} \trianglelefteq_{\vartheta^2} |S|_{\Pi^1_3}$ holds. We have $T \subseteq^{\Sigma^1_3}_{\Pi^1_3}$

S, and Π^1_3-$\mathsf{RFN}(S)$ is ACA_0-provably equivalent to a Π^1_3-statement. Hence, $S + \sigma \vdash \Pi^1_3$-$\mathsf{RFN}(S)$ for some true Σ^1_3-sentence σ, so we get

$$S + \sigma \vdash \Pi^1_3\text{-}\mathsf{RFN}(S + \sigma).$$

However, no Π^1_3-sound Σ^1_3-definable $S \supseteq \mathsf{ACA}_0$ cannot prove its own Π^1_3-reflection by Proposition A.6.4, a contradiction. □

5.4 3-Pseudoptykes and 3-Growth comparison

In this section, we make a preparation before proving the characterization of Σ^1_4-consequences comparison. One of the main issues is that we do not have an ordinal characteristic representing Σ^1_4-consequences of a theory; If we have a faithful notion of '2-climax' $\mathsf{Clim}^2(P)$, an ordinal representing a complexity of a 2-ptyx P, then we should be able to define a 3-climax of a 3-pseudoptykes as

$$\mathsf{Clim}^3(Q) = \min\{\mathsf{Clim}^2(P) \mid \neg\mathsf{WO}(Q(P))\}.$$

Then we could define, for a Σ^1_4-sound theory T extending $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$,

$$s^1_4(T) = \sup\{\mathsf{Clim}^3(Q) \mid T \vdash \neg\mathsf{Ptyx}^3(Q)\}.$$

Unfortunately, we do not have 2-climaxes, so there is no ordinal characteristic $s^1_4(T)$ representing the Σ^1_4-consequences of T. Hence, we take a different approach: The main idea is that we pretend $\trianglelefteq_{\vartheta^2}$-least 2-ptyx π making $P(\pi)$ ill-founded as a "3-climax" of P, and compare the two least such 2-ptykes to compare two 3-pseudoptykes. We start from the following lemma:

Lemma 5.4.1 (Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *For a 3-pseudoptyx P, we can find a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx π making $P(\pi)$ ill-founded, in the sense that*

$$\neg\mathsf{WO}(P(\pi)) \wedge \forall^1\rho \lhd_{\vartheta^2} \pi\mathsf{WO}(P(\rho)). \tag{5.5}$$

Moreover, a 2-ptyx π satisfying (5.5) is unique up to $\equiv_{\vartheta^2}$.

Proof. Fix a 3-pseudoptyx P. Suppose the contrary that for every 2-ptyx π such that $\neg\mathsf{WO}(P(\pi))$, we can find a 2-ptyx $\rho \lhd_{\vartheta^2} \pi$. That is,

$$\forall^1\pi\exists^1\rho\big[\neg\mathsf{WO}(P(\pi)) \to \neg\mathsf{WO}(P(\rho)) \wedge \rho \lhd_{\vartheta^2} \pi\big].$$

The bracketed formula is equivalent to a Σ^1_3-formula, so by Σ^1_3-DC_0, we have a sequence of 2-preptykes $\langle \rho_n \mid n \in \mathbb{N}\rangle$ such that $\rho_0 = \pi$ and

$$\rho_0 \rhd_{\vartheta^2} \rho_1 \rhd_{\vartheta^2} \rho_2 \rhd_{\vartheta^2} \cdots.$$

It contradicts Proposition 5.2.6 that $\trianglelefteq_{\vartheta^2}$ is well-founded.

For the furthermore part, suppose that π and π' are two 2-ptykes satisfying (5.5). We have either $\pi \lhd_{\vartheta^2} \pi'$, $\pi \rhd_{\vartheta^2} \pi'$, or $\pi \equiv_{\vartheta^2} \pi'$, but the first two imply the contradiction. □

Now let us describe how to compare two 3-pseudoptykes over Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det. Let P and Q be two 3-pseudoptykes, and π_P and π_Q be $\trianglelefteq_{\vartheta^2}$-least 2-ptykes making P and Q ill-founded respectively. We can see that

$$\pi_P \trianglelefteq_{\vartheta^2} \pi_Q \iff \exists^1\rho \in \mathsf{Ptyx}^2\big[\rho \trianglelefteq_{\vartheta^2} \pi_Q \wedge \mathsf{WO}(P(\rho))\big].$$

Hence $\pi_P \trianglelefteq_{\vartheta^2} \pi_Q$ is equivalent to

$$\forall^1\pi\Big[\big[\mathsf{Ptyx}^2(\pi) \wedge \neg\mathsf{WO}(Q(\pi)) \wedge \forall^1\rho \in \mathsf{LCPrePtyx}^2(\rho \lhd_{\vartheta^2} \pi \to \mathsf{WO}(Q(\pi)))\big] \\ \to \exists^1\rho \in \mathsf{LCPrePtyx}^2\big[\rho \trianglelefteq_{\vartheta^2} \pi \wedge \mathsf{WO}(P(\rho))\big]\Big], \quad (5.6)$$

where $\mathsf{LCPrePtyx}^2$ is the class of all locally well-founded 2-preptykes, which is a Π^1_1-class of reals. Here, we do not need to require ρ to be a 2-ptyx; ρ will be a 2-ptyx

by Lemma 5.2.7. Let us check the complexity of (5.6): Note that $\lhd_{\vartheta^2}$ and $\trianglelefteq_{\vartheta^2}$ are $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2$-Det-provably Δ^1_3, so the complexity of (5.6) is

$$\forall^1[[\Pi^1_3 \wedge \Sigma^1_1 \wedge \forall^1(\Pi^1_1 \to \Delta^1_3 \to \Pi^1_1)] \to \exists^1[\Pi^1_1 \to \Delta^1_3 \to \Pi^1_1]],$$

which is $\forall^1[\Pi^1_3 \to \Sigma^1_3]$. Hence (5.6) is equivalent to a Π^1_4-formula, and let us denote (5.6) by $P \trianglelefteq_{\mathsf{Grow}^3} Q$. We also write

- $P \equiv_{\mathsf{Grow}^3} Q$ if $P \trianglelefteq_{\mathsf{Grow}^3} Q$ and $Q \trianglelefteq_{\mathsf{Grow}^3} P$.
- $P \lhd_{\mathsf{Grow}^3} Q$ if $P \trianglelefteq_{\mathsf{Grow}^3} Q$ and $Q \not\trianglelefteq_{\mathsf{Grow}^3} P$.

There is also a Σ^1_4-formulation of $P \trianglelefteq_{\mathsf{Grow}^3} Q$, whose details are left to the reader.

Proposition 5.4.2 (Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det)**.** *1. $\trianglelefteq_{\mathsf{Grow}^3}$ is a linear preorder over the class of 3-pseudoptykes.*

2. *If P is a 3-pseudoptyx and Q is a 3-ptyx, then $P + Q \equiv_{\mathsf{Grow}^3} P$.*
3. *If P is a 3-preptyx, Q is a 3-pseudoptyx, and $P \trianglelefteq_{\mathsf{Grow}^3} Q$, then P is a 3-pseudoptyx.*
4. (Σ^1_4-$\mathsf{AC}_0 + \Sigma^1_3$-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det) *There is no sequence of 3-pseudoptykes $\langle P_n \mid n \in \mathbb{N}\rangle$ such that $\neg\mathsf{Grow}_{\Pi^1_4}(P_n, P_{n+1})$ for every $n \in \mathbb{N}$.*

Proof. The first statement is immediate by definition.

2. It follows from that the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx π making $P(\pi)$ ill-founded is the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx π making $(P + Q)(\pi)$, and vice versa.
3. $P \trianglelefteq_{\mathsf{Grow}^3} Q$ implies if π is a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx making $Q(\pi)$ ill-founded, then there is a locally well-founded 2-preptyx $\rho \trianglelefteq_{\vartheta^2} \pi$ such that $P(\rho)$ is ill-founded. ρ is also a 2-ptyx by Lemma 5.2.7, so P is a 3-pseudoptyx.

4. Suppose that we have a sequence $\langle P_n \mid n \in \mathbb{N}\rangle$ of 3-pseudoptykes such that $P_n \rhd_{\mathsf{Grow}^3} P_{n+1}$ holds for every $n \in \mathbb{N}$. For each n, there is a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx π making $P_n(\pi)$ ill-founded; That is,

$$\forall^0 n \exists^1 \pi \big[\mathsf{Ptyx}^2(\pi) \wedge \neg \mathsf{WO}(P(\pi)) \wedge \forall^1 \rho \lhd_{\vartheta^2} \pi \mathsf{WO}(P(\rho))\big].$$

The bracketed formula is Π^1_3. Hence by Σ^1_4-AC_0, we can find a sequence $\langle \pi_n \mid n \in \mathbb{N}\rangle$ of reals such that

$$\forall^0 n \big[\mathsf{Ptyx}^2(\pi_n) \wedge \neg \mathsf{WO}(P(\pi_n)) \wedge \forall^1 \rho \lhd_{\vartheta^2} \pi_n \mathsf{WO}(P(\rho))\big].$$

$\pi_n \trianglelefteq_{\vartheta^2} \pi_{n+1}$ implies $P_n \trianglelefteq_{\mathsf{Grow}^3} P_{n+1}$, contradicting the assumption. Hence $\pi_n \rhd_{\vartheta^2} \pi_{n+1}$, so we get an infinite $\trianglelefteq_{\vartheta^2}$-descending sequence of 2-ptykes. This contradicts Proposition 5.2.6. □

Since we do not have $s^1_4(T)$, we need to find another object isolating Σ^1_4-consequences of T. We construct a desired object by taking a countable disjunction of T-provably recursive 3-pseudoptykes. First, the following proposition follows by mimicking the proof of Lemma 1.1.6:

Proposition 5.4.3 (ACA_0)**.** *Let $\langle P_n \mid n \in \mathbb{N}\rangle$ be a countable sequence of 3-predendrograms. We can find a 3-predendrogram $P = \bigvee_{n\in\mathbb{N}} P_n$ such that for every 2-ptyx π,*

$$\mathsf{WO}(P(\pi)) \iff \exists n < \omega \mathsf{WO}(P(\pi)).$$

Furthermore, if $\langle P_n \mid n \in \mathbb{N}\rangle$ is recursive, then so is $\bigvee_{n\in\mathbb{N}} P_n$. □

Now, let us claim that there is a notion of proof-theoretic 3-pseudoptyx. It is not uniquely determined for a given theory, but we can still determine it up to 3-Growth equivalence:

Proposition 5.4.4 (Σ^1_4-$\mathsf{AC}_0 + \Sigma^1_3$-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det)**.** *Let T be a Σ^1_4-sound theory extending Σ^1_4-$\mathsf{AC}_0 + \Sigma^1_3$-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det. Then we can find a* proof-theoretic 3-pseudoptyx $\|T\|_{\Sigma^1_4}$ of T *satisfying the following:*

1. *If P is a recursive 3-preptyx such that $T \vdash \neg\mathsf{Ptyx}^3(P)$, then $P \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$.*
2. *If Q is a 3-pseudoptyx such that $P \trianglelefteq_{\mathsf{Grow}^3} Q$ holds for every T-provably recursive 3-pseudoptyx P, then $\|T\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} Q$.*
3. *If T is recursive, then so is $\|T\|_{\Sigma^1_4}$.*

In particular, if Q and Q' are proof-theoretic 3-pseudoptykes of T, then $Q \equiv_{\mathsf{Grow}^3} Q'$.

Proof. Let us fix an enumeration $\langle P_n \mid n \in \mathbb{N}\rangle$ of T-provably recursive 3-pseudoptykes, and take $\|T\|_{\Sigma^1_4} := \bigvee_{n\in\mathbb{N}} P_n$. The claims follow immediately by Proposition 5.4.3. □

Proposition 5.4.5 (Σ^1_4-AC_0)**.** *Let T be a extension of ACA_0. Then Σ^1_4-$\mathsf{RFN}(T)$ if and only if $\|T\|_{\Sigma^1_4}$ is not a 3-ptyx.*

Proof. For one direction, suppose that Σ^1_4-$\mathsf{RFN}(T)$ holds. Then for every recursive 3-preptyx P, $T \vdash \neg\mathsf{Ptyx}^3(P)$ implies P is not a 3-ptyx. That is,

$$\forall^0 P \in \mathcal{X} \exists^1 \pi \big[\mathsf{Ptyx}^2(\pi) \wedge \neg\mathsf{WO}(P(\pi))\big],$$

where $\mathcal{X}$ is the set of all recursive 3-preptykes P such that $T \vdash \neg\mathsf{Ptyx}^3(P)$. By Σ^1_4-AC_0, there is a real A such that

$$\forall^0 P \in \mathcal{X} \big[\mathsf{Ptyx}^2((A)_P) \wedge \neg\mathsf{WO}(P((A)_P))\big],$$

If we take ρ be an ordered sum $\sum_{P\in\mathcal{X}}(A)_P$, then $\|T\|_{\Sigma^1_4}(\rho)$ is ill-founded.

Conversely, suppose that $\|T\|_{\Sigma^1_4}$ is not a 3-ptyx. If $\|T\|_{\Sigma^1_4}(\pi)$ is ill-founded for a 2-ptyx π, then $P(\pi)$ is ill-founded for every recursive 3-preptyx P such that $T \vdash \neg\mathsf{Ptyx}^3(P)$ by definition. □

We can define $\int^3 P$ for a 3-preptyx P by mimicking Definition 5.1.7. Also, mimicking the proof of Proposition 5.1.8 shows that every thread of $\int^3 P$ takes the form $\int^2 \pi$ for a thread π of P. Moveover, $\int^3 P$ is recursive if P is. These facts are provable in ACA_0. We also have

Proposition 5.4.6 (Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$)**.** *For a 3-pseudoptyx P, $P \lhd_{\mathsf{Grow}^3} \int^3 P$.*

Proof. Let π be the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx such that $P(\pi)$ is ill-founded, which exists by Proposition 5.4.2. We clearly have that $(\int^3 P)(\int^2 \pi)$ is ill-founded. Now we claim that $\int^2 \pi$ is the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx witnessing $\int^3 P$ being not a 3-ptyx. That is,

$$\forall^1 \rho \lhd_{\vartheta^2} \int^2 \pi [\mathsf{WO}((\int^3 P)(\rho))].$$

Suppose the contrary that there is $\rho \lhd_{\vartheta^2} \int^2 \pi$ such that $(\int^3 P)(\rho)$ is ill-founded. Then there is $\rho' \subseteq \rho$ isomorphic to a 2-ptyx thread of $\int^3 P$, say, $\int^2 \theta$ for some 2-ptyx thread θ of P. Since $P(\theta)$ is ill-founded, we have $\pi \trianglelefteq_{\vartheta^2} \theta$. We have

$$\pi \trianglelefteq_{\vartheta^2} \theta \lhd_{\vartheta^2} \int^2 \theta \cong \rho' \trianglelefteq_{\vartheta^2} \rho \lhd_{\vartheta^2} \int^2 \pi,$$

where the second inequality holds by Proposition 5.2.8. Hence $\pi \lhd_{\vartheta^2} \rho' \lhd_{\vartheta^2} \int^2 \pi$, contradicting Proposition 5.2.9.

This shows $\int^2 \pi$ is the $\trianglelefteq_{\vartheta^2}$-least 2-ptyx witnessing $\int^3 P$ not being a 3-ptyx. Hence $\pi \lhd_{\vartheta^2} \int^2 \pi$ implies $P \lhd_{\mathsf{Grow}^3} \int^3 P$. □

Corollary 5.4.7. *Suppose that T is a Σ^1_4-sound extension of Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-$\mathsf{Det}$ and P is a 3-preptyx such that $T \vdash \neg\mathsf{Ptyx}^3(P)$. Then $P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$.*

Proof. T also proves $\int^3 P$ is not a 3-ptyx, and $P \lhd_{\mathsf{Grow}^3} \int^3 P \trianglelefteq \|T\|_{\Sigma^1_3}$. □

The following fact will not be used explicitly in this chapter. However, we need this result to generalize the results to Π^1_{2n+1}- and Σ^1_{2n+2}-consequence comparisons. Hence, we record a proof of the following:

Proposition 5.4.8 (Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det)**.** *Let P and Q be 3-pseudoptykes. If $P \lhd_{\mathsf{Grow}^3} Q$, then $\int^3 P \trianglelefteq_{\mathsf{Grow}^3} Q$.*

Proof. Suppose that π and π' are $\trianglelefteq_{\vartheta^2}$-least 2-ptykes such that $P(\pi)$ and $Q(\pi')$ are ill-founded. Then $\pi \lhd_{\vartheta^2} \pi'$ implies $\int^2 \pi \trianglelefteq_{\vartheta^2} \pi'$, and $\int^2 \pi$ is a $\trianglelefteq_{\vartheta^2}$-least 2-ptyx ρ such that $(\int^3 P)(\rho)$ is ill-founded. □

5.5 Case Σ^1_4

Throughout this section, we take Σ^1_4-$\mathsf{AC}_0 + \Sigma^1_3$-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det as the metatheory.

Theorem 5.5.1. *Let S and T be Σ^1_4-sound extension of Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det. Then we have*

$$S \subseteq^{\Pi^1_4}_{\Sigma^1_4} T. \iff \|S\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$$

We first prove the following lemma:

Lemma 5.5.2. *Let T be a Σ^1_4-sound extension of Σ^1_3-$\mathsf{DC}_0 + \mathbf{\Delta}^1_2$-Det and P be a recursive 3-preptyx. Then*

$$T \vdash^{\Pi^1_4} \neg\mathsf{Ptyx}^3(P) \iff P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}.$$

Proof. For one direction, suppose that there is a recursive 3-ptyx Q such that $T + \mathsf{Ptyx}^3(Q) \vdash \neg\mathsf{Ptyx}^3(P)$, so we have $T \vdash \neg\mathsf{Ptyx}^3(Q) \wedge \neg\mathsf{Ptyx}^3(P)$. This implies $T \vdash \neg\mathsf{Ptyx}^3(P + Q)$, so

$$P \equiv_{\mathsf{Grow}^3} P + Q \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}.$$

Conversely, suppose that $P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$. Then by the second item of Proposition 5.4.4, there is a recursive 3-pseudoptyx Q such that $P \lhd_{\mathsf{Grow}^3} Q$ and $T \vdash \neg\mathsf{Ptyx}^3(Q)$. $P \lhd_{\mathsf{Grow}^3} Q$ is a true Π^1_4-sentence, and $T + P \lhd_{\mathsf{Grow}^3} Q$ implies $\neg\mathsf{Ptyx}^3(P)$ by Proposition 5.4.2. □

Proof of Theorem 5.5.1. For one direction, suppose that $S \subseteq^{\Pi^1_4}_{\Sigma^1_4} T$ holds. We claim that for every 3-pseudoptyx P such that $P \lhd_{\mathsf{Grow}^3} \|S\|_{\Sigma^1_4}$, we have $P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$. It gives a desired result by taking $P = \|T\|_{\Sigma^1_4}$ and deriving a contradiction.

Suppose that $P \lhd_{\mathsf{Grow}^3} \|S\|_{\Sigma^1_4}$. By Proposition 5.4.4, there is a recursive 3-pseudoptyx Q such that $P \lhd_{\mathsf{Grow}^3} Q$ and $S \vdash \neg\mathsf{Ptyx}^3(Q)$. By the assumption, $T \vdash^{\Pi^1_4} \neg\mathsf{Ptyx}^3(Q)$, which is equivalent to $Q \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$ by Lemma 5.5.2. Hence, we have $P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$.

Conversely, suppose that $\|S\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$ holds, and suppose that $S \vdash^{\Pi^1_4} \neg\mathsf{Ptyx}^3(P)$ for a recursive 3-preptyx P. This implies $P \lhd_{\mathsf{Grow}^3} \|S\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$, so $P \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$. This implies $T \vdash^{\Pi^1_4} \neg\mathsf{Ptyx}^3(P)$ by Proposition 5.4.4. □

Theorem 5.5.3. *For* $\mathsf{Bool}(\Pi^1_3)$*-definable* Σ^1_4*-sound extensions* S *and* T *of* $\Sigma^1_4\text{-}\mathsf{AC}_0 + \Sigma^1_3\text{-}\mathsf{DC}_0 + \mathbf{\Delta}^1_2\text{-}\mathsf{Det}$*, we have*

$$\|S\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4} \iff \Sigma^1_4\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_4} \Sigma^1_4\text{-}\mathsf{RFN}(T) \to \Sigma^1_4\text{-}\mathsf{RFN}(S).$$

Furthermore, we have

$$\|S\|_{\Sigma^1_4} \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4} \iff T \vdash^{\Pi^1_4} \Sigma^1_4\text{-}\mathsf{RFN}(S). \tag{5.7}$$

Proof. For one implication, suppose that $\|S\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$, which implies $S \subseteq^{\Pi^1_4}_{\Sigma^1_4} T$. Then consider the sentence

$$\theta :\equiv \forall^0\phi \in \Sigma^1_4[S \vdash \phi \to T \vdash^{\Pi^1_4} \phi].$$

θ is a true sentence of form $\forall^0\Pi^1_4$, so we can find a Π^1_4-sentence θ' such that $\mathsf{ACA}_0 \vdash \theta \to \theta'$ and Σ^1_4-$\mathsf{AC}_0 \vdash \theta' \to \theta$. We also have

$$\mathsf{ACA}_0 + \theta \vdash \Sigma^1_4\text{-}\mathsf{RFN}(T) \to \Sigma^1_4\text{-}\mathsf{RFN}(S),$$

so

$$\Sigma^1_4\text{-}\mathsf{AC}_0 + \theta' \vdash \Sigma^1_4\text{-}\mathsf{RFN}(T) \to \Sigma^1_4\text{-}\mathsf{RFN}(S).$$

To see the converse, we instead prove the forward direction of (5.7). From $\|S\|_{\Sigma^1_4} \lhd_{\mathsf{Grow}^3} \|T\|_{\Sigma^1_4}$, we can find a recursive 3-pseudoptyx P such that $T \vdash^{\Pi^1_4} \neg\mathsf{Ptyx}^3(P)$ and $\|S\|_{\Sigma^1_4} \lhd_{\mathsf{Grow}^3} P$. The assertion "$\|S\|_{\Sigma^1_4} \lhd_{\mathsf{Grow}^3} P$" is a true Π^1_4-sentence, so we have

$$T \vdash^{\Pi^1_4} [\forall^0 Q \in \mathsf{RecPrePtyx}^3[S \vdash \mathsf{Ptyx}^3(Q) \implies Q \lhd_{\mathsf{Grow}^3} P]],$$

which implies $T \vdash^{\Pi^1_4} \Sigma^1_4\text{-}\mathsf{RFN}(S)$.

It remains to show the backward direction of (5.7). Suppose that $T+\sigma \vdash \Sigma^1_4\text{-}\mathsf{RFN}(S)$ holds for some true Π^1_4-sentence σ, but $\|T\|_{\Sigma^1_4} \trianglelefteq_{\mathsf{Grow}^3} \|S\|_{\Sigma^1_4}$ also holds. The latter implies

$$\tau :\equiv \forall^0\phi \in \Sigma^1_4[T \vdash \phi \to S \vdash^{\Pi^1_4} \phi]$$

holds, which is Σ^1_4-AC_0-provably equivalent to a Π^1_4-sentence. We can derive

$$T + \sigma + \tau \vdash \Sigma^1_4\text{-}\mathsf{RFN}(T + \sigma + \tau),$$

which is impossible. □

5.6 Concluding remarks

Although we described the results for Π^1_3 and Σ^1_4 pointclasses, the results are generalizable into Π^1_{2n+1} and Σ^1_{2n+2} for $n \geq 1$ by the almost identical argument. For the odd case,

we define the thread comparison game $\mathcal{G}_{\vartheta^{2n}}(P,Q)$ for two countable $2n$-ptykes P and Q: In this game, each players play a thread of P and Q respectively and build the associated $(2n-1)$-predilator d_{I} and d_{II} respectively. We say Player I wins if $d_{\mathrm{I}} \rhd_{\mathsf{Grow}^{2n}} d_{\mathrm{II}}$. One can see that $\mathcal{G}_{\vartheta^{2n}}(P,Q)$ is a $\mathbf{\Delta}^1_{2n}$-game over Σ^1_{2n}-$\mathsf{AC}_0 + \Sigma^1_{2n-1}$-$\mathsf{DC}_0 + \mathbf{\Delta}^1_{2n-2}$-$\mathsf{Det}$, and if we define

$$P \trianglelefteq_{\vartheta^{2n}} Q \iff \text{Player II has a winning strategy in } \mathcal{G}_{\vartheta^{2n}}(P,Q),$$

then Σ^1_{2n+1}-$\mathsf{DC}_0 + \mathbf{\Delta}^1_{2n}$-$\mathsf{Det}$ proves $\trianglelefteq_{\vartheta^{2n}}$ is a prewellorder over the class of $(2n+1)$-ptykes. We also have

Theorem 5.6.1. *Let $n \geq 2$, and S, T be Π^1_{2n+1}-sound extensions of $\mathsf{ACA}_0 + \mathbf{\Delta}^1_{2n}$-$\mathsf{Det}$. Then we have*

$$S \subseteq^{\Sigma^1_{2n+1}}_{\Pi^1_{2n+1}} T \iff |S|_{\Pi^1_{2n+1}} \trianglelefteq_{\vartheta^{2n}} |T|_{\Pi^1_{2n+1}}.$$

Furthermore, if S and T are $\mathsf{Bool}(\Sigma^1_{2n})$-definable, then we have

$$S \subseteq^{\Sigma^1_{2n+1}}_{\Pi^1_{2n+1}} T \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_{2n+1}} \Pi^1_{2n+1}\text{-}\mathsf{RFN}(T) \to \Pi^1_{2n+1}\text{-}\mathsf{RFN}(S).$$

We also have

$$|S|_{\Pi^1_{2n+1}} \triangleleft_{\vartheta^{2n}} |T|_{\Pi^1_{2n+1}} \iff T \vdash^{\Sigma^1_{2n+1}} \Pi^1_{2n+1}\text{-}\mathsf{RFN}(S). \qquad \square$$

Similarly, we can define $\trianglelefteq_{\mathsf{Grow}^{2n+1}}$ for $(2n+1)$-pseudptykes by comparing $\trianglelefteq_{\vartheta^{2n}}$-least $2n$-ptykes witnessing pseudoptyxness, and we get the following:

Theorem 5.6.2. *Let $n \geq 2$, and S, T be Σ^1_{2n+2}-sound extensions of Σ^1_{2n+1}-$\mathsf{DC}_0 + \mathbf{\Delta}^1_{2n}$-$\mathsf{Det}$. Then we have*

$$S \subseteq^{\Pi^1_{2n+2}}_{\Sigma^1_{2n+2}} T \iff \|S\|_{\Sigma^1_{2n+2}} \trianglelefteq_{\mathsf{Grow}^{2n+1}} \|T\|_{\Sigma^1_{2n+2}}.$$

Furthermore, if S and T are $\mathsf{Bool}(\Sigma^1_{2n})$-definable, then we have

$$S \subseteq^{\Pi^1_{2n+2}}_{\Sigma^1_{2n+2}} T \iff \Sigma^1_{2n+2}\text{-}\mathsf{AC}_0 \vdash^{\Pi^1_{2n+2}} \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(T) \to \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(S).$$

We also have

$$\|S\|_{\Sigma^1_{2n+2}} \lhd_{\mathsf{Grow}^{2n+1}} \|T\|_{\Sigma^1_{2n+2}} \iff T \vdash^{\Pi^1_{2n+2}} \Sigma^1_{2n+2}\text{-}\mathsf{RFN}(S). \qquad \square$$

The next question we can ask is whether we can go higher; Steel [92] raised the following observation:

Phenomenon 5.6.3. Let S and T be natural theories of consistency strength at least that of $\mathsf{ZFC}+$"there are infinitely many Woodin cardinals with a measurable cardinal above them all." Then either $\mathrm{Th}(L(\mathbb{R}))^S \subseteq \mathrm{Th}(L(\mathbb{R}))^T$ or $\mathrm{Th}(L(\mathbb{R}))^S \supseteq \mathrm{Th}(L(\mathbb{R}))^T$, where

$$\mathrm{Th}(L(\mathbb{R}))^T = \{\phi \mid T \vdash L(\mathbb{R}) \vDash \phi\}.$$

In projective levels, n-ptykes represent Π^1_{n+1}-statements. We do not have objects faithfully representing $L(\mathbb{R})$-statements, but let me suggest a potential object that may capture inductive (dually, coinductive) statements.

Definition 5.6.4. A set $X \subseteq \mathbb{R}$ is *inductive* if there is a positive Σ^1_n-operator Γ and a real r such that, if Γ^∞ is the fixed point of Γ, then $x \in X$ iff $\langle r, x\rangle \in \Gamma^\infty$. If the real r is recursive, then we call X an *absolutely inductive set*. A *coinductive set* is the complement of an inductive set. X is *hyperprojective* if it is both inductive and coinductive.

Moschovakis [67, Theorem 7C.4] proved that a set $X \subseteq \mathbb{R}$ is absolutely inductive if there is a Σ^1_1 set P such that for every real x,

$$x \in X \iff \exists^1 y_0 \forall^1 y_1 \exists^1 y_2 \forall^1 y_3 \cdots \exists n[\langle x, y_0, \cdots, y_{n-1}\rangle \in P].$$

Here, we understand an infinite string of quantifiers on reals as an infinite game with reals. To get the right-hand-side like formula in terms of ptykes, let us make the following calculations: We can write

- $\mathsf{Dil}(P) \equiv \forall^1 Q_0[\mathsf{WO}(P(Q_0)) \vee \neg\mathsf{WO}(Q_0)]$.

- $\mathsf{Ptyx}^2(P) \equiv \forall^1 Q_0 \exists^1 Q_1 [\mathsf{WO}(P(Q_0)) \vee [\neg\mathsf{WO}(Q_0(Q_1)) \wedge \mathsf{WO}(Q_1)]]$.
- $\mathsf{Ptyx}^3(P) \equiv \forall^1 Q_0 \exists^1 Q_1 \forall^1 Q_2 \Big[\mathsf{WO}(P(Q_0)) \vee \Big[\neg\mathsf{WO}(Q_0(Q_1)) \wedge [\mathsf{WO}(Q_1(Q_2)) \vee \neg\mathsf{WO}(Q_2)]\Big]\Big]$.

We can imagine continuing the process, and get the following infinite formula:

$$\forall^1 Q_0 \exists^1 Q_1 \forall^1 Q_2 \exists^1 Q_3 \cdots \Big[\mathsf{WO}(P(Q_0)) \vee \Big[\neg\mathsf{WO}(Q_0(Q_1)) \wedge \big[\mathsf{WO}(Q_1(Q_2)) \vee \cdots\big]\Big]\Big]$$

By distributing conjunctions and disjunctions, we get the following:

$$\mathsf{Ptyx}^\omega(P) :\equiv \forall^1 Q_0 \exists^1 Q_1 \forall^1 Q_2 \exists^1 Q_3 \cdots \bigwedge_{n<\omega} \Big[\mathsf{WO}(P(Q_0)) \vee \mathsf{WO}(Q_0(Q_1)) \vee \cdots \vee \mathsf{WO}(Q_{n-2}(Q_{n-1})) \vee \neg\mathsf{WO}(Q_{n-1})\Big].$$

To make the previous definition sensible, we need a structure that generalizes n-preptykes, allowing for the application of the same type. That is, we need a class of objects such that if P and Q are in the class, then $P(Q)$ is well-defined and a linear order. Let me introduce a possible candidate for the class:

Definition 5.6.5. The *category of ω-preptykes* is the inverse limit of the category of n-preptykes $\mathsf{PrePtyx}^n$ for every $n < \omega$ under the map $\pi_{n,m} \colon \mathsf{PrePtyx}^n \to \mathsf{PrePtyx}^m$ for $n \geq m$ such that $\pi_{n,m}(P)$ is obtained from P by forgetting every term whose arity is not a finite $(m-1)$-preptyx.

The category of ω-preptykes should coincide with the direct limit, pullback closure of *the category of finite ω-preptykes*, which is the direct limit of the category of finite n-preptykes under the canonical embedding.

If the notion of ω-(pre)ptykes is correct, it should represent coinductive sets. Here we confront another technical issue: Girard's normal form theorem for n-ptykes is about second-order statements, so it is expressible in a second-order arithmetic. (Co)inductive

pointclasses and $\mathsf{Ptyx}^{\omega}(P)$ require an infinite string of second-order quantifiers, whose description requires a third-order quantifier. Hence, we must move from a second-order arithmetic to a higher-order arithmetic to correctly state claims about (co)inductive sets and $\mathsf{Ptyx}^{\omega}(P)$. I do not want to delve into which higher-order arithmetic is appropriate to formulate facts about ω-ptykes, so I will state the following conjecture as ZFC-statements:

Conjecture 5.6.6. For every absolutely coinductive statement $\phi(X)$, we can find an ω-preptyx P_X recursive in X such that ZFC proves $\forall^1 X[\phi(X) \leftrightarrow \mathsf{Ptyx}^{\omega}(P_X)]$.

It looks like coinductive sets correspond to the ptyx side, so the inductive sets should correspond to the pseudoptyx side. It is known [67, Exercise 7C. 15] that the class of inductive sets has the prewellordering property, so comparing inductive consequences should resemble comparing $(2n-1)$-pseudoptykes. It is unclear to me how the comparison of ω-pseudoptykes should work.

Part II

Appendix

APPENDIX A

PRELIMINARIES

In this chapter, we introduce the terminology and results used in several other chapters.

A.1 Partial and Linear orders

In this section, we review the necessary details about partial and linear orders. The notion of partial and linear orders is standard, so we do not introduce their definitions. We often say $X \leq Y$ for two linear orders X and Y, which means there is an embedding from X to Y.

Throughout the dissertation, we will seldom see non-linear partial orders. One of the exceptions is a free distributive lattice with finite generators, which we will use to define semidilators and semiptykes:

Definition A.1.1. A structure $(X, \leq, \wedge, \vee)$ is a *lattice* if for every $x, y \in X$, $x \wedge y$ is the greatest lower bound of x and y, and $x \vee y$ is the least upper bound of x and y. A lattice X is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in X$.

Free distributive lattices can be defined in a category-theoretic manner. Note that, unlike some texts, *we do not allow joins and meets of the empty set* in the definition of a free distributive lattice.

Definition A.1.2. For $n \in \mathbb{N}$, let us denote the free distributive lattice of n generators by $\mathtt{L}_n$. We denote its generators by natural numbers (say, $0, 1, \cdots, n-1$.) We also denote the top element $0 \vee \cdots \vee (n-1)$ and the bottom element $0 \wedge \cdots \wedge (n-1)$ by $\top$ or $\bot$ if there is no confusion.

Note that $\mathtt{L}_n$ exists: It is folklore that $\mathtt{L}_n$ is the distributive lattice of non-constant[1] *monotonic Boolean functions* $f\colon 2^n \to 2$ (if we understand 2 as the least Boolean algebra) with pointwise operators. We will use $\mathtt{L}_n$ to index the intersections and unions of given sets.

Now let us review notions about linear orders:

Definition A.1.3. Let X be a linear order and $Y \subseteq X$. The *insertion map (or inclusion map) from Y to X* is the map $i\colon Y \to X$ given by $i(x) = x$.

Remark A.1.4. Every increasing map between two linear orders appearing in this book is strictly increasing; i.e., $x < y$ implies $f(x) < f(y)$.

It is well-known that every finite linear order is a well-order. Hence, we can enumerate elements of a finite linear order increasingly:

Definition A.1.5. Let a be a finite linear order. Define en_a to be the unique order isomorphism $\mathrm{en}_a\colon |a| \to a$.

Thus, we can identify every finite linear order a with its increasing enumeration en_a, and we write $a(i)$ to mean the ith element of a (put differently, $\mathrm{en}_a(i)$.) Conversely, from an element $x \in a$ of a finite linear order a, we can track its *position* $\mathrm{en}_a^{-1}(x)$. That is, if x is the ith least element of a, then the position of x in a is i.

Definition A.1.6. Let X be a linear order and $T \subseteq X^{<\omega}$ a tree. The *Kleene-Brouwer order* $<_{\mathsf{KB}}$ over T is defined as follows: For $\sigma, \tau \in X^{<\omega}$, $\sigma <_{\mathsf{KB}} \tau$ iff either

1. $\sigma \supsetneq \tau$, or
2. There is $n < \min(|\sigma|, |\tau|)$ such that $\sigma \restriction n = \tau \restriction n$ and $\sigma(n) < \tau(n)$.

Proposition A.1.7 (ACA_0). *Suppose that X is a well-order and $T \subseteq X^{<\omega}$ is a tree. Then $(T, <_{\mathsf{KB}})$ is ill-founded iff T has an infinite branch.*

[1] If we allow empty joins and meets, we can drop the non-constant requirement.

Proof. One direction is clear. For the other direction, suppose that we are given a $<_{\mathsf{KB}}$-descending sequence $\langle \sigma^i \mid i \in \mathbb{N} \rangle$ over T. Observe that for a fixed $n \in \mathbb{N}$, the ordered set

$$(\{\sigma \in T \mid |\sigma| \leq n\}, <_{\mathsf{KB}})$$

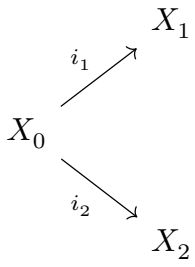
is well-founded, so the length of σ_i is unbounded for i. Hence we may assume that $|\sigma^0| < |\sigma^1| < |\sigma^2| < \cdots$. Let us write $\sigma^i = \langle \sigma^i_j \mid j < |\sigma^i| \rangle$, and claim that for every j, $\langle \sigma^i_j \mid i \in \mathbb{N} \rangle$ is eventually constant. Suppose not, let j is the least natural number such that $\langle \sigma^i_j \mid i \in \mathbb{N} \rangle$ is not eventually constant. If $\sigma^i_j \neq \sigma^{i+1}_j$, then we have $\sigma^i_j > \sigma^i_{j+1}$ since $\sigma^i >_{\mathsf{KB}} \sigma^{i+1}$ and $\sigma^i_k = \sigma^{i+1}_k$ for every $k < j$. It means a subsequence of $\langle \sigma^i_j \mid i \in \mathbb{N} \rangle$ is decreasing, a contradiction.

Hence for each $j \in \mathbb{N}$, we have $\sigma^\bullet_j$ such that $\langle \sigma^i_j \mid i \in \mathbb{N} \rangle$ eventually has value $\sigma^\bullet_j$. For each j, we can find i such that $\langle \sigma^\bullet_k \mid k < j \rangle \subseteq \sigma^i$, so T being a tree implies $\langle \sigma^\bullet_k \mid k < j \rangle \in T$. Therefore, T has an infinite branch $\langle \sigma^\bullet_j \mid j \in \mathbb{N} \rangle$. □

Note that Proposition A.1.7 fails when X is not a well-order.

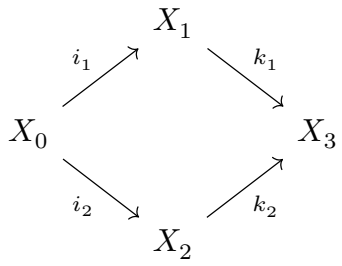
We finish this section with the *amalgamation theorem for linear orders*, which allows us to take a 'union' of two linear orders that are not subsets of another linear order:

Theorem A.1.8 (Amalgamation theorem for linear orders, ACA_0). *Let X_0, X_1, X_2 be linear orders and suppose that we have a diagram*



then we can find a new linear order X_3 with embeddings $k_1 \colon X_1 \to X_3$, $k_1 \colon X_2 \to X_3$

such that the following diagram is a pullback diagram and $\operatorname{ran} k_1 \cup \operatorname{ran} k_2 = X_3$*:*



(A.1)

Furthermore, if all of X_0*,* X_1*,* X_2*,* i_1*,* i_2 *are finite (recursive), then so are* X_3*,* k_1*,* k_2*.*

Proof. By replacing the elements of X_1 and X_2 if necessary, we may assume that $X_0 \subseteq X_1, X_2$ and i_1 and i_2 are inclusion maps. Then let us define a linear order over $X_3 := X_1 \cup X_2$ as follows: If both x and y are in the same set, follow the comparison over the set. If $x \in X_1 \setminus X_2$ and $y \in X_2 \setminus X_1$, define

$$X_3 \vDash x < y \iff \exists z \in X_0[X_1 \vDash x < z \text{ and } X_2 \vDash z < y].$$

Then the resulting order is transitive as follows: Suppose that $X_3 \vDash x < y < z$. We have $X_3 \vDash x < z$ if all of x, y, z are in either X_1 or X_2. We also have $X_3 \vDash x < z$ when x, y or y, z are in the same set. Hence it remains to check the following two cases:

1. $x, z \in X_1 \setminus X_2$ and $y \in X_2 \setminus X_1$.
2. $x, z \in X_2 \setminus X_1$ and $y \in X_1 \setminus X_2$.

In the first case, we have

1. $X_3 \vDash x < y \iff \exists u \in X_0[X_1 \vDash x < u \wedge X_2 \vDash u < y]$.
2. $X_3 \vDash y < z \iff \forall v \in X_0[X_2 \vDash v < y \implies X_1 \vDash v \leq z]$.

Combining these two, we have $X_1 \vDash x < z$. Similarly, the second case also implies $X_2 \vDash x < z$. □

A.2 Second-order arithmetic

We will use various subsystems of second-order arithmetic. We mostly follow Simpson's book [89], so the reader should refer to it for details about the second-order arithmetic.

The standard terminology is to denote natural numbers as lowercase letters and real numbers as uppercase letters. I tried to follow this convention as much as possible in this dissertation, but not every letter follows. For example, we denote linear orders or well-orders by lowercase Greek letters α, β, γ, etc., following the notation of ordinals via these letters. We distinguish number quantifiers and set quantifiers by superscript, so $\forall^0$, $\exists^0$ denote number quantifiers, and $\forall^1$, $\exists^1$ denote set quantifiers.

There are two notions of definability of reals; For a complexity class Γ, we say a real R is Γ-*definable* if there is a Γ-formula $\phi(m)$ such that $R = \{n \in \mathbb{N} \mid \phi(m)\}$. A real R is Γ-*singleton* (or *implicitly* Γ-*definable*) if there is a Γ-formula $\phi(X)$ such that R is the unique real satisfying $\phi(R)$. Let me warn that in set theory, the definability of a set mostly means the definability in the latter sense; That is, we say a set a is definable if there is a formula $\psi(x)$ in the language of set theory such that a is the unique set satisfying $\psi(a)$. At least in this dissertation, we do not talk about the definability of the reals in this set-theoretic sense.

A.3 Admissible set-theoretic facts

In this section, we briefly review standard facts in admissible set theory. The standard material for admissible set theory is Barwise [9], although we do not strictly follow Barwise's convention. KP is a set theory comprising Extensionality, Pairing, Union, Infinity,

Δ_0-Separation, Δ_0-Collection, and full Set Induction

$$\forall x[(\forall y \in x \phi(y)) \to \phi(x)] \implies \forall x \phi(x)$$

for every formula ϕ in the language of set theory. Some material restricts Set Induction to Π_1-formulas, but we do not follow this convention and let ϕ be an arbitrary formula. An *admissible set* is a transitive model of KP, and KPi is the system KP plus "Every set is contained in an admissible set." We also define the admissibility properties of ordinals and sets as follows:

Definition A.3.1. An ordinal α is *admissible* if L_α is a model of KP. α is *recursively inaccessible* if L_α is a model of KPi.

A transitive set M is *admissible* if M is a model of KP.

Note that if M is admissible, then $L^M = M \cap L$ is also admissible.

We will also use weaker subsystems like PRS and KPℓ: PRS is the set theory for primitive recursive set functions[2], whose details are introduced in Freund's thesis [21, Ch. 1] under the name PRSω. Freund's PRSω and our PRS differ in the following manner:

1. Freund's PRSω in [21] has a richer language, including symbols for every primitive recursive set function. PRS has the standard language $\{\in\}$.
2. Freund's PRSω only has the Set Induction scheme for bounded formulas. PRS has the full Set Induction. (This difference is not important in this dissertation.)

KPℓ is obtained from PRS by adding "Every set is contained in an admissible set." ℓ in KPℓ denotes "limit," and one can see that if $M = \bigcup_{n<\omega} M_n$, Ord^M is a strict supremum of Ord^{M_n}, and each M_n is admissible, then M models KPℓ.

[2] It is different from primitive recursive functions.

Now, let us review facts about projective ordinals.

Definition A.3.2. δ^1_n is the supremum of the order-types of all Δ^1_n-definable countable well-orders.

Proposition A.3.3 ([9, Corollary V.7.9]). *δ^1_2 is the set of all Σ_1-definable ordinals over L without parameters. Similarly, $L_{\delta^1_2}$ is the set of all Σ_1-definable sets over L without parameters.* □

Proposition A.3.4 ([9, Corollary V.8.3]). *δ^1_2 is the least stable ordinal, that is, δ^1_2 is the least σ such that $L_\sigma \prec_{\Sigma_1} L$.* □

Proposition A.3.5. *Every Π^1_1-statement is equivalent to a Σ_1-statement over $L_{\omega_1^{\mathsf{CK}}}$. Every Σ^1_2-statement is equivalent to a Σ_1-statement over $L_{\delta^1_2}$.*

Proof. For a real number X, let ω_1^X be the least ordinal α such that $L_\alpha[X] \vDash \mathsf{KP}$. Then we can see the following: For every Π^1_1-formula $\phi(X)$ with only real free variable X displayed, we can find a Σ_1-formula $\psi(X)$ such that for every real X,

$$\phi(X) \iff L_{\omega_1^X}[X] \vDash \psi(X). \tag{A.2}$$

To see this, we first appeal to the Kleene normal form theorem to find a recursive linear order α_X such that $\phi(X)$ and $\mathsf{WO}(\alpha_X)$ are equivalent. Then we take

$$\psi(X) \equiv \exists \xi \in \mathrm{Ord} \exists f [f \text{ is an isomorphism from } \xi \text{ to } \alpha_X]$$

The right-to-left implication of (A.2) is clear. For the left-to-right implication, fix X and suppose that α_X is well-founded. Since α_X is recursive in X, $\alpha_X \in L_{\omega_1^X}[X]$. By mimicking the proof of Σ-recursion theorem [9, Theorem I.6.4], we can see that if F is a Σ-definable function over $L_{\omega_1^X}[X]$, one can define a $\Sigma_1^{L_{\omega_1^X}[X]}$-definable $G \colon \alpha_X \to L_{\omega_1^X}[X]$ such that for every $i \in \alpha_X$,

$$G(i) = F(\langle (j, G(j)) \mid \alpha_X \vDash j < i \rangle).$$

We take F by $F(x) = \min(\mathrm{Ord} \setminus \mathrm{ran}\, x)$, which is uniformly Σ-definable over an admissible set. We can also see that the resulting Σ-recursively defined function G gives an embedding from α_X to ω_1^X whose range is an ordinal, and Σ-Replacement over $L_{\omega_1^X}[X]$ implies $\xi := \mathrm{ran}\, G$ is in $L_{\omega_1^X}[X]$.

This completes the proof of the first part of the proposition. To see the remaining part, suppose that $\phi(X)$ is a Π^1_1-formula, and let $\psi(X)$ be a Σ_1-formula satisfying (A.2). By Shoenfield absoluteness, we have

$$\exists^1 X \phi(X) \iff L \vDash \exists^1 X \phi(X) \iff L \vDash \exists X \in \mathbb{R}[L_{\omega_1^X}[X] \vDash \psi(X)].$$

Observe that the function $X \mapsto \omega_1^X$ is Σ_1-definable in L, together with the function $\alpha \mapsto L_\alpha$. Hence the statement "$\exists X \in \mathbb{R}[L_{\omega_1^X}[X] \vDash \psi(X)]$" is Σ_1. $L_{\delta^1_2} \prec_{\Sigma_1} L$ by Proposition A.3.4, so we have

$$\exists^1 X \psi(X) \iff L_{\delta^1_2} \vDash \exists X \in \mathbb{R}[L_{\omega_1^X}[X] \vDash \psi(X)]. \qquad \square$$

δ^1_n for $n \geq 3$ also has a similar characterization, but it involves canonical inner models with Woodin cardinals. See [91] for details.

A.4 Miscellaneous Notations

I will denote $\langle i, x\rangle$ by $\mathsf{in}_i(x)$ when we want to stress the role of $\langle i, x\rangle$ as an element of label i, instead of viewing $\langle i, x\rangle$ as a tuple.

A.5 Sets-as-trees interpretation

In this section, we review standard facts about β-models of ATR_0 and its set-theoretic formulation $\mathsf{ATR}_0^{\mathsf{set}}$ discussed in [89, Ch. VII]. Most explanations in this section appear in [45] in verbatim or with some modification.

Definition A.5.1. $\mathsf{ATR}_0^{\mathsf{set}}$ is a theory in the language of set theory with the following axioms:

1. Axioms of primitive recursive set theory:
   (a) Extensionality
   (b) Foundation
   (c) Infinity
   (d) An axiom defining each primitive recursive set function[3]
2. Axiom Beta: A relation r is said to be *regular* if every non-empty set has an r-minimal element:
   $$\forall u \neq \emptyset \exists x \in u \forall y \in u(\langle y, x\rangle \notin r).$$
   Beta states for every regular relation r there is a function f such that $\mathrm{dom}\, f = \mathrm{field}(r)$ and for every $x \in \mathrm{field}(r)$,
   $$f"[x] = \{f"[y] \mid \langle y, x\rangle \in r\}.$$
   We call such f the *collapsing function of* r.
3. Axiom of countability: Every set is hereditarily countable.

[3] See [21, Ch. 1.1] for a definition. Note that the definition in [21, Ch. 1.1] uses an expanded language for PRS, but we stick into the language of set theory $\{\in\}$. The choice of PRS is arbitrary, and we could replace PRS with its weaker fragment, like Bowler-Mathias' Provident set theory [62].

It is not too hard to prove that every regular relation has at most one collapsing function. As the name suggests, $\mathsf{ATR}_0^{\mathsf{set}}$ is closely connected to the theory ATR_0. In fact, $\mathsf{ATR}_0^{\mathsf{set}}$ is bi-interpretable with ATR_0, via the *sets-as-trees interpretation*. First, we can see that $\mathsf{ATR}_0^{\mathsf{set}}$ is strong enough to prove axioms of ATR_0 if we naturally interpret them:

Theorem A.5.2 ([89, Theorem VII.3.9]). *If we interpret second-order arithmetic over* $\mathsf{ATR}_0^{\mathsf{set}}$ *by interpreting numbers as finite ordinals and sets as subsets of* ω*, then* $\mathsf{ATR}_0^{\mathsf{set}}$ *proves every axiom of* ATR_0.

Before defining the interpretation, let us define *suitable trees* that will code sets.

Definition A.5.3. Working over ATR_0, we say a non-empty set $\mathtt{T} \subseteq \mathbb{N}^{<\omega}$ is a *suitable tree* if it satisfies the following:

1. $\mathtt{T}$ is a tree: If $s \in \mathtt{T}$ and t is an initial segment of s, then $t \in \mathtt{T}$.
2. $\mathtt{T}$ is well-founded: There is no $f \colon \mathbb{N} \to \mathbb{N}$ such that $f \upharpoonright n \in \mathtt{T}$ for every $n \in \mathbb{N}$.

For a suitable tree $\mathtt{T}$ and $s \in \mathtt{T}$, define $\mathtt{T} \upharpoonright s = \{t \in \mathbb{N}^{<\omega} \mid s^\frown t \in \mathtt{T}\}$. Then $\mathtt{T} \upharpoonright s$ is again a suitable tree. We understand the root of a suitable tree as an encoded set and the childhood relation over a tree as a membership relation. That is, for a suitable tree $\mathtt{T}$, we understand $s^\frown\langle n\rangle \in \mathtt{T}$ as "$\mathtt{T} \upharpoonright s^\frown\langle n\rangle$ is an element of $\mathtt{T} \upharpoonright s$." Different trees may represent the same set: For example, both $\mathtt{T}_0 = \{\langle\rangle, \langle 0\rangle\}$, $\mathtt{T}_1 = \{\langle\rangle, \langle 1\rangle\}$, and $\mathtt{T}_{01} = \{\langle\rangle, \langle 0\rangle, \langle 1\rangle\}$ represent the same set (namely, $1 = \{0\}$). It will complicate defining the 'equality' and the 'membership relation' between suitable trees.

We define a suitable tree over $\mathsf{ATR}_0^{\mathsf{set}}$ in the same manner. We can decode a suitable tree into a set in the following way:

Definition A.5.4. We can see that a well-founded tree $\mathtt{T}$ induces a regular relation given by

$$r_{\mathtt{T}} = \{\langle s^\frown\langle n\rangle, s\rangle \mid \sigma^\frown\langle n\rangle \in \mathtt{T}\}.$$

By Beta, we have the collapsing function c_T of r_T. For a suitable tree $\mathtt{T}$, we define

$$|\mathtt{T}| = c_{\mathtt{T}}"\{\langle n\rangle \mid \langle n\rangle \in \mathtt{T}\}.$$

Let me remark that other references (e.g., [89, 45]) use italic uppercase S, T, U, etc., to denote suitable trees. We use typewriter uppercases $\mathtt{S}$, $\mathtt{T}$, etc., to denote suitable trees since we need italic uppercases to denote formal theories.

We can define two relations $=^*$ and $\in^*$ between two suitable trees such that $\mathtt{S} =^* \mathtt{T}$ iff $|\mathtt{S}| = |\mathtt{T}|$ and $\mathtt{S} \in^* \mathtt{T}$ iff $|\mathtt{S}| \in |\mathtt{T}|$. We briefly review the definition of $=^*$:

Proposition A.5.5 (ATR_0**, [89, Lemma VII.3.17]).** *Let* $\mathtt{T}$ *be a suitable tree. We define* $\mathsf{Iso}(X, \mathtt{T})$ *as the following assertion: For every* $\langle s, t\rangle \in \mathtt{T}$, $\langle s, t\rangle \in X$ *iff*

1. $\forall^0 m[s^\frown\langle m\rangle \in \mathtt{T} \to \exists^0 n\langle s^\frown\langle m\rangle, t^\frown\langle n\rangle\rangle \in X]$, *and*
2. $\forall^0 n[t^\frown\langle n\rangle \in \mathtt{T} \to \exists^0 m\langle s^\frown\langle m\rangle, t^\frown\langle n\rangle\rangle \in X]$.

Roughly speaking, $\mathsf{Iso}(X, T)$ *says* X *encodes an extensional equality over a set* T *codes. Then for each suitable tree* $\mathtt{T}$*, there is a unique* X *such that* $\mathsf{Iso}(X, \mathtt{T})$*. Furthermore, such* X *is hyperarithmetically definable from* $\mathtt{T}$. □

Then we define

1. $\mathtt{S} =^* \mathtt{T} \iff \exists^1 X[\mathsf{Iso}(X, \mathtt{S} \oplus \mathtt{T}) \wedge \langle\langle 0\rangle, \langle 1\rangle\rangle \in X]$,
2. $\mathtt{S} \in^* \mathtt{T} \iff \exists^1 X[\mathsf{Iso}(X, \mathtt{S} \oplus \mathtt{T}) \wedge \exists^0 n\langle\langle 0\rangle, \langle 1, n\rangle\rangle \in X]$.

Then by [89, Lemma VII.3.14], $\mathtt{S} =^* \mathtt{T}$ iff $|\mathtt{S}| = |\mathtt{T}|$, and $\mathtt{S} \in^* \mathtt{T}$ iff $|\mathtt{S}| \in |\mathtt{T}|$.

Then let us define the tree interpretation $\mathfrak{t}\colon \mathsf{ATR}_0^{\mathsf{set}} \to \mathsf{ATR}_0$ as follows: For a formula ϕ over the language of set theory, associate $\phi^{\mathfrak{t}}$ over the language of second-order arithmetic recursively as follows:

1. $(x = y)^{\mathfrak{t}} \equiv (x =^* y)$.
2. $(x \in y)^{\mathfrak{t}} \equiv (x \in^* y)$.
3. $\mathfrak{t}$ respects logical connectives; e.g., $(\phi \wedge \psi)^{\mathfrak{t}} \equiv \phi^{\mathfrak{t}} \wedge \psi^{\mathfrak{t}}$.
4. $(\forall x \phi(x))^{\mathfrak{t}} \equiv \forall x[x \text{ is a suitable tree} \to \phi^{\mathfrak{t}}(x)]$.
5. $(\exists x \phi(x))^{\mathfrak{t}} \equiv \exists x[x \text{ is a suitable tree} \wedge \phi^{\mathfrak{t}}(x)]$.

Note that [89] uses $|\phi|$ for interpretations instead of $\phi^{\mathfrak{t}}$. However, we already use the notation $|\mathtt{T}|$ to denote collapses of suitable trees in Definition A.5.4). Our notation $\phi^{\mathfrak{t}}$ is commonly used for interpretations (c.f. [19, 100]). There is also a natural interpretation $\mathfrak{i} : \mathsf{ATR}_0 \to \mathsf{ATR}_0^{\mathsf{set}}$ interpreting natural numbers as finite ordinals and sets of naturals as subsets of ω. The following lemmas show that $\mathfrak{t}$ and $\mathfrak{i}$ give bi-interpretations between $\mathsf{ATR}_0^{\mathsf{set}}$ and ATR_0:

Lemma A.5.6 ([89, Lemma VII.3.16]). *Let ϕ be any formula of set theory with n free variables. Then $\mathsf{ATR}_0^{\mathsf{set}}$ proves the following: For suitable trees $\mathtt{T}_0, \cdots, \mathtt{T}_{n-1}$,*

$$\phi(|\mathtt{T}_0|, \cdots, |\mathtt{T}_{n-1}|) \leftrightarrow (\phi^{\mathfrak{t}})^{\mathfrak{i}}(\mathtt{T}_0, \cdots, \mathtt{T}_{n-1}).$$

In particular, we have $\mathsf{ATR}_0^{\mathsf{set}} \vdash \phi \leftrightarrow (\phi^{\mathfrak{t}})^{\mathfrak{i}}$ for every sentence *ϕ, so $\mathfrak{i} \circ \mathfrak{t}$ is the identity interpretation.*

Before stating the reverse direction, let us define the canonical suitable tree for a set of natural numbers:

Definition A.5.7. Working in ATR_0, for $X \subseteq \mathbb{N}$, define X^* to be the suitable tree consisting of the empty node and all $\langle m_0, \cdots, m_{k-1} \rangle$ such that $m_0 \in X$ and $m_0 > \cdots > m_{k-1}$. For a natural number n, define $n^* = \{0, 1, \cdots, n-1\}^*$.

It is possible to prove $|X^*| = X$ over $\mathsf{ATR}_0^{\mathsf{set}}$, and we have the following:

Lemma A.5.8 ([89, Lemma VII.3.19]). *Let ϕ be a formula of second-order arithmetic with n number variables and m set variables. Then ATR_0 proves*

$$\phi(x_0, \cdots, x_{n-1}, X_0, \cdots, X_{m-1}) \leftrightarrow (\phi^{\mathfrak{i}})^{\mathfrak{t}}(x_0^*, \cdots x_{n-1}^*, X_0^*, \cdots, X_{m-1}^*).$$

In particular, we have $\mathsf{ATR}_0 \vdash \phi \leftrightarrow (\phi^{\mathfrak{i}})^{\mathfrak{t}}$ if ϕ is a sentence*, so $\mathfrak{t} \circ \mathfrak{i}$ is the identity interpretation.*

The interpretation $\mathfrak{t}$ and $\mathfrak{i}$ generalize to extensions of ATR_0. For a given extension T_0 of ATR_0, let us define its set-theoretic counterpart:

Definition A.5.9. Let $T_0 \supseteq \mathsf{ATR}_0$ be any theory in the language of second-order arithmetic. Define

$$T_0^{\mathsf{set}} = \mathsf{ATR}_0^{\mathsf{set}} \cup \{\phi^{\mathfrak{i}} \mid \phi \in T_0\}.$$

By Lemma A.5.6 and Lemma A.5.8, we get the following:

Theorem A.5.10 ([89, Theorem VII.3.34]). *Let T_0 be an extension of ATR_0. Then $\mathfrak{t} \colon T_0^{\mathsf{set}} \to T_0$ and $\mathfrak{i} \colon T_0 \to T_0^{\mathsf{set}}$ jointly form a bi-interpretation.*

A.5.1 Model-theoretic interpretation

Definition A.5.11. Let A be a model of $\mathsf{ATR}_0^{\mathsf{set}}$. There are uniform definitions φ_+ and $\varphi_\times$ of ordinal addition and multiplication in the language of $\mathsf{ATR}_0^{\mathsf{set}}$

- $\mathfrak{i}(A)$ has domain $(\mathbb{N}^A, \mathcal{P}(\mathbb{N})^A)$
- $+^{\mathfrak{i}(A)} = \varphi_+^A$
- $\times^{\mathfrak{i}(A)} = \varphi_\times^A$
- $0^{\mathfrak{i}(A)} = \emptyset^A$

- $1^{\mathfrak{i}(A)} = \{\emptyset\}^A$
- $<^{\mathfrak{i}(A)} = \in^A$

Then we have the following:

Theorem A.5.12 ([89, Theorem VII.3.27]). *Let A be a model of $\mathsf{ATR}_0^{\mathsf{set}}$. Then*

1. *$\mathfrak{i}(A)$ is a model of ATR_0.*
2. *$\mathfrak{i}(A)$ is isomorphic to a β-model iff A is well-founded.*

In the other direction, we can extract a model of $\mathsf{ATR}_0^{\mathsf{set}}$ from each model of ATR_0:

Definition A.5.13 ([89, Definition VII.3.28])**.** Let M be a model of ATR_0, and let $\mathcal{T}_M$ be the set of all suitable trees in M:

$$\mathcal{T}_M = \{\mathtt{T} \in M \mid M \vDash \mathtt{T} \text{ is a suitable tree}\}.$$

For each $\mathtt{T} \in \mathcal{T}_M$, associate $[\mathtt{T}]^* = \{\mathtt{T}' \in \mathcal{T}_M \mid M \vDash \mathtt{T} =^* \mathtt{T}'\}$, and define

$$A = \{[\mathtt{T}]^* \mid T \in \mathcal{T}_M\}.$$

For $\mathtt{S}, \mathtt{T} \in \mathcal{T}_M$, define $[\mathtt{S}]^* \in_A [\mathtt{T}]^*$ iff $M \vDash \mathtt{S} \in^* \mathtt{T}$. Then $(A, \in_A)$ is a model over the language of set theory, and define $\mathfrak{t}(M) = (A, \in_A)$.

We will want to "internalize" $\mathfrak{i}$ and $\mathfrak{t}$ over transitive models of $\mathsf{ATR}_0^{\mathsf{set}}$, so it is necessary to guarantee that the construction of $\mathfrak{t}(M)$ works over a transitive model of $\mathsf{ATR}_0^{\mathsf{set}}$. In fact, $\mathsf{ATR}_0^{\mathsf{set}}$ proves that we can construct $\mathfrak{t}(M)$ from a model M of ATR_0:

The following fact is standard:

Lemma A.5.14. *$\mathsf{ATR}_0^{\mathsf{set}}$ proves that if M be a model of ATR_0, then $\mathfrak{t}(M)$ exists.*

The following theorem follows from the fact that $\mathfrak{t}$ and $\mathfrak{i}$ form a bi-interpretation:

Theorem A.5.15 ([89, Theorem VII.3.29]). *Let M be a model of ATR_0. Then $\mathfrak{t}(M)$ is a model of $\mathsf{ATR}_0^{\mathsf{set}}$. Furthermore, there is a natural isomorphism between $\mathfrak{i}(\mathfrak{t}(M))$ and M. Conversely, if A is a model of $\mathsf{ATR}_0^{\mathsf{set}}$, then there is a natural isomorphism between $\mathfrak{t}(\mathfrak{i}(A))$ and A.*

A.6 Proof theory and reflections

A.6.1 Complexity class and Γ-Reflection

Many parts of this subsection are from [39] in verbatim.

A *complexity class* is a collection of formulas. If a formula or a sentence is a member of Γ, then we call it Γ-*formula* or Γ-*sentence* respectively. Also, for a complexity class Γ, we define $\check{\Gamma}$ by

$$\check{\Gamma} = \{\neg\phi \mid \phi \in \Gamma\}.$$

It is known that ACA_0 proves there is a universal Π^1_n-formula for $n \geq 1$. This enables us to talk about the truth of Π^1_n- or Σ^1_n-formulas.

Proposition A.6.1. *Let Γ be one of Π^1_n or Σ^1_n for $n \geq 1$. Then there is a Γ-formula $\phi(i, X)$ such that for every Γ-formula $\psi(X)$, we can find a natural number n such that*

$$\forall^1 X(\phi(n, X) \leftrightarrow \psi(X)).$$ □

We denote a fixed universal Γ-formula by $\vDash_\Gamma (\phi)$, where ϕ is a natural number code for a Γ-formula. Then we formulate the Γ-*reflection of a theory T* as follows:

$$\Gamma\text{-}\mathsf{RFN}(T) \equiv \forall^0\sigma \in \Gamma[T \vdash \sigma \implies \vDash_\Gamma \sigma].$$

For an arithmetical T and $\Gamma \supseteq \Pi^1_1$ we consider in this dissertation (for example, $\Gamma = \Pi^1_1, \Sigma^1_2, \Pi^1_3, \Sigma^1_4, \cdots$), Γ-$\mathsf{RFN}(T)$ is a Γ-statement. The following lemma says adding a true $\check{\Gamma}$ sentence does not change the Γ-reflection of a theory T:

Lemma A.6.2. *Suppose that ϕ is in $\check{\Gamma}$. If ϕ is true, then Γ-$\mathsf{RFN}(T) \to \Gamma$-$\mathsf{RFN}(T + \phi)$.*

Proof. Suppose that $T + \phi \vdash \psi$ holds for a Γ-sentence ψ. Then $T \vdash \phi \to \psi$, and $\phi \to \psi$ is equivalent to a Γ-sentence, whose equivalence is provable over the first-order logic. By Γ-$\mathsf{RFN}(T)$, $\phi \to \psi$ is true. Since ϕ is true, ψ is also true. □

The following lemma says being Γ-sound is transitive: That is, if T is Γ-sound and if $T \vdash S$ is Γ-sound, then S is also Γ-sound.

Lemma A.6.3. *Let S, T be extensions of ACA_0 such that T is Γ-sound and S is r.e. If $T \vdash \Gamma$-$\mathsf{RFN}(S)$, then Γ-$\mathsf{RFN}(S)$ is true.*

Proof. If $\Gamma = \Pi^1_n$, then one can easily check that the statement Π^1_n-$\mathsf{RFN}(S)$ is Π^1_n. However, this line of argument does not work when $\Gamma = \Sigma^1_n$ since Σ^1_n-$\mathsf{RFN}(S)$ takes of the form

$$\forall^0 \phi \in \Sigma^1_n [(S \vdash \phi) \to \vDash_{\Sigma^1_n} \phi]$$

whose complexity takes of the form $\forall^0(\Sigma^0_1 \to \Sigma^1_n)$. However, we can still prove Σ^1_n-$\mathsf{RFN}(S)$ from $T \vdash \Sigma^1_n$-$\mathsf{RFN}(S)$ as follows: Let $\phi \in \Sigma^1_n$ be a (natural number coding a) Σ^1_n-formula. If $S \vdash \phi$, then we have $T \vdash (S \vdash \phi)$ since "$S \vdash \phi$" is a true Σ^0_1 statement. Since $T \vdash \Sigma^1_n$-$\mathsf{RFN}(S)$, we have $T \vdash \vDash_{\Sigma^1_n} \phi$, which implies $\vDash_{\Sigma^1_n} \phi$ by Σ^1_n-soundness of T. □

The following theorem expresses the Γ-reflection version of Gödel's second incompleteness theorem, which is a modification of [97, Theorem 2.9]

Proposition A.6.4. *Suppose that* Γ *is either* Π^1_n *or* Σ^1_n *for* $n \geq 1$. *If* T *is a* Γ*-sound,* $\check{\Gamma}$*-definable extension of* ACA_0*, then* T *does not prove* Γ-$\mathsf{RFN}(T)$.

Proof. Suppose that $T \vdash \Gamma$-$\mathsf{RFN}(T)$, and let ϕ be a conjunction of all statements in T employed in the proof of Γ-$\mathsf{RFN}(T)$ from T. The statement $T \vdash \phi$ is RCA_0-provably equivalent to a $\check{\Gamma}$-assertion, and we have $\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”} \vdash \Gamma$-$\mathsf{RFN}(T)$.

Working in the theory $\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”}$, we know Γ-$\mathsf{RFN}(T)$ holds. T proves ϕ, so Γ-$\mathsf{RFN}(\mathsf{ACA}_0 + \phi)$. Since “$T \vdash \phi$” is equivalent (both over the metatheory and ACA_0-provably) to a $\check{\Gamma}$-assertion, we have $\mathsf{RFN}(\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”})$. In summary, we get

$$\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”} \vdash \Gamma\text{-}\mathsf{RFN}\big(\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”}\big).$$

In particular, $\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”}$ proves its own consistency. Observe that the theory $\mathsf{ACA}_0 + \phi + \text{“}T \vdash \phi\text{”}$ is recursive, so we get a contradiction by Gödel's second incompleteness theorem. □

The class of Σ^1_n-formulas is not closed under number quantifiers. That is, for a Σ^1_n-formula $\phi(m)$, there is no guarantee to see that $\forall^0 m\phi(m)$ is equivalent to a Σ^1_n-formula. If we have Σ^1_n-AC_0, however, we can reduce $\forall^0 m\phi(m)$ to a Σ^1_n-formula. Moreover, ACA_0 proves the reduction of $\forall^0 m\phi(m)$ implies $\forall^0 m\phi(m)$. We state this fact as follows:

Lemma A.6.5. *Let* $\phi(m)$ *be a* Σ^1_n*-formula. Then we can effectively find a* Σ^1_n*-formula* ψ *such that*

1. ACA_0 *proves* $\psi \to \forall^0 m\phi(m)$.
2. Σ^1_n-AC_0 *proves* $\forall^0 m\phi(m) \to \psi$.

We call ψ *a* reduction of $\forall^0 m\phi(m)$.

Proof. Let $\phi(m) \equiv \exists^0 m\theta(m, x)$ for a Π^1_{n-1}-formula θ. Then consider the following statement:

$$\psi \equiv \exists^1 F \forall^0 m\theta(m, F(m)). \qquad \square$$

A.6.2 Π^1_1-proof theory

In this subsection, we briefly review a more technical aspect of Π^1_1-proof theory that we did not explain in the introduction.

Proof theorists gauged the strength of a given theory by examining how far the given theory can be capable of recursive well-orders.

Definition A.6.6. Let T be a Π^1_1-sound extension of ACA_0. We define its *proof-theoretic ordinal* by

$$|T|_{\Pi^1_1} = \sup\{\alpha \in \mathsf{RWO} \mid T \vdash \mathsf{WO}(\alpha)\},$$

where RWO is the set of all recursive well-orders, and $\mathsf{WO}(\alpha)$ is the assertion that a linear order α is well-ordered.

The subscript Π^1_1 of $|T|_{\Pi^1_1}$ means the proof-theoretic ordinal of T gauges its robustness via gauging its Π^1_1-consequences. The Kleene normal form theorem justifies the reason for using primitive recursive well-orders. Note that the definition of $|T|_{\Pi^1_1}$ is strict supremum since if $T \vdash \mathsf{WO}(\alpha)$, then $T \vdash \mathsf{WO}(\alpha + 1)$.

We claim that the assertion '$|T|_{\Pi^1_1}$ is well-ordered' is equivalent to the Π^1_1-reflection of T, but this statement itself is problematic since $|T|_{\Pi^1_1}$ is already an ordinal, which is always well-founded. To make the statement precise, we define the following subsidiary linear order, which is isomorphic to the proof-theoretic ordinal:

Definition A.6.7. Let T be a Π^1_1-sound extension of ACA_0. We define $\|T\|_{\Pi^1_1}$ by the ordered sum of T-provably recursive well-orders. That is, we define

$$\|T\|_{\Pi^1_1} := \sum\{\alpha \in \mathsf{RWO} \mid T \vdash \mathsf{WO}(\alpha)\}.$$

To make the definition precise, we assume that T in the previous definition is not a set of theorems, but an enumeration $\langle \sigma_n \mid n \in \mathbb{N} \rangle$ of theorems of ordertype $\mathbb{N}$. For a recursive or arithmetically definable theory as a set, we can find such an enumeration of the theorems of the same definitional complexity. The ordered sum is indexed by the index for the statement $\mathsf{WO}(\alpha)$ in the enumeration.

Theorem A.6.8. *Let T be a Π^1_1-sound extension of ACA_0.*

1. *The ordertype of $\|T\|_{\Pi^1_1}$ is $|T|_{\Pi^1_1}$.*
2. *If T is recursive or arithmetically definable, then so is $\|T\|_{\Pi^1_1}$.*
3. *For an arithmetically definable T,*

$$\mathsf{ACA}_0 \vdash \mathsf{WO}(\|T\|_{\Pi^1_1}) \leftrightarrow \Pi^1_1\text{-}\mathsf{RFN}(T).$$

Proof. 1. To see $|T|_{\Pi^1_1} \leq \|T\|_{\Pi^1_1}$, suppose that $T \vdash \mathsf{WO}(\alpha)$ for a recursive linear order α. Then $\|T\|_{\Pi^1_1}$ has a component isomorphic to α, so $\alpha \leq \|T\|_{\Pi^1_1}$. Then the definition of $|T|_{\Pi^1_1}$ yields the desired result. For $\|T\|_{\Pi^1_1} \leq |T|_{\Pi^1_1}$, observe that if $\|T\|_{\Pi^1_1} = \sum_{i\in\mathbb{N}} \alpha_i$ for recursive α_i such that $T \vdash \mathsf{WO}(\alpha_i)$, we have that $T \vdash \mathsf{WO}(\sum_{i<n} \alpha_i)$ for every n. The ordertype of $\|T\|_{\Pi^1_1}$ is the supremum of the ordertype of $\sum_{i<n} \alpha_i$ for $n \in \mathbb{N}$, which is less than or equal to $|T|_{\Pi^1_1}$. This finishes the proof.

2. Clear.

3. Let us work in ACA_0. Suppose that $\|T\|_{\Pi^1_1}$ is a well-order and $T \vdash \phi$ for a Π^1_1-sentence ϕ. By the Kleene normal form theorem, we can find a recursive linear order α such that ϕ and $\mathsf{WO}(\alpha)$ are equivalent (over the metatheory) and $\mathsf{ACA}_0 \vdash \phi \leftrightarrow \mathsf{WO}(\alpha)$. Hence $T \vdash \mathsf{WO}(\alpha)$, so α embeds to $\|T\|_{\Pi^1_1}$ and so α is a well-order. This shows ϕ is true. In summary, we have that every Π^1_1-consequence of T is true. Conversely, if Π^1_1-$\mathsf{RFN}(T)$ holds, then each additive component of $\|T\|_{\Pi^1_1}$ is a well-order. Hence $\|T\|_{\Pi^1_1}$ is also a well-order. □

Also, the following folklore result credited to Kriesel states that adding a set of true Σ^1_1-sentences does not change the proof-theoretic ordinal. (See [103, Proposition 2.1] for its proof.)

Theorem A.6.9. *Let T be a Π^1_1-sound extension of ACA_0, and X be a set of true Σ^1_1-sentences. Then $|T|_{\Pi^1_1} = |T + X|_{\Pi^1_1}$.* □

Walsh provided equivalent conditions for comparing proof-theoretic ordinals in [102] The following did not appear in [102], so we provide its proof:

Theorem A.6.10. *Let S, T be arithmetically definable Π^1_1-sound extensions of ACA_0. Then*

$$|S|_{\Pi^1_1} < |T|_{\Pi^1_1} \iff T \vdash^{\Sigma^1_1} \Pi^1_1\text{-}\mathsf{RFN}(S).$$

Proof. For one direction, suppose that $|S|_{\Pi^1_1} < |T|_{\Pi^1_1}$. One can see that $|T|_{\Pi^1_1}$ is a limit ordinal, so there is $\alpha \in \mathsf{PRWO}$ such that $T \vdash \mathsf{WO}(\alpha)$ and $|S|_{\Pi^1_1} \leq \alpha$. Hence the following Σ^1_1 sentence is true:

$$\sigma \equiv \exists^1 F[F \text{ is an embedding from } \|S\|_{\Pi^1_1} \text{ to } \alpha.]$$

We need the arithmetical definability of S to guarantee σ is Σ^1_1. We have $T + \sigma \vdash \mathsf{WO}(\|S\|_{\Pi^1_1})$, so Theorem A.6.8 implies $T + \sigma \vdash \Pi^1_1$-$\mathsf{RFN}(S)$. For the other direction,

suppose that $T \vdash^{\Sigma^1_1} \Pi^1_1\text{-}\mathsf{RFN}(S)$, so we can find a true Σ^1_1-sentence σ such that $T + \sigma \vdash \mathsf{WO}(\|S\|_{\Pi^1_1})$. This implies

$$|S|_{\Pi^1_1} \cong \|S\|_{\Pi^1_1} < |T + \sigma|_{\Pi^1_1} = |T|_{\Pi^1_1}. \qquad \square$$

A.6.3 Π^1_2-proof theory

The proof-theoretic ordinal gauges the Π^1_1-consequences of a theory. We need a more complicated object than ordinals to describe Π^1_2-consequences of a theory in analogy to the proof-theoretic ordinal. It turns out that dilators are the right objects to describe Π^1_2-consequences. The following definition is due to Aguilera-Pakhomov [2]:

Definition A.6.11. Let T be a Π^1_2-sound extension of ACA_0. Then $|T|_{\Pi^1_2}$ is the dilator unique up to bi-embeddability satisfying the following conditions:

1. For a recursive predilator D, if $T \vdash \mathsf{Dil}(D)$ then D embeds into $|T|_{\Pi^1_2}$.
2. (Universality) If a dilator $\hat{D}$ satisfies (1), then $|T|_{\Pi^1_2}$ embeds into $\hat{D}$.

Aguilera-Pakhomov [2] showed that every Π^1_2-sound extension of ACA_0 has a proof-theoretic dilator, and is recursive if T is. In fact, the ordered sum of all recursive predilators that are T-provably dilators satisfies the criteria for $|T|_{\Pi^1_2}$. Furthermore, if T is r.e., then we can find a recursive dilator satisfying the criteria for $|T|_{\Pi^1_2}$. We may try to define $|T|_{\Pi^1_2}$ for Π^1_2-unsound theories. In this case, however, $|T|_{\Pi^1_2}$ will not be a dilator, and there is no guarantee that $|T|_{\Pi^1_2}$ satisfies the universality condition.

Also, just as proof-theoretic ordinals are associated with the Π^1_1-reflection, proof-theoretic dilators are related to Π^1_2-reflection:

Theorem A.6.12 (Aguilera-Pakhomov [2, Theorem 7]). ACA_0 *proves the following: If* T *is a* Π^1_2*-sound r.e. extension of* ACA_0*, then* $\mathsf{Dil}(|T|_{\Pi^1_2})$ *is equivalent to* $\Pi^1_2\text{-}\mathsf{RFN}(T)$. $\square$

$|T|_{\Pi^1_2}$ has the following extensional description for ordinals less than ω_1^{CK}:

Theorem A.6.13 (Aguilera-Pakhomov [2, Theorem 9]). *Let T be a Π^1_2-sound extension of ACA_0. If α is a recursive well-order, then*

$$|T|_{\Pi^1_2}(\alpha) = |T + \mathsf{WO}(\alpha)|_{\Pi^1_1}. \qquad \square$$

The following theorem shows an extensional description of some proof-theoretic dilators. Some of the results are available in [71], and some others are unpublished results by Aguilera and Pakhomov:

Theorem A.6.14.

1. *Let ε^+ be the* next epsilon dilator*, so $\varepsilon^+(\alpha)$ is isomorphic to the next epsilon number greater than α. Then $|\mathsf{ACA}_0|_{\Pi^1_2} = \varepsilon^+$.*
2. *Let φ^+_2 be a dilator such that $\varphi^+_2(\alpha)$ is isomorphic to the least Velben number of the form $\varphi_2(\xi)$ greater than α. Then $|\mathsf{ACA}^+_0|_{\Pi^1_2} = \varphi^+_2$.*
3. *Let Γ^+ be a dilator such that $\Gamma^+(\alpha)$ is isomorphic to the least fixpoint for Veblen function greater than α. Then $|\mathsf{ATR}_0|_{\Pi^1_2} = \Gamma^+$.* $\square$

$\Pi^1_1[R]$-consequences of a theory for a Σ^1_2-singleton real R is linearly comparable modulo true $\Sigma^1_1[R]$-consequences as proved in subsection 3.1.3. However, [5] found that comparing the Π^1_2-consequences of two Π^1_2-sound theories modulo true Σ^1_2-sentences is not linear:

Theorem A.6.15 (Aguilera-Pakhomov [5]). *There is a Π^1_2-sound recursive extension T of ATR_0 and two Π^1_2-sentences ϕ_0, ϕ_1 such that $T + \phi_i$ is Π^1_2-sound for $i = 0, 1$, but neither $T \vdash^{\Sigma^1_2} \phi_0 \to \phi_1$ nor $T \vdash^{\Sigma^1_2} \phi_1 \to \phi_0$ hold.* $\square$

APPENDIX B

DILATORS

Non si non la.

Maurice Scève

From the remaining chapters, we cover basic facts about dilators and ptykes needed in this thesis. Most of the content in this part appears verbatim in the book draft [42] work in progress. Most results in the remaining part, except for dendrograms in Appendix D and latter sections of Appendix E, are not new and have appeared in different literature (in particular, that of Girard [26, 29]). Despite this, I believe it is worth presenting the known details, as there is little accessible, organized literature on the theory of dilators and ptykes. Most results in this part are provable in RCA_0 or a weak set theory like provident set theory, but there are a few exceptions. I will put explanations for results requiring more than RCA_0 (like Ramsey's theorem for pairs) or non-trivial set-theoretic axioms.

In this chapter, we define the notion of dilators, which Girard first defined for his Π^1_2-ordinal analysis. Girard defined dilators as an endofunctor of the category WO of well-orders. However, we often confront dilator-like objects that are yet unknown to be a dilator called *predilators* and *semidilators*. Although Girard's definition is easy to state, it is the hardest form of definition to handle in practice. Thus, we will define dilators based on the idea of *dilators-as-denotation-systems*, which is also noticed by Girard and frequently used in other references on dilators (e.g., Kechris-Woodin [47] or Aguilera-Pakhomov [2]). Meanwhile, Freund (e.g. [22]) uses a different formulation for dilators, which combines an endofunctor over WO with a *support transformation*. Freund's definition is quite useful when we check whether an object we constructed is a

(semi)dilator, so we also include a section for blending denotation systems with Freund-style dilators. Most results in this chapter appeared in an earlier version of [39, §2], and most contents from B.1 to B.4 appeared verbatim.

B.1 Denotation system

One way to understand a dilator is to understand it as an *denotation system* for a class ordinal. Let us give a simple example to illustrate what dilators are: Let us construct a class ordinal Ord + Ord. There is no way to express Ord + Ord as a transitive class, but there is a way to express its ordertype: We can understand Ord + Ord as a collection of pairs of the form $(\xi, 0)$ or $(\xi, 1)$ for $\xi \in$ Ord, and compare them as follows:

- $(\xi, 0) < (\eta, 0)$ iff $\xi < \eta$. The same holds for $(\xi, 1)$ and $(\eta, 1)$.
- $(\xi, 0) < (\eta, 1)$ always hold.

Moreover, the denotation system is *uniform* in the sense that the same construction over α instead of Ord gives an ordertype of $\alpha + \alpha$. It turns out that handling the denotation system for class ordinals directly as dilators is more fruitful than considering their ordertypes. In terms of dilators, the expression for Ord + Ord corresponds to the dilator $\mathsf{Id} + \mathsf{Id}$, where Id is the identity dilator.

The previous explanation illustrates how to understand a dilator D as a class ordinal $D(\text{Ord})$. However, $D(\text{Ord})$ for a denotational system D may not be well-ordered. We can see that $D(\text{Ord})$ is well-ordered when $D(\alpha)$ is well-ordered for every well-order α, but not every denotation system satisfies it. To embrace ill-founded denotation systems, we first define *semidilators* and *predilators*, then define dilators as semidilators (or predilators) preserving well-orders. Although predilators are better objects and behave more

similarly to linear orders, we define semidilators first since defining predilators requires defining semidilators.

Informally, a denotation system D is a set of terms with a comparison rule between terms. Each term t comes with an *arity* $n = \text{arity}(t) \in \mathbb{N}$, and we understand t as a 'function' taking an increasing sequence of length n, or alternatively, a subset of size n. We will form a new linear order $D(\alpha)$ from a given linear order α, which takes the form

$$D(\alpha) = \{t(a \mid t \text{ is a } D\text{-term and } a \in [\alpha]^{\text{arity}(t)}\}.$$

We assume that $s(a) \neq t(b)$ if $s \neq t$. Comparing $s(a)$ and $t(b)$ only depends on the isomorphism type of $(a \cup b; <, a, b)$. Let us introduce some preliminary notions before defining semidilators as denotation systems.

Definition B.1.1. An *arity diagram* Д is a commutative diagram over the category Nat of natural numbers with strictly increasing maps of the form

$$\text{Д} = \begin{array}{ccc} a_\cap & \longrightarrow & a_1 \\ \downarrow & & \downarrow{\scriptstyle g} \\ a_0 & \xrightarrow[f]{} & a_\cup \end{array} \tag{B.1}$$

such that the above diagram is a pullback and $\operatorname{ran} f \cup \operatorname{ran} g = \operatorname{field} a_\cup$. We say an arity diagram is *trivial* if $a_\cap = a_0 = a_1 = a_\cup$ (so all arrows in an arity diagram are the identity map.) For an arity diagram Д of the form (B.1), the diagram $-$Д is a diagram obtained by switching the order of a_0 and a_1:

$$-\text{Д} = \begin{array}{ccc} a_\cap & \longrightarrow & a_0 \\ \downarrow & & \downarrow{\scriptstyle f} \\ a_1 & \xrightarrow[g]{} & a_\cup \end{array}$$

An arity diagram is a way to describe the relative order between elements of a_0 and those of a_1. A typical example of an arity diagram is induced from an inclusion diagram,

for example,

$$\begin{array}{ccc} \{1,3\} & \xrightarrow{\subseteq} & \{0,1,3\} \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ \{1,2,3,4\} & \xrightarrow[\subseteq]{} & \{0,1,2,3,4\} \end{array}$$

We can see that the above diagram is isomorphic to

$$\begin{array}{ccc} \{\mathbf{0},\mathbf{1}\} & \xrightarrow{h} & \{0,\mathbf{1},\mathbf{2}\} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ \{\mathbf{0},1,\mathbf{2},3\} & \xrightarrow[f]{} & \{0,1,2,3,4\} \end{array}$$

where h and k are maps sending boldface numbers to boldface numbers in an increasing manner, $f(n) = n + 1$, and $g(0) = 0$, $g(1) = 1$, $g(2) = 3$.

Remark B.1.2. We may view an arity diagram as a functor from $\mathtt{L}_2$ to Nat with certain conditions. From this perspective, let us denote components of an arity diagram Д by

$$\text{Д} = \begin{array}{ccc} \text{Д}(\bot) & \longrightarrow & \text{Д}(1) \\ \downarrow & & \downarrow{\scriptstyle \text{Д}(1,\top)} \\ \text{Д}(0) & \xrightarrow[\text{Д}(0,\top)]{} & \text{Д}(\top) \end{array}$$

We will use this notion when we do not designate specific terms for components of an arity diagram.

The following notion will be used to define the linearity condition of a denotation system:

Definition B.1.3. Let $a_0, \cdots, a_{m-1}$ be natural numbers and let $\mathtt{L}_m$ be the free distributive lattice generated by $\{0, 1, \cdots, m-1\}$. An *IU diagram* **Д** *for* $a_0, \cdots, a_{m-1}$ (abbreviation of *Intersection-Union diagram*) is a collection of objects $\{a_i \mid i \in \mathtt{L}_m\}$ and strictly increasing functions $f_{ij} \colon a_i \to a_j$ for $i \leq j$ such that the following holds:

1. f_{ii} is the identity map.
2. For $i \leq j \leq k$, $f_{ik} = f_{jk} \circ f_{ij}$.

3. For each $i, j \in \mathtt{L}_m$, the following diagram

$$\text{Д}[i,j] = \begin{array}{ccc} a_{i\wedge j} & \xrightarrow{f_{i\wedge j,j}} & a_j \\ {\scriptstyle f_{i\wedge j,i}}\downarrow & & \downarrow{\scriptstyle f_{j,i\vee j}} \\ a_i & \xrightarrow[f_{i,i\vee j}]{} & a_{i\vee j} \end{array}$$

is a pullback and ran $f_{i,i\vee j} \cup$ ran $f_{j,i\vee j} = a_{i\vee j}$.

Alternatively, we can think of an IU diagram as a functor from $\mathtt{L}_n$ as a category to the category of linear orders LO. If every object in Д is in a class $A \subseteq \mathsf{LO}$, then we say Д is an *IU diagram over* A.

We may think of arity diagrams as IU diagrams for two objects over $\mathbb{N}$. Like arity diagrams, we also use the notion $\text{Д}(i)$ and $\text{Д}(i,j)$ to denote objects and morphisms of an IU diagram Д. We can see that every IU diagram over finite linear orders is uniquely isomorphic to an IU diagram over $\mathbb{N}$. We will use this fact frequently with no mention.

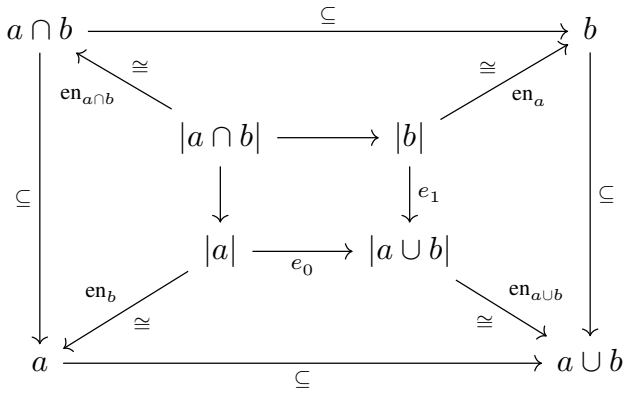
Arity diagrams and IU diagrams over $\mathbb{N}$ are means to express the inclusion relation between finitely many finite linear orders. The following notion describes how we capture IU diagrams over $\mathbb{N}$ from the inclusion diagram for finitely many finite linear orders:

Definition B.1.4. Let $a, b \subseteq X$ be two finite suborders of a linear order X. *The diagram* $\text{Diag}_X(a,b)$ *of* a, b *over* X is the unique arity diagram isomorphic to the inclusion diagram

$$\begin{array}{ccc} a \cap b & \xrightarrow{\subseteq} & b \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ a & \xrightarrow[\subseteq]{} & a \cup b \end{array}$$

and more precisely, $\text{Diag}_X(a,b)$ is the innermost diagram in the below commutative

diagram, where $\mathrm{en}_a : |a| \to a$ is the unique order isomorphism for finite linear order a.



For finite suborders $a_0, \cdots, a_{n-1} \subseteq X$ of a linear order X, $\mathrm{Diag}_X(a_0, \cdots, a_{n-1})$ is the unique IU diagram over $\mathbb{N}$ isomorphic to the inclusion diagram $\mathcal{I} : \mathtt{L}_n \to \mathcal{P}(a_0 \cup \cdots \cup a_{n-1})$, where $\mathcal{I}$ can be thought of a lattice homomorphism

$$(\mathtt{L}_n, \wedge, \vee) \to (\mathcal{P}(a_0 \cup \cdots \cup a_{n-1}), \cap, \cup)$$

satisfying $\mathcal{I}(i) = a_i$ for every $i < n$. We will often drop X and simply write $\mathrm{Diag}(a_0, \cdots, a_{n-1})$ if X is clear by context.

In practice, we turn inclusion diagrams into IU diagrams over $\mathbb{N}$ to handle inclusion relations in the context of the definition of semidilators. Conversely, when we construct a new IU diagram over $\mathbb{N}$, we instead work with inclusion diagrams and calculate their diagram. Now, let us define denotation systems, which we will call semidilators.

Definition B.1.5. A *semidilator* D consists of its field, the *arity function* $\mathrm{arity}(t) \in \mathbb{N}$, with comparison relations. We call a member $t \in D$ a *D-term*. Also, for each two D-terms t_0, t_1 and an arity diagram

$$\text{Д} = \begin{array}{ccc} a_\cap & \longrightarrow & a_1 \\ \downarrow & & \downarrow \\ a_0 & \longrightarrow & a_\cup \end{array}$$

such that $a_i = \mathrm{arity}(t_i)$, we have the relation $t_0 <_{\text{Д}} t_1$ between two terms. We call Д an *arity diagram for t_0 and t_1*. Then D satisfies the following:

1. (Irreflexivity) If $t_0 = t_1$ and Д is trivial, then $t_0 <_{Д} t_0$ does not hold.
2. (Linearity) If $t_0 \neq t_1$ or Д is not trivial, then one of $t_0 <_{Д} t_1$ or $t_1 <_{-Д} t_0$ must hold.
3. (Transitivity) For three D-terms t_0, t_1, t_2 such that $\text{arity}(t_i) = a_i$, and an IU diagram **Д** for t_0, t_1, t_2, if $t_0 <_{\mathbf{Д}[0,1]} t_1$ and $t_1 <_{\mathbf{Д}[1,2]} t_2$, then $t_0 <_{\mathbf{Д}[0,2]} t_2$.

We write

$$t_0 \leq_{Д} t_1 \iff t_0 <_{Д} t_1 \vee [Д \text{ is trivial and } t_0 = t_1].$$

We say D is *(primitive) recursive* or *countable* if the set of D-terms and the set of comparison rules are (primitive) recursive or countable, respectively.

Remark B.1.6. The reader should be warned that materials call a semidilator in different ways: Aguilera-Pakhomov [2] call it a predilator, but this terminology collides with the standard definition of a predilator we will introduce later. Girard [29] call semidilator *paleodilator*, Freund [21] call it *prae-dilator*, and Catlow [16] call it *1-functor*.

Remark B.1.7. We can first-order axiomatize the definition of a semidilator over a multi-sorted language. We may think a semidilator D as a multi-sorted structure whose element $t \in D$ has sort $n = \text{arity}\, t$, and the language of D is

$$\mathcal{L}^1 = \{<_{Д} | \; Д \text{ is an arity diagram}\}.$$

The axioms for semidilators comprise the schema of irreflexivity, linearity, and transitivity.

Note that the disjunction on the linearity is exclusive.

Proposition B.1.8. *Let D be a semidilator, $t_0, t_1 \in \text{field}(D)$, and Д an arity diagram for t_0, t_1. Then neither $t_0 <_{Д} t_1$ nor $t_1 <_{-Д} t_0$ both hold.*

Proof. Suppose not, assume that $t_0 <_{Д} t_1 <_{-Д} t_0$. Assume that Д has of the form (B.1), and consider the IU diagram $\mathbf{Д} = \mathrm{Diag}_{a_\cup}(\mathrm{ran}\, f, \mathrm{ran}\, g, \mathrm{ran}\, f)$. Then we have $\mathbf{Д}[0,1] = Д$, $\mathbf{Д}[1,2] = -Д$, and $\mathbf{Д}[0,2]$ is the trivial diagram for a_0. By transitivity of a semidilator, we have $t_0 <_{\mathbf{Д}[0,2]} t_0$, so $t_0 <_{Ж} t_0$ for a trivial diagram Ж, contradicting with the irreflexivity. □

Every first-order relational structure admits its subsets as substructures, and it is the same for semidilators:

Definition B.1.9. Let D be a semidilator and $X \subseteq \mathrm{field}(D)$. A *sub-semidilator* $D \restriction X$ *induced by* X is a semidilator whose field is X with the same arity function and comparison rules for D.

B.2 Category-theoretic aspects of semidilators

We want to understand semidilators as a functor over the category of linear orders LO with strictly increasing functions as morphisms. Hence, let us define the application of semidilators.

Definition B.2.1. From a given semidilator D and a linear order X, let us define $D(X)$, the *application of D to X* by

$$D(X) = \{(t, a) \mid t \text{ is a } D\text{-term}, a \subseteq X, \text{and } |a| = \mathrm{arity}(t)\}. \tag{B.2}$$

We write $t(a)$ instead of (t, a), and we identify a with a finite increasing sequence over X. The order of $D(X)$ is given by

$$D(X) \vDash s(a) < t(b) \iff D \vDash s <_{\mathrm{Diag}_X(a,b)} t$$

For a strictly increasing function $f\colon X \to Y$, consider the map $D(f)\colon D(X) \to D(Y)$ given by

$$D(f)(t, a) = (t, f[a]), \quad \text{where } f[a] = \{f(x) \mid x \in a\}.$$

We can see the following facts without difficulty:

Proposition B.2.2. *Let D be a semidilator and X be a linear order. Then $D(X)$ is a linear order. Also, D induces a functor* $\mathsf{LO} \to \mathsf{LO}$.

Proof. Let us prove that $D(X)$ is a linear order. The irreflexivity and the linearity of a semidilator imply that $<_{D(X)}$ satisfies the irreflexivity and the linearity, respectively. To see $<_{D(X)}$ is transitive, suppose that we have $s_0(a_0) <_{D(X)} s_1(a_1) <_{D(X)} s_2(a_2)$. Now let Д be the IU diagram over $\mathbb{N}$ isomorphic to a diagram induced by intersections and unions of a_0, a_1, a_2, and the insertion maps. In particular, $\text{Д}(i, j)$ is the induced diagram $\mathrm{Diag}_X(a_i, a_j)$ from a_i and a_j for $i \neq j < 2$. Then we have

$$s_0 <_{\text{Д}(0,1)} s_1 \wedge s_1 <_{\text{Д}[0,2]} s_2.$$

Thus we have $s_0 <_{\text{Д}[0,2]} s_2$ by the transitivity of a semidilator, which implies $D(X) \vDash s_0(a_0) < s_2(a_2)$. To see that D induces a functor, it suffices to show that the following holds:

1. $D(\mathsf{Id}_X) = \mathsf{Id}_{D(X)}$,
2. $D(f \circ g) = D(f) \circ D(g)$.

Both of them are easy to verify, so we leave their proof to the reader. □

Like strictly increasing maps between linear orders, we can define a map between semidilators preserving their structures:

Definition B.2.3. Let D and E be two semidilators. A map $\iota\colon D \to E$ is an *embedding* or a *semidilator morphism* if it satisfies the following conditions:

1. ι is a function field D to field E.
2. ι preserves the arity: i.e., $\mathrm{arity}(\iota(t)) = \mathrm{arity}(t)$ for every $t \in \mathrm{field}\, D$,
3. For each two terms $t_0, t_1 \in \mathrm{field}\, D$ and an arity diagram Д between them, we have $t_0 <_{\text{Д}} t_1$ iff $\iota(t_0) <_{\text{Д}} \iota(t_1)$.

An embedding ι is an *isomorphism* if whenever $\iota\colon \mathrm{field}\, D \to \mathrm{field}\, E$ is a bijection and ι^{-1} is also an embedding. We denote $D \cong E$ if there is an isomorphism between D and E. We also write $D \leq E$ if there is an embedding from D to E.

We did not require an embedding between two semidilators to be one-to-one, but we can see that every embedding between two semidilators must be injective:

Lemma B.2.4. *If $\iota\colon D \to E$ is an embedding, then $\iota\colon \mathrm{field}\, D \to \mathrm{field}\, E$ is injective.*

Proof. Suppose the contrary that we have two different D-terms t_0 and t_1 such that $\iota(t_0) = \iota(t_1)$. Then $\mathrm{arity}\, t_0 = \mathrm{arity}\, \iota(t_0) = \mathrm{arity}\, t_1$, and let Д be a trivial diagram for t_0. Then either $t_0 <_{\text{Д}} t_1$ or $t_1 <_{\text{Д}} t_0$ holds, so we get $\iota(t_0) <_{\text{Д}} \iota(t_1)$ or $\iota(t_1) <_{\text{Д}} \iota(t_0)$. It contradicts the irreflexivity of E. □

Moreover, every bijective embedding is an isomorphism:

Lemma B.2.5. *If $\iota\colon D \to E$ is an embedding and a bijection, then $\iota^{-1}\colon E \to D$ is an embedding.*

Proof. Suppose that $E \vDash t_0 <_{\text{Д}} t_1$. Then we can find $s_0, s_1 \in \mathrm{field}(D)$ such that $\iota(s_0) = t_0$ and $\iota(s_1) = t_1$. If $D \vDash s_1 <_{-\text{Д}} s_0$, then $E \vDash t_1 <_{-\text{Д}} t_0$, so $E \vDash t_1 <_{-\text{Д}} t_0 <_{\text{Д}} t_1$, which is impossible. Hence $D \vDash s_0 <_{\text{Д}} s_1$. □

We can view an embedding between two semidilators as a natural transformation in the following manner:

Definition B.2.6. For an embedding $\iota\colon D \to E$ and a linear order X, define $\iota_X\colon D(X) \to E(X)$ by

$$\iota_X(t, a) = (\iota(t), a).$$

Proposition B.2.7. *Let $\iota\colon D \to E$ be an embedding between two predilators D and E. Then $\iota_X\colon D(X) \to E(X)$ is a strictly increasing map. Furthermore, if $f\colon X \to Y$ is a strictly increasing map between two linear orders X and Y, then the following diagram commutes:*

$$\begin{array}{ccc} D(X) & \xrightarrow{D(f)} & D(Y) \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ E(X) & \xrightarrow[E(f)]{} & E(Y) \end{array}$$

Proof. For ι_X being strictly increasing, suppose that we have $s(a) <_{D(X)} t(b)$, and let Д be the induced diagram from a and b. Then we have $s <_{\text{Д}} t$, so $\iota(s) <_{\text{Д}} \iota(t)$. Since arity $s =$ arity $\iota(s)$ and arity $t =$ arity $\iota(t)$, $(\iota(s), a), (\iota(t), b)$ are in $E(X)$ and moreover we have $\iota(s)(a) <_{E(X)} \iota(t)(b)$, which is $\iota_X(s(a)) <_{E(X)} \iota_X(t(b))$. The commutativity is easy to check. □

B.3 Predilators and Dilators

In this section, we define predilators and dilators, which have the role of linear orders and well-orders at the semidilator level. The reader might wonder why we do not use semidilators directly instead of introducing a new notion called predilators. One of the reasons is that every semidilator appearing in the application is a predilator. Another reason is that finite predilators will turn out to be *dilators*, but not every finite semidilator

is a dilator, as will be illustrated in Example B.6.8. We expect the role between predilators and dilators to be similar to that between linear orders and well-orders, so predilators are a more natural dilator counterpart of linear orders than semidilators.

Definition B.3.1. Let us say an arity diagram

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ n & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{B.3}$$

is *monotone if* $e_0(i) \leq e_1(i)$ for each $i < n$.

A semidilator D is a *predilator* if it satisfies the *monotonicity condition*: For every $t \in \text{field}(D)$ and a non-trivial monotone arity diagram Д of the form (B.3), $t <_\text{Д} t$ holds.

The definition of a predilator indicates we can formulate a first-order theory for predilators over the multi-sorted language, comprising the axioms of semidilators and extra axioms $t <_\text{Д} t$ for each non-trivial monotone arity diagram Д. Girard defined predilators as semidilators preserving the pointwise dominance of two increasing maps. The next lemma says Girard's formulation of predilators is equivalent to our formulation. For two strictly increasing maps $f, g \colon X \to Y$, let us say $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

Proposition B.3.2. *A semidilator D is a predilator iff for every linear order X, Y and strictly increasing maps $f, g \colon X \to Y$, if $f \leq g$ then $D(f) \leq D(g)$.*

Proof. For the left-to-right direction, suppose that $f, g \colon X \to Y$ are two strictly increasing maps such that $f \leq g$. We claim that for each $x \in D(X)$, $D(f)(x) \leq D(g)(x)$. Every element of $D(X)$ has the form $t(a)$ for some $t \in \text{field}(D)$ and $a \subseteq X$ such that $|a| = \text{arity}\, t$. Then

$$D(Y) \vDash D(f)(t(a)) \leq D(g)(t(a)) \iff D(Y) \vDash t(f[a]) \leq t(g[a]).$$

If $f[a] = g[a]$, then $D(Y) \vDash t(f[a]) \leq t(g[a])$ clearly holds. Now consider the case $f[a] \neq g[a]$: Let $n = \text{arity}\, t$ and Д be the induced diagram from $f[a]$ and $g[a]$. Then Д is not trivial. Moreover, if Д has the form (B.3), then

$$e_0(i) \leq e_1(i) \iff f(\text{en}_a(i)) = \text{en}_{f[a]}(i) \leq \text{en}_{g[a]}(i) = g(\text{en}_a(i)),$$

Hence $e_0 \leq e_1$, so the monotonicity condition implies $D \vDash t <_{\text{Д}} t$. Therefore $D(Y) \vDash t(f[a]) < t(g[a])$.

For the right-to-left direction, let Д be a non-trivial monotone arity diagram of the form (B.3), and t be a D-term of arity n. Then $e_0 \leq e_1$ implies $D(e_0) \leq D(e_1)$, so

$$D(n_\cup) \vDash D(e_0)(t(\{0, 1, \cdots, n-1\})) \leq D(e_1)(t(\{0, 1, \cdots, n-1\})).$$

Observe that $D(e_i)(t(\{0, 1, \cdots, n-1\})) = t(\text{ran}\, e_i)$, so we get $D(n_\cup) \vDash t(\text{ran}\, e_0) \leq t(\text{ran}\, e_1)$. By the non-triviality of Д, we have $\text{ran}\, e_0 \neq \text{ran}\, e_1$, so $D(n_\cup) \vDash t(\text{ran}\, e_0) < t(\text{ran}\, e_1)$. Since the induced diagram from $\text{ran}\, e_0$ and $\text{ran}\, e_1$ is again Д, we get $D \vDash t <_{\text{Д}} t$. □

Now, let us define dilators as semidilators preserving well-orderedness:

Definition B.3.3. A semidilator D is a *dilator* if whenever for every well-order α, $D(\alpha)$ is also well-ordered.

It turns out that checking D being a dilator only requires checking the well-orderedness of $D(\alpha)$ for countable α. Note that the following proof requires the axiom of countable choice, which we can avoid when D is countable due to the well-order over D derived from the countability of D.

Proposition B.3.4. *Let D be a predilator. Then D is dilator iff for every countable well-order α, $D(\alpha)$ is well-ordered.*

Proof. Let us only prove the non-trivial direction. Suppose that D is a predilator, and there is an ordinal α such that $D(\alpha)$ is ill-founded. That is, we can find a sequence of D-terms $\langle t_n \mid n < \omega \rangle$ and ordinals $\langle \vec{\xi}_n \mid n < \omega \rangle$ such that $\vec{\xi}_n$ is a finite tuple of elements of α and

$$t_0(\vec{\xi}_0) >_{D(\alpha)} t_1(\vec{\xi}_1) >_{D(\alpha)} t_2(\vec{\xi}_2) >_{D(\alpha)} \cdots.$$

Now consider the well-order β given by the union of all $\vec{\xi}_n$ for $n < \omega$. Then β is countable, and we have

$$t_0(\vec{\xi}_0) >_{D(\beta)} t_1(\vec{\xi}_1) >_{D(\beta)} t_2(\vec{\xi}_2) >_{D(\beta)} \cdots.$$

Hence $D(\beta)$ is ill-founded. □

Note that we need the axiom of countable choice to ensure the countability of β. The previous proposition has no meaning if we work over a subsystem of second-order arithmetic in which every object is countable.

We will see that every dilator is a predilator. The following proof is more coherent with relevant results in Appendix E, and it gives a better bound for a *climax* of a non-predilator semidilators (See Proposition 1.3.5 for more details.)

Proposition B.3.5. *If D is a dilator, then D is a predilator.*

Proof. Suppose that D is a semidilator that is not a predilator. Then we have a non-trivial monotone arity diagram Д of the form (B.3) and a D-term t of arity n such that $D \vDash t \not<_{Д} t$. We begin the proof by decomposing Д into other monotone arity diagrams.

Let $\{k_0 < \cdots < k_{N-1}\} \subseteq n$ be set of natural numbers such that $e_0(k) < e_1(k)$. For each $i \leq N$, define

$$f_i(j) = \begin{cases} e_0(j), & \text{If } j \leq k_{N-1-i} \\ e_1(j), & \text{If } j > k_{N-1-i}. \end{cases}$$

(For technical convenience let $k_{-1} = -1$.) Then $f_0 = e_0$, $f_N = e_1$, and $f_0 \leq f_1 \leq \cdots \leq f_N$. In fact, f_i and f_{i+1} differs at just one point, namely k_{N-1-i}. Hence the induced diagram $\mathrm{Diag}_{n_\cup}(\mathrm{ran}\, f_i, \mathrm{ran}\, f_{i+1})$ is also a non-trivial monotone diagram.

Now define the IU diagram Д for $N+1$ copies of n over $\mathbb{N}$ by the unique diagram isomorphic to the inclusion diagram generated by the intersection and union of $\mathrm{ran}\, f_i$ for $i = 0, 1, \cdots, N$. That said, we define an inclusion diagram $\text{Д}' : \mathbb{L}_{N+1} \to \mathcal{P}(n_\cup)$ by $\text{Д}'(i) = \mathrm{ran}\, f_i$, $\text{Д}'(i \wedge j) = \text{Д}'(i) \cap \text{Д}'(j)$, and $\text{Д}'(i \vee j) = \text{Д}'(i) \cup \text{Д}'(j)$, and Д is the unique IU diagram over $\mathbb{N}$ isomorphic to $\text{Д}'$. Then $\text{Д}[i, i+1] = \mathrm{Diag}_{n_\cup}(\mathrm{ran}\, f_i, \mathrm{ran}\, f_{i+1})$ is monotone and non-trivial, and $\text{Д}[0, N] = \mathrm{Diag}_{n_\cup}(\mathrm{ran}\, e_0, \mathrm{ran}\, e_1) = \text{Д}$. Hence $D \vDash t \not<_\text{Д} t$ implies there is $i < N$ such that $D \vDash t \not<_{\text{Д}[i,i+1]} t$, and let $\text{Ж} = \text{Д}[i, i+1]$. Since f_i and f_{i+1} differ only at one point, Ж must take the form

$$\text{Ж} = \begin{array}{ccc} n-1 & \longrightarrow & n \\ \downarrow & & \downarrow{\scriptstyle g_1} \\ n & \xrightarrow[g_0]{} & n+1 \end{array}$$

and there is a unique $k < n$ such that $g_0(k) < g_1(k)$.

Now we claim that $D(\omega + (n-k))$ is ill-founded. For each $j \in \mathbb{N}$, define $a_i = \{0, \cdots, k-1, k+i, \omega, \cdots, \omega + n - k - 1\}$. Then $|a_i| = n$ and $\mathrm{Diag}(a_i, a_j) = \text{Ж}$ for every $i < j \in \mathbb{N}$. Hence $D(\omega + n - k) \vDash t(a_i) > t(a_j)$ for $i < j$, so $D(\omega + n - k)$ has an infinite decreasing sequence. □

Note that the previous proof shows the following: If D is a semidilator that is not a predilator, then there is $\alpha < \omega + \omega$ such that $D(\alpha)$ is ill-founded. Conversely, we prove that every finite prediltor is a dilator. Let us start with the following lemma:

Lemma B.3.6. *For $n \in \mathbb{N}$, there are finitely many arity diagrams between n and n.*

Proof. First, let us observe that we can identify an arity diagram (B.3) a pair

$(\operatorname{ran} e_0, \operatorname{ran} e_1)$ of subsets of $n_\cup$: For a given $X_0, X_1 \subseteq n_\cup$, we can consider the induced arity diagram $\mathrm{Diag}_{n_\cup}(X_0, X_1)$. Furthermore, the map $Д \mapsto (n_\cup, \operatorname{ran} e_0, \operatorname{ran} e_0 \cap \operatorname{ran} e_1)$ is a bijection from the set of arity diagrams for two copies of n to the set $\{(m, A, B) \mid n \leq m \leq 2n \wedge B \subseteq A \subseteq m\}$. Hence, the number of arity diagrams for two copies of n is equal to $\sum_{n \leq k \leq 2n} \binom{k}{n}\binom{n}{2n-k}$. □

Now let us prove that every finite predilator is a dilator. Its proof uses the infinite Ramsey theorem for pairs, unlike other propositions in this section, so the following proof does not work over RCA_0 or ECA_0:

Proposition B.3.7. *Every finite predilator is a dilator.*

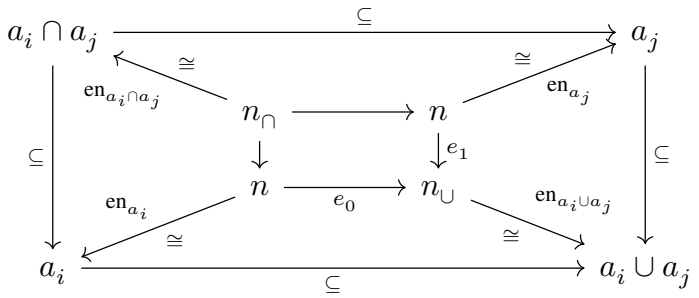
Proof. Suppose that D is a finite semidilator that is not a dilator. We claim that D is not a predilator. Fix a countable well-order α such that $D(\alpha)$ is ill-founded, so we have a sequence of D-terms $\langle t_i \mid i \in \mathbb{N} \rangle$ and finite subsets $\langle a_i \mid i \in \mathbb{N} \rangle$ of α such that

$$D(\alpha) \vDash t_0(a_0) > t_1(a_1) > t_2(a_2) > \cdots.$$

By the Pigeonhole principle, we may assume that all of t_i are equal to t whose arity is n. Now consider the coloring F of domain $[\mathbb{N}]^2$ defined by $F(i, j) = \mathrm{Diag}_\alpha(a_i, a_j)$. Then the range of F is finite by Lemma B.3.6. Hence, by Ramsey's theorem for pairs, we have an infinite homogeneous set $H \subseteq \mathbb{N}$ for F with value Д.

For $i, j \in H$, $t(a_i) >_{D(\alpha)} t(a_j)$ implies $t \not<_Д t$. Д is not trivial since otherwise we have $a_i = a_j$ for $i, j \in H$, so $t(a_i) = t(a_j)$. Now let us claim that Д witnesses D is not a predilator: Suppose that Д has the form (B.3). We claim that $e_0 \leq e_1$ holds. Suppose not, let $k < n$ be a natural number such that $e_0(k) > e_1(k)$. Recall that we have the unique isomorphism $\mathrm{en}_{a_i} : n \to a_i$. Since Д is the induced diagram from a_i and a_j, we

have the following commutative diagram:



Then $e_0(k) > e_1(k)$ implies $a_i(k) > a_j(k)$. It leads to a contradiction since H is infinite, so we get an infinite descending sequence over α. Thus, we have a monotone arity diagram Д of the form (B.3) and $t \in \mathrm{field}(D)$ of arity n such that $D \vDash t \not<_{\text{Д}} t$. Hence, D is not a predilator. □

The previous proposition gives the following 'local' characterization of a predilator:

Corollary B.3.8. *For a semidilator D, D is a predilator iff for each $t \in \mathrm{field}(D)$, the semidilator $D\restriction\{t\}$ obtained from D by restricting the field to the singleton $\{t\}$ is a dilator.*

Proof. By the definition of a predilator, a semidilator D is a predilator iff $D\restriction\{t\}$ is a predilator for each $t \in \mathrm{field}(D)$. Hence Proposition B.3.7 implies the desired result. □

We know that there is at most one isomorphism between two well-orders. The same holds for dilators.

Proposition B.3.9. *Let D be a dilator and $f\colon D \to D$ be an isomorphism. Then f is the identity. As a corollary, if there is an isomorphism from a dilator D to another dilator E, then it is unique.*

Proof. Suppose that an isomorphism $f\colon D \to D$ is not the identity. Then we have $t \in \mathrm{field}(D)$ such that $t \neq f(t)$. For the trivial diagram Д for t, we have either $t <_{\text{Д}} f(t)$

or $t >_{\text{Д}} f(t)$. In the former case, we have

$$D \vDash t >_{\text{Д}} f^{-1}(t) >_{\text{Д}} f^{-2}(t) >_{\text{Д}} f^{-3}(t) >_{\text{Д}} \cdots,$$

so $D(\text{arity}\, t)$ is ill-founded, a contradiction. Similarly, $t <_{\text{Д}} f(t)$ also implies the contradiction. Hence f must be the identity. Now let $f, g \colon D \to E$ be isomorphisms between two dilators. Then $f \circ g^{-1}$ must be the identity. □

In fact, the proof of the previous proposition gives the following stronger conclusion:

Proposition B.3.10. *If D is a semidilator and $D(n)$ is a well-order for every $n \in \mathbb{N}$, then every isomorphism $f \colon D \to D$ is the identity.* □

B.4 Semidilators as functors with the support transformation

There are different ways to define predilators and dilators. Girard [26] defined dilators as autofunctors $F \colon \mathsf{WO} \to \mathsf{WO}$ over the category of well-orders preserving direct limits and pullbacks. Freund [22] defined dilators and predilators as functors $F \colon \mathsf{LO} \to \mathsf{LO}$ admitting the support transformation $\text{supp} \colon F \to [\cdot]^{<\omega}$, a transformation taking an element $t(a) \in F(X)$ and returning the element part a. Freund also proved that his definitions of a predilator and a dilator are equivalent to Girard's.

It is natural to ask if our definitions of a predilator and a dilator are equivalent to those of Freund, and the answer is yes. We prove the equivalence of our definition and Freund's definition in this subsection.

Freund defined two distinct but equivalent notions of predilators: a usual predilator and a coded predilator. Let us introduce the definition for both for the reader's convenience. Let LO be the category of all linear orders with strictly increasing functions. We

say a functor $F\colon \mathsf{LO} \to \mathsf{LO}$ is *monotone* if $f \leq g$ implies $D(f) \leq D(g)$. For a given set X, $[X]^{<\omega}$ is the set of all finite subsets of X. We can turn $[\cdot]^{<\omega}$ to a functor by defining $[f]^{<\omega}(a) = \{f(x) \mid x \in a\}$ for $f\colon X \to Y$.

Definition B.4.1. An *F-semidilator* (F abbreviates 'Freund') is a pair (D, supp), where $F\colon \mathsf{LO} \to \mathsf{LO}$ is a functor and $\mathrm{supp}\colon F \to [\cdot]^{<\omega}$ is a natural transformation satisfying the *support condition*

$$\{\sigma \in F(Y) \mid \mathrm{supp}_Y(\sigma) \subseteq \mathrm{ran}(f)\} \subseteq \mathrm{ran}(F(f)) \tag{B.4}$$

for every morphism $f\colon X \to Y$. We often omit supp and simply write F to mean (F, supp). If F is monotone, then we call F an *F-predilator*.

Definition B.4.2. The *category of F-semidilators* SDil_F is the category whose objects are F-semidilators, and whose morphisms are natural transformations.

We can see that a natural transformation between two F-semidilators preserves the support transformation. The following proof is due to [24, Lemma 2.19]:

Proposition B.4.3. *Suppose that F, G are F-semidilators and $\iota\colon F \to G$ is a natural transformation. Then* $\mathrm{supp}^G \circ \iota = \mathrm{supp}^F$.

Proof. Let X be a linear order and $\sigma \in F(X)$. We also let $\epsilon_\sigma\colon \mathrm{supp}^F_X(\sigma) \to X$ be the inclusion map and $\mathrm{en}_\sigma = \mathrm{en}_{\mathrm{supp}^F_X(\sigma)}$. For $n = |\mathrm{supp}^F_X(\sigma)|$, the support condition implies the existence of $\sigma_0 \in F(n)$ satisfying $\sigma = F(\epsilon_\sigma \circ \mathrm{en}_\sigma)(\sigma_0)$. By the naturality of ι and supp^G, we have

$$\begin{aligned}\mathrm{supp}^G_X(\iota_X(\sigma)) &= \mathrm{supp}^G_X(\iota_X \circ F(\epsilon_\sigma \circ \mathrm{en}_\sigma)(\sigma_0)) = \mathrm{supp}^G_X(G(\epsilon_\sigma \circ \mathrm{en}_\sigma) \circ \iota_n(\sigma_0)) \\ &= [\epsilon_\sigma \circ \mathrm{en}_\sigma]^{<\omega}(\mathrm{supp}^G_n(\iota_n(\sigma_0))) \subseteq \mathrm{ran}\,\epsilon_\sigma = \mathrm{supp}^F_X(\sigma).\end{aligned}$$

For the remaining inclusion, suppose that the inclusion fails and there is $x \in \mathrm{supp}^F_X(\sigma) \setminus \mathrm{supp}^G_X(\iota_X(\sigma))$. Fix $k \in n$ such that $x = \mathrm{en}_\sigma(k)$, then we have $k \in \mathrm{supp}^F_n(\sigma_0) \setminus$

$\mathrm{supp}_n^G(\iota_n(\sigma_0))$ since

$$\mathrm{en}_\sigma(k) = (\epsilon_\sigma \circ \mathrm{en}_\sigma)(k) \in \mathrm{supp}_X^G(\iota_X(\sigma)) = [\epsilon_\sigma \circ \mathrm{en}_\sigma]^{<\omega}[\mathrm{supp}_n^G(\iota_n(\sigma))] \iff k \in \mathrm{supp}_n^G(\iota_n(\sigma)).$$

Now consider $f_0, f_1 \colon n \to n+1$ given by

$$f_0(i) = \begin{cases} i & \text{if } i \leq k, \\ i+1 & \text{if } i > k, \end{cases} \qquad f_1(i) = \begin{cases} i & \text{if } i < k, \\ i+1 & \text{if } i \geq k. \end{cases}$$

Then, on the one hand, we have

- $k = f_0(k) \in [f_0]^{<\omega}[\mathrm{supp}_n^F(\sigma_0)] = \mathrm{supp}_{n+1}^F(F(f_0)(\sigma_0))$, and
- $k \notin \mathrm{ran}\, f_1 \supseteq [f_1]^{<\omega}[\mathrm{supp}_n^F(\sigma_0)] = \mathrm{supp}_{n+1}^F(F(f_1)(\sigma_0))$.

Hence $F(f_0)(\sigma_0) \neq F(f_1)(\sigma_0)$. By the injectivity of ι_{n+1}, we have

$$G(f_0)(\iota_n(\sigma_0)) = \iota_{n+1} \circ F(f_0)(\sigma_0) \neq \iota_{n+1} \circ F(f_1)(\sigma_0) = G(f_1)(\iota_n(\sigma_0)).$$

On the other hand, the support condition also implies the existence of $\tau_0 \in G(|\mathrm{supp}_n^G(\iota_n(\sigma_0))|)$ satisfying

$$\iota_n(\sigma_0) = G(\epsilon_{\iota_n(\sigma_0)} \circ \mathrm{en}_{\iota_n(\sigma_0)})(\tau_0)$$

for the inclusion map $\epsilon_{\iota_n(\sigma_0)} \colon \mathrm{supp}_n^G(\iota_n(\sigma_0)) \to n$ and $\mathrm{en}_{\iota_n(\sigma_0)} = \mathrm{en}_{\mathrm{supp}_n^G(\iota_n(\sigma_0))}$. Since $k \notin \mathrm{ran}\, \epsilon_{\iota_n(\sigma_0)} = \mathrm{supp}_n^G(\iota_n(\sigma_0))$, we have $f_0 \circ \epsilon_{\iota_n(\sigma_0)} \circ \mathrm{en}_{\iota_n(\sigma_0)} = f_1 \circ \epsilon_{\iota_n(\sigma_0)} \circ \mathrm{en}_{\iota_n(\sigma_0)}$. It implies

$$\begin{aligned} G(f_0)(\iota_n(\sigma_0)) &= G(f_0 \circ \epsilon_{\iota_n(\sigma_0)} \circ \mathrm{en}_{\iota_n(\sigma_0)})(\tau_0) \\ &= G(f_1 \circ \epsilon_{\iota_n(\sigma_0)} \circ \mathrm{en}_{\iota_n(\sigma_0)})(\tau_0) = G(f_1)(\iota_n(\sigma_0)), \end{aligned}$$

contradicting the previous inequality. □

The main goal of this section is to construct a canonical category equivalence between the category SDil of semidilators and SDil_F. We first construct an associated semidilator from an F-coded semidilator. It starts by extracting a denotation system from an F-coded semidilator:

Definition B.4.4. The *trace* of an F-semidilator F is given by

$$\mathrm{Tr}(F) = \{(n, \sigma) \mid \sigma \in F(n) \wedge \mathrm{supp}_n(\sigma) = n\}.$$

Roughly, $\mathrm{Tr}(F)$ will be the set of all elements of the form $t(0, 1, \cdots, n-1)$ for a F-term t of arity n. This description will be the key to a later proof. Then we will define a semidilator $\mathfrak{f}(F)$ as follows:

Proposition B.4.5. *For an F-semidilator F, if we define $\mathfrak{f}(F)$ by* $\mathrm{field}(\mathfrak{f}(F)) = \mathrm{Tr}(F)$, $\mathrm{arity}^{\mathfrak{f}(F)}(n, t) = n$, *and for an arity diagram*

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{B.5}$$

and $(n_0, t_0), (n_1, t_1) \in \mathrm{Tr}(F)$, *let us define*

$$(n_0, t_0) <_{\text{Д}} (n_1, t_1) \iff F(n_\cup) \vDash F(e_0)(t_0) < F(e_1)(t_1). \tag{B.6}$$

Then $\mathfrak{f}(F)$ is a semidilator.

Proof. It is easy to check that the comparison rule of $\mathfrak{f}(F)$ satisfies asymmetry and linearity. For transitivity, suppose that we have an IU diagram **Д** for three objects and $t_0, t_1, t_2 \in \mathrm{field}(\mathfrak{f}(F))$ such that $(n_i, t_i) \in \mathrm{Tr}(F)$ for $i = 0, 1, 2$ and

$$\mathfrak{f}(F) \vDash t_0 <_{\text{Д}[0,1]} t_1 <_{\text{Д}[1,2]} t_2.$$

Let n_i ($i \in \mathtt{L}_3$) be objects of **Д**, and $e_{ij} \colon n_i \to n_j$ for $i \leq j \in \mathtt{L}_3$ be morphisms of **Д**. Then we have

1. $F(n_{0\vee 1}) \vDash F(e_{0,0\vee 1})(t_0) < F(e_{1,0\vee 1})(t_1)$, and
2. $F(n_{1\vee 2}) \vDash F(e_{1,1\vee 2})(t_1) < F(e_{2,1\vee 2})(t_2)$.

By applying $F(e_{0\vee 1,0\vee 1\vee 2})$ to the left side, we have

$$F(n_{0\vee 1\vee 2}) \vDash F(e_{0,0\vee 1\vee 2})(t_0) < F(e_{1,0\vee 1\vee 2})(t_1).$$

Similarly, we can see

$$F(n_{0\vee 1\vee 2}) \vDash F(e_{1,0\vee 1\vee 2})(t_1) < F(e_{2,0\vee 1\vee 2})(t_2).$$

Thus we have

$$F(n_{0\vee 1\vee 2}) \vDash F(e_{0,0\vee 1\vee 2})(t_0) < F(e_{2,0\vee 1\vee 2})(t_2).$$

Then by subtracting $F(e_{0\vee 2,0\vee 1\vee 2})$, we get

$$F(n_{0\vee 2}) \vDash F(e_{0,0\vee 2})(t_0) < F(e_{2,0\vee 2})(t_2),$$

which is equivalent to $D \vDash t_0 <_{\text{Д}[0,2]} t_2$. □

We can turn $\mathfrak{f}$ into a functor $\mathsf{SDil}_{\mathsf{F}} \to \mathsf{SDil}$ by defining $\mathfrak{f}(\iota)$ for a natural transformation $\iota \colon F \to G$ by

$$\mathfrak{f}(\iota)(n, t) = (n, \iota_n(t)).$$

Lemma B.4.6. *$\mathfrak{f}(\iota)$ is a semidilator embedding from $\mathfrak{f}(F)$ to $\mathfrak{f}(G)$.*

Proof. $\mathfrak{f}(\iota)$ preserves an arity of a term. To see the comparison rule preservation, suppose we are given an arity diagram Д of the form (B.5), and assume that we are given $\mathfrak{f}(F) \vDash (n_0, t_0) <_{\text{Д}} (n_1, t_1)$, or equivalently,

$$F(n_\cup) \vDash F(e_0)(t_0) < F(e_1)(t_1).$$

Then let us apply $\iota_{n_\cup} \colon F(n_\cup) \to G(n_\cup)$. Since ι is a natural transformation, we have $\iota_{n_\cup} \circ F(e_0) = G(e_0) \circ \iota_{n_0}$ and $\iota_{n_\cup} \circ F(e_1) = G(e_1) \circ \iota_{n_1}$. Hence

$$G(n_\cup) \vDash G(e_0)(\iota_{n_0}(t_0)) < G(e_1)(\iota_{n_1}(t_1)),$$

which is the definition of $\mathfrak{f}(G) \vDash \mathfrak{f}(\iota)(n_0, t_0) <_{\text{Д}} \mathfrak{f}(\iota)(n_1, t_1)$. □

Conversely, we can construct a functor $\mathfrak{a}(D)$ from a semidilator in a natural way:

Definition B.4.7. Let D be a semidilator. Let us define $\mathfrak{a}(D)(X) = D(X)$ for a linear order X. For increasing $f \colon X \to Y$, let us define $\mathfrak{a}(D)(f) \colon \mathfrak{a}(D)(X) \to \mathfrak{a}(D)(Y)$ by

$$\mathfrak{a}(D)(f)(t(a)) = t(f[a]).$$

Then $\mathfrak{a}(D) \colon \mathsf{LO} \to \mathsf{LO}$ is a functor. Also, for a semidilator embedding $\iota \colon D \to E$, let us define the natural transformation $\mathfrak{a}(\iota) \colon \mathfrak{a}(D) \to \mathfrak{a}(E)$ by

$$\mathfrak{a}(\iota)_X(t(a)) = \iota(t)(a).$$

for $t \in \text{field}(D)$ and $a \in [X]^{\text{arity}\, t}$. To turn $\mathfrak{a}(D)$ into an F-semidilator, we need a support transformation: Let us define $\text{supp}^{\mathfrak{a}(D)} \colon \mathfrak{a}(D) \to [\cdot]^{<\omega}$ by

$$\text{supp}^{\mathfrak{a}(D)}_X(t(a)) = a.$$

Lemma B.4.8. $\text{supp}^{\mathfrak{a}(D)}$ *satisfies the support condition, so* $\mathfrak{a}(D)$ *is an F-semidilator.*

Proof. Let X, Y be linear orders and $f \colon X \to Y$ be an increasing function. Suppose that $t \in D$ and $a \in [Y]^{\text{arity}\, t}$, and

$$\text{supp}^{\mathfrak{a}(D)}_X(t(a)) = a \subseteq \text{ran}\, f.$$

Since f is one-to-one, there is $b \in [X]^{<\omega}$ such that $a = f^{"}[b]$, so $t(a) = \mathfrak{a}(D)(f)(t(b))$ by definition. □

So far we constructed functors $\mathfrak{f} \colon \mathsf{SDil}_{\mathsf{F}} \to \mathsf{SDil}$ and $\mathfrak{a} \colon \mathsf{SDil} \to \mathsf{SDil}_{\mathsf{F}}$. We prove they are category equivalences:

Theorem B.4.9. *There is a natural isomorphism* $\mathsf{eaf} \colon \mathfrak{a} \circ \mathfrak{f} \to \mathsf{Id}_{\mathsf{SDil}_{\mathsf{F}}}$.

Proof. Let us describe $\mathfrak{a}(\mathfrak{f}(F))$ for an F-semidilator F: $\mathfrak{a}(\mathfrak{f}(F))$ is a functor from LO to LO, which maps a linear order X to a linear order of field

$$\mathfrak{a}(\mathfrak{f}(F))(X) = \{(n,t)(a) \mid (n,t) \in \mathrm{Tr}(F) \wedge a \in [X]^n\}.$$

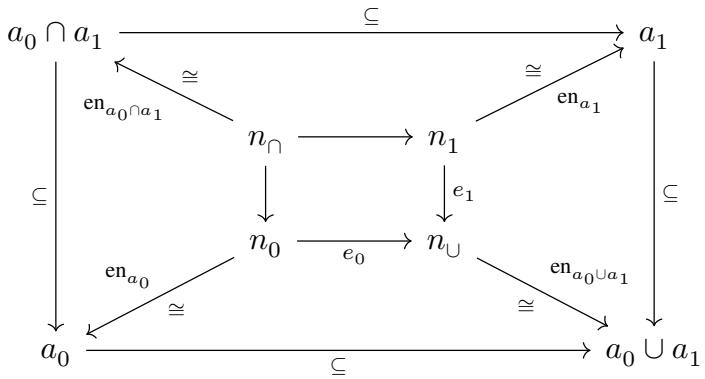
The comparison rule over $\mathfrak{a}(\mathfrak{f}(F))(X)$ is as follows: For $(n_0,t_0),(n_1,t_1) \in \mathrm{Tr}(F)$ and $a_0, a_1 \in [X]^{<\omega}$,

$$\begin{aligned}\mathfrak{a}(\mathfrak{f}(F))(X) \vDash (n_0,t_0)(a_0) < (n_1,t_1)(a_1) \\ \iff \text{For } Д = \mathrm{Diag}_X(a_0,a_1),\ \mathfrak{f}(F) \vDash (n_0,t_0) <_Д (n_1,t_1).\end{aligned}$$

and if $Д$ takes the form (B.5),

$$\mathfrak{f}(F) \vDash (n_0,t_0) <_Д (n_1,t_1) \iff F(n_\cup) \vDash F(e_0)(t_0) < F(e_1)(t_1).$$

Note that $Д = \mathrm{Diag}_X(a_0,a_1)$ implies $n_\cap = |a_0 \cap a_1|$ and $n_\cup = |a_0 \cup a_1|$ and the following diagram commutes:



Hence applying $F(\mathrm{en}_{a_0 \cup a_1})$ to $F(n_\cup) \vDash F(e_0)(t_0) < F(e_1)(t_1)$ gives

$$\mathfrak{f}(F) \vDash (n_0,t_0) <_Д (n_1,t_1) \iff F(a_0 \cup a_1) \vDash F(\mathrm{en}_{a_0})(t_0) < F(\mathrm{en}_{a_1})(t_1).$$

Hence

$$\mathfrak{a}(\mathfrak{f}(F))(X) \vDash (n_0, t_0)(a_0) < (n_1, t_1)(a_1) \\ \iff F(a_0 \cup a_1) \vDash F(\mathrm{en}_{a_0})(t_0) < F(\mathrm{en}_{a_1})(t_1).$$

Now let us define a natural transformation $\mathsf{eaf}_F \colon (\mathfrak{a} \circ \mathfrak{f})(F) \to F$ as follows: For each linear order X, $\mathsf{eaf}_{F,X} \colon (\mathfrak{a} \circ \mathfrak{f})(F)(X) \to F(X)$ is the map

$$\mathsf{eaf}_{F,X}\big((n, t)(a)\big) = F(\mathrm{en}_a)(t).$$

Then $\mathsf{eaf}_{F,X}$ is an isomorphism: It is order-preserving since

$$\begin{aligned} &(\mathfrak{a} \circ \mathfrak{f})(F)(X) \vDash (n_0, t_0)(a_0) < (n_1, t_1)(a_1) \\ \iff &F(a_0 \cup a_1) \vDash F(\mathrm{en}_{a_0})(t_0) < F(\mathrm{en}_{a_1})(t_1) \\ \iff &F(X) \vDash F(\mathrm{en}_{a_0})(t_0) < F(\mathrm{en}_{a_1})(t_1). \end{aligned}$$

$\mathsf{eaf}_{F,X}$ is onto since for every $u \in F(X)$, if we let $a = \mathrm{supp}^F_X(u)$ and $\sigma = F(\mathrm{en}_a^{-1})(u)$, then $(|a|, t) \in \mathrm{Tr}(F)$ and $\mathsf{eaf}_{F,X}((|a|, t)(a)) = u$.

$$\mathfrak{a}\big(\mathfrak{f}(F)\big)(f)\big((n, t)(a)\big) = (n, t)(f[a]).$$

Hence for $(n, t)(a) \in \mathfrak{a}(\mathfrak{f}(F))(X)$,

$$\begin{aligned} F(f) \circ \mathsf{eaf}_{F,X}\big((n, t)(a)\big) &= F(f)(F(\mathrm{en}_a)(t)) = F(f \circ \mathrm{en}_a)(t) \\ &= F(\mathrm{en}_{f[a]})(t) = \mathsf{eaf}_{F,Y}\big((n, t)(f[a])\big) \\ &= \big(\mathsf{eaf}_{F,Y} \circ \mathfrak{a}(\mathfrak{f}(F))(f)\big)\big((n, t)(a)\big), \end{aligned}$$

proving the naturality of eaf_F. However, we also need that ϵ itself is natural in F. To see this, we claim that the following diagram commutes for every natural transformation

$\iota\colon F \to G$ and a linear order X:

$$\begin{array}{ccc} \mathfrak{a}(\mathfrak{f}(F))(X) & \xrightarrow{\mathfrak{a}(\mathfrak{f}(\iota))_X} & \mathfrak{a}(\mathfrak{f}(G))(X) \\ \Big\downarrow{\scriptstyle \mathsf{eaf}_{F,X}} & & \Big\downarrow{\scriptstyle \mathsf{eaf}_{G,X}} \\ F(X) & \xrightarrow[\iota_X]{} & G(X) \end{array}$$

For each $(n,t)(a) \in \mathfrak{a}(\mathfrak{f}(F))(X)$, note that $n = |a|$, so

$$\begin{aligned} \mathsf{eaf}_{G,X}(\mathfrak{a}(\mathfrak{f}(\iota))_X((n,t)(a))) &= \mathsf{eaf}_{G,X}(\mathfrak{f}(\iota)(n,t)(a)) = \mathsf{eaf}_{G,X}((n,\iota_n(t))(a)) \\ &= G(\mathsf{en}_a)(\iota_{|a|}(t)) = \iota_X(F(\mathsf{en}_a)(t)) \\ &= \iota_X(\mathsf{eaf}_{F,X}((n,t)(a))), \end{aligned}$$

here the linebreaked equality follows from $G(\mathsf{en}_a) \circ \iota_{|a|} = \iota_X \circ F(\mathsf{en}_a)$ since $\iota\colon F \to G$ is a natural transformation. It finishes the proof. □

Theorem B.4.10. *There is a natural isomorphism* $\mathsf{efa}\colon \mathsf{Id}_{\mathsf{SDil}} \to \mathfrak{f} \circ \mathfrak{a}$.

Proof. Let us observe the structure of $\mathfrak{f}(\mathfrak{a}(D))$ for a semidilator D. Its field is

$$\begin{aligned} \mathrm{Tr}(\mathfrak{a}(D)) &= \{(n,t) \mid n \in \mathbb{N} \wedge t \in D(n) \wedge \mathrm{supp}_n^{\mathfrak{a}(D)}(t) = n\} \\ &= \{(n,t(n)) \mid n \in \mathbb{N} \wedge t \in \mathrm{field}(D) \wedge \mathrm{arity}^D(t) = n\} \end{aligned}$$

and $\mathrm{arity}^{\mathfrak{f}(\mathfrak{a}(D))}(n,t(n)) = n$. For an arity diagram Д of the form (B.5),

$$\begin{aligned} &\mathfrak{f}(\mathfrak{a}(D)) \vDash (n_0, t_0(n_0)) <_{\text{Д}} (n_1, t_1(n_1)) \\ \iff &D(n_\cup) \vDash D(e_0)(t_0(n_0)) < D(e_1)(t_1(n_1)) \\ \iff &D(n_\cup) \vDash t_0(e_0[n_0]) < t_1(e_1[n_1]), \end{aligned}$$

and observe that $\mathrm{Diag}_{n_\cup}(e_0[n_0], e_1[n_1]) = \text{Д}$, so

$$D(n_\cup) \vDash t_0(e_0[n_0]) < t_1(e_1[n_1]) \iff D \vDash t_0 <_{\text{Д}} t_1.$$

That is, $\mathfrak{f}(\mathfrak{a}(D)) \vDash (n_0, t_0(n_0)) <_{\text{Д}} (n_1, t_1(n_1)) \iff D \vDash t_0 <_{\text{Д}} t_1$. Hence if we let $\mathsf{efa}_D\colon D \to \mathfrak{f}(\mathfrak{a}(D))$ by $\mathsf{efa}_D(t) = (\mathrm{arity}\, t, t(\mathrm{arity}\, t))$, then efa_D is a semidilator isomorphism. The naturality of efa follows from an easy computation. □

Combining all the results, we have

Corollary B.4.11. SDil *and* $\mathsf{SDil}_{\mathsf{F}}$ *are category equivalent.* □

Hence, we can equate the two categories SDil and $\mathsf{SDil}_{\mathsf{F}}$.

Remark B.4.12. Throughout this book, we will often conflate denotation systems with semidilators with support transformation. For example, we will frequently construct an F-semidilator F instead of a denotation system. In this case, we identify F with the corresponding denotation system $\mathfrak{f}(F)$ (or a semidilator isomorphic to $\mathfrak{f}(F)$). Similarly, we often call a denotation system D an F-semidilator instead, which means $\mathfrak{a}(D)$ instead of D itself. Throughout the book, we will introduce various equivalent but different notions of semidilators or predilators. In each case, we introduce the category equivalences between the two different notions, and the equivalences provide information on how to identify them.

Also, let us observe that the construction of $\mathfrak{f}(F)$ does not require the whole F. In fact, we can construct $\mathfrak{f}(F)$ from the restriction of F to the category Nat of natural numbers. Hence, we can construct a semidilator not only from a whole F-semidilator but also from the following object:

Definition B.4.13. An *F-coded semidilator* is a pair (F, supp), where $F : \mathbb{N} \to \mathsf{LO}$ is a functor and supp is a natural transformation with the support condition (B.4). If F is monotone, we say F is *F-coded predilator*.

The previous proofs give the following corollary:

Corollary B.4.14. *If F is an F-coded semidilator, then we can construct a semidilator $\mathfrak{f}(F)$. If F is an F-semidilator, then $\mathfrak{f}(F) = \mathfrak{f}(F \restriction \mathsf{Nat})$ and $F \cong \mathfrak{a}(\mathfrak{f}(F \restriction \mathsf{Nat}))$.* □

We finish this section with the recursiveness of an F-semidilator:

Proposition B.4.15. *Suppose that F is an F-coded semidilator that is coded by a (primitive) recursive subset of $\mathbb{N}$. Then $\mathfrak{f}(F)$ is (primitive) recursive.*

Proof. We can see that the set of terms, the arity function, and the comparison rules of $\mathfrak{f}(F)$ are primitive recursive in F, so the desired result follows. □

Remark B.4.16. Fruend-style definition of semidilators does not require an F-semidilator F to be *inclusive* in the sense that if $X \subseteq Y$, then $F(X) \subseteq F(Y)$. However, every functor induced from a denotational system is inclusive, and we proved that F is naturally isomorphic to $\mathfrak{a}(\mathfrak{f}(F))$. Hence, we also have the following: Every F-semidilator is naturally isomorphic to an inclusive F-semidilator.

B.5 Semidilators with multiple arguments

We will see that the sum is a dilator in the sense that $X \mapsto X + Y$ and $Y \mapsto X + Y$ are dilators. In this section, we develop a notion of a multi-variable semidilator. We will only consider binary semidilators, but all definitions and arguments we introduce can be easily generalized to semidilators with multiple arguments. Throughout the section, we will omit proofs for statements whose proof is similar to the unary case.

Definition B.5.1. A *binary semidilator* is a structure D, whose field is called the set of *D-terms*, and field(D) denotes the set of all D-terms. D equips with two arity functions arity$_0$ and arity$_1$. For two arity diagrams Д and Ж of the form

$$\text{Д} = \begin{array}{ccc} m_\cap & \longrightarrow & m_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ m_0 & \xrightarrow[e_0]{} & m_\cup \end{array} \qquad \text{Ж} = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ n_0 & \xrightarrow[f_0]{} & n_\cup \end{array} \tag{B.7}$$

we have a comparison relation $<_{\text{Д,Ж}}$. For two D-terms t_0, t_1, we say $(\text{Д}, \text{Ж})$ is *a pair of arity diagrams for* t_0, t_1 if arity$_0(t_0) = m_0$, arity$_0(t_1) = m_1$, arity$_1(t_0) = n_0$, and

$\text{arity}_1(t_1) = n_1$. Then D satisfies the following axioms:

1. (Irreflexivity) If both Д and Ж are trivial and they are arity diagrams for t, t, then $t \not<_{\text{Д,Ж}} t$.
2. (Linearity) If $(\text{Д}, \text{Ж})$ is a pair of arity diagrams for t_0, t_1, and if either $t_0 \neq t_1$ or one of Д or Ж is not trivial, then either $t_0 <_{\text{Д,Ж}} t_1$ or $t_1 <_{-\text{Д},-\text{Ж}} t_0$.
3. (Transitivity) Let **Д** and **Ж** be two IU diagrams. If $(\mathbf{Д}(i, i+1), \mathbf{Ж}(i, i+1))$ is a pair of arity diagrams for t_i, t_{i+1} for $i = 0, 1$, and if we have $t_0 <_{\mathbf{Д}[0,1],\mathbf{Ж}[0,1]} t_1 <_{\mathbf{Д}[1,2],\mathbf{Ж}[1,2]} t_2$, then $t_0 <_{\mathbf{Д}[0,2],\mathbf{Ж}[0,2]} t_2$.

We can also define the application of a binary semidilator:

Definition B.5.2. Let D be a binary semidilator and X, Y be linear orders. Then let us define $D(X,Y) = \{(t,a,b) \mid a \in [X]^{\text{arity}_0(t)} \wedge b \in [Y]^{\text{arity}_1(t)}\}$. We write $t(a;b)$ instead of (t,a,b), and let us define the order relation as follows: For two $t_0(a_0;b_0)$ and $t_1(a_1;b_1)$, let $\text{Д} = \text{Diag}_X(a_0, a_1)$ and $\text{Ж} = \text{Diag}_Y(b_0, b_1)$. Then we define

$$t_0(a_0;b_0) < t_1(a_1;b_1) \iff D \vDash t_0 <_{\text{Д,Ж}} t_1.$$

Proposition B.5.3. *If D is a binary semidilator and X, Y are linear orders, then $D(X,Y)$ is a linear order. Furthermore, for $f\colon X \to X'$, $g\colon Y \to Y'$, if we define $D(f,g)\colon D(X,Y) \to D(X',Y')$ by*

$$D(f,g)(t(a;b)) = t(f[a]; g[b]),$$

then D induces a functor from LO^2 to LO; That is,

1. $D(f,g) \circ D(f',g') = D(f \circ f', g \circ g')$.
2. $D(\mathsf{Id}_X, \mathsf{Id}_Y) = \mathsf{Id}_{D(X,Y)}$.

Similar to the unary case, we can define an embedding between two binary semidilators:

Definition B.5.4. Let D and E be two binary semidilators. An *embedding* or a *binary semidilator morphism* $\iota\colon D \to E$ is a function $\iota\colon \text{field}(D) \to \text{field}(E)$ with the following conditions:

1. ι preserves arity: That is, $\text{arity}_0(\iota(t)) = \text{arity}_0(t)$ and $\text{arity}_1(\iota(t)) = \text{arity}_1(t)$ for every $t \in \text{field}(D)$.
2. For each $t_0, t_1 \in \text{field}(D)$ and a pair of arity diagrams Д, Ж for t_0, t_1, $t_0 <_{\text{Д,Ж}} t_1$ implies $\iota(t_0) <_{\text{Д,Ж}} \iota(t_1)$.

Proposition B.5.5. *Suppose that $\iota\colon D \to E$ is an embedding between two binary semidilators D and E.*

1. *$\iota\colon \text{field}(D) \to \text{field}(E)$ is injective.*
2. *If ι is a bijection, then $\iota^{-1}\colon E \to D$ is an embedding.*
3. *For two linear orders X, Y, if we define $\iota_{X,Y}\colon D(X,Y) \to E(X,Y)$ by*

$$\iota_{X,Y}(t(a;b)) = \iota(t)(a;b)$$

and furthermore if $f\colon X \to X'$, $g\colon Y \to Y'$ are increasing maps, then the following diagram commutes:

$$\begin{array}{ccc} D(X,Y) & \xrightarrow{D(f,g)} & D(X',Y') \\ {\scriptstyle \iota_{X,Y}}\downarrow & & \downarrow{\scriptstyle \iota_{X',Y'}} \\ E(X,Y) & \xrightarrow[E(f,g)]{} & E(X',Y') \end{array}$$

We can also define a binary version of predilator and dilator:

Definition B.5.6. A binary semidilator D is a *binary predilator* if for every monotone arity diagrams Д, Ж, whenever if (Д, Ж) is a pair of arity diagrams for $t \in \mathrm{field}(D)$ and t itself, then $t <_{Д,Ж} t$.

A binary semidilator D is a *binary dilator* if whenever X, Y are well-orders, then so is $D(X, Y)$.

Proposition B.5.7. *A binary semidilator D is a binary predilator iff for every $f, f' \colon X \to X'$ and $g, g' \colon Y \to Y'$, if whenever $f \leq f'$ and $g \leq g'$, then $D(f, f') \leq D(g, g')$.*

Proposition B.5.8. *Every binary dilator is a binary predilator.*

Sketch of the proof. We mimic the proof of Proposition B.3.5: Suppose that $D \vDash t \not<_{Д,Ж} t$ for some monotone Д and Ж. Like we did in the proof of Proposition B.3.5, let us decompose Д and Ж into IU diagrams **Д** and **Ж** with $N+1$ and $M+1$ objects respectively. Especially, we have $\mathbf{Д}[0, N] = Д$ and $\mathbf{Ж}[0, M] = Ж$.

If

$$D \vDash t <_{\mathbf{Д}[0,1],\mathbf{Ж}[0,1]} t <_{\mathbf{Д}[1,2],\mathbf{Ж}[0,1]} \cdots <_{\mathbf{Д}[N-1,N],\mathbf{Ж}[0,1]} t <_{\mathbf{Д}[0,N],\mathbf{Ж}[1,2]} \cdots \cdots <_{\mathbf{Д}[0,N],\mathbf{Ж}[M-1,M]} t,$$

then $D \vDash t <_{Д,Ж} t$. Hence we have $i < N$ such that $D \vDash t \not<_{\mathbf{Д}[i.i+1],\mathbf{Ж}[0,1]} t$, or $j < M$ such that $D \vDash t \not<_{\mathbf{Д}[0,N],\mathbf{Ж}[j,j+1]} t$. In either case, we can see that $D(\omega + \omega, \omega + \omega)$ is ill-founded. □

The following proposition follows from a similar proof of Proposition B.3.7:

Proposition B.5.9. *Every finite binary predilator is a binary dilator.*

We can also formulate a Freund-style binary semidilator as follows:

Definition B.5.10. We say a functor $F\colon \mathsf{LO} \times \mathsf{LO} \to \mathsf{LO}$ is an *binary F-semidilator* if it admits two natural transformations $0\text{-}\operatorname{supp}^F_{X,Y}\colon F(X,Y) \to [X]^{<\omega}$ and $1\text{-}\operatorname{supp}^F_{X,Y}\colon F(X,Y) \to [Y]^{<\omega}$ such that for every $f\colon X \to X'$, $g\colon Y \to Y'$, we have the *support condition*

$$\{\sigma \in F(X',Y') \mid 0\text{-}\operatorname{supp}^F_{X',Y'}(\sigma) \subseteq \operatorname{ran} f \wedge 1\text{-}\operatorname{supp}^F_{X',Y'}(\sigma) \subseteq \operatorname{ran} g\} \subseteq \operatorname{ran} F(f,g). \tag{B.8}$$

The *category of binary F-semidilators* is a category whose objects are binary semidilators, and morphisms are natural transformations $\iota\colon F \to G$ satisfying

$$i\text{-}\operatorname{supp}^G_{X,Y} \circ \iota_{X,Y} = i\text{-}\operatorname{supp}^F_{X,Y}$$

for $i = 0, 1$ and linear orders X, Y.

We will see that the category of binary semidilators SDil_2 and the category of $\mathsf{SDil}_{\mathsf{F},2}$ are equivalent. Its proof is more or less similar to the unary case. To define a functor $\mathfrak{f}\colon \mathsf{SDil}_{\mathsf{F},2} \to \mathsf{SDil}_2$, we need the trace of an F-binary semidilator:

Definition B.5.11. Let F be a binary F-semidilator. Define its *trace* by

$$\operatorname{Tr}(F) = \{(n_0, n_1, \sigma) \mid \sigma \in F(n_0,n_1) \wedge 0\text{-}\operatorname{supp}^F_{n_0,n_1}(\sigma) = n_0 \wedge 1\text{-}\operatorname{supp}^F_{n_0,n_1}(\sigma) = n_1\}.$$

Proposition B.5.12. *For a binary F-semidilator F, if we define $\mathfrak{f}(F)$ by* $\operatorname{field}\mathfrak{f}(F) = \operatorname{Tr}(F)$, $\operatorname{arity}^{\mathfrak{f}(F)}_i(n_0,n_1,t) = n_i$ *for $i = 0, 1$. Also, for a pair of arity diagrams*

$$Д^0 = \begin{array}{ccc} n^0_\cap & \longrightarrow & n^0_1 \\ \downarrow & & \downarrow{\scriptstyle e^0_1} \\ n^0_0 & \xrightarrow[e^0_0]{} & n^0_\cup \end{array} \qquad \textit{and} \qquad Д^1 = \begin{array}{ccc} n^1_\cap & \longrightarrow & n^1_1 \\ \downarrow & & \downarrow{\scriptstyle e^1_1} \\ n^1_0 & \xrightarrow[e^1_0]{} & n^1_\cup \end{array}$$

let us define

$$(n^0_0, n^0_1, t_0) <_{Д^0, Д^1} (n^1_0, n^1_1, t_1) \iff F(n^0_\cup, n^1_\cup) \vDash F(e^0_0, e^1_0)(t_0) < F(e^0_1, e^1_1)(t_1).$$

Then $\mathfrak{f}(F)$ *is a binary semidilator.*

Again, let us turn $\mathfrak{f}$ to a functor $\mathsf{SDil}_{\mathsf{F},2} \to \mathsf{SDil}$ by defining

$$\mathfrak{f}(\iota)(n_0, n_1, t) = (n_0, n_1, \iota(t))$$

for a natural transformation $\iota \colon F \to G$.

Lemma B.5.13. $\mathfrak{f}(\iota)$ *is a binary semidilator embedding from* $\mathfrak{f}(F)$ *to* $\mathfrak{f}(G)$.

We can also define a functor $\mathfrak{a}(D)$ from a binary semidilator:

Definition B.5.14. Let D be a binary semidilator. Let us define $\mathfrak{a}(D)(X, Y) = D(X, Y)$ for linear orders X and Y. For increasing $f \colon X \to X'$ and $g \colon Y \to Y'$, let us define $\mathfrak{a}(D)(f, g) \colon \mathfrak{a}(D)(X, Y) \to \mathfrak{a}(D)(X', Y')$ by

$$\mathfrak{a}(P)(f, g)(t(a; b)) = t(f[a]; g[b]).$$

Then $\mathfrak{a}(D) \colon \mathsf{LO}^2 \to \mathsf{LO}$ is a functor. Also, for a binary semidilator embedding $\iota \colon D \to E$, let us define the natural transformation $\mathfrak{a}(\iota) \colon \mathfrak{a}(D) \to \mathfrak{a}(E)$ by

$$\mathfrak{a}(\iota)_{X,Y}(t(a; b)) = \iota(t)(a; b).$$

for $t \in \mathrm{field}(D)$, $a \in [X]^{\mathrm{arity}_0 t}$, and $b \in [X]^{\mathrm{arity}_1 t}$. To turn $\mathfrak{a}(D)$ into a binary F-semidilator, we need support transformations: Let us define $0\text{-}\mathrm{supp}^{\mathfrak{a}(D)} \colon \mathfrak{a}(D) \to [\cdot]^{<\omega}$ and $1\text{-}\mathrm{supp}^{\mathfrak{a}(D)} \colon \mathfrak{a}(D) \to [\cdot]^{<\omega}$ by

1. $0\text{-}\mathrm{supp}^{\mathfrak{a}(D)}_{X,Y}(t(a; b)) = a$,
2. $1\text{-}\mathrm{supp}^{\mathfrak{a}(D)}_{X,Y}(t(a; b)) = b$.

Lemma B.5.15. $i\text{-}\mathrm{supp}^{\mathfrak{a}(D)}$ *satisfies the support condition, so* $\mathfrak{a}(D)$ *is an F-semidilator.*

Again, we can show that $\mathfrak{f}\colon \mathsf{SDil}_{\mathsf{F},2} \to \mathsf{SDil}_2$ and $\mathfrak{a}\colon \mathsf{SDil}_2 \to \mathsf{SDil}_{\mathsf{F},2}$ are category equivalences:

Theorem B.5.16. *There are natural isomorphisms* $\mathsf{eaf}_2\colon \mathfrak{a}\circ\mathfrak{f} \to \mathsf{Id}_{\mathsf{SDil}_{\mathsf{F},2}}$ *and* $\mathsf{efa}_2\colon \mathfrak{f}\circ\mathfrak{a} \to \mathsf{Id}_{\mathsf{SDil}_{\mathsf{F}}}$.

Proof. We only provide the definition of eaf_2 and efa_2: For a binary F-semidilator F and linear orders X and Y,

$$(\mathsf{eaf}_2)_{F,X,Y}((n_0,n_1,t))(a;b) = F(\mathrm{en}_a,\mathrm{en}_b)(t).$$

Also, for a binary semidilator D, let us define

$$(\mathsf{efa}_2)_D(t) = (\mathrm{arity}_0(t), \mathrm{arity}_1(t), t(\mathrm{arity}_0(t), \mathrm{arity}_1(t))). \qquad \square$$

It shows that SDil_2 and $\mathsf{SDil}_{\mathsf{F},2}$ are category equivalent. As we did for unary F-semidilators, we identify a binary F-semidilator F with $\mathfrak{f}(F)$, so we view F as a binary semidilator with its field $\mathrm{Tr}(F)$. Like the unary case, we can recover the full binary semidilator from a binary F-coded semidilator:

Definition B.5.17. A *binary F-coded semidilator* is a triple $(F, 0\text{-}\,\mathrm{supp}, 1\text{-}\,\mathrm{supp})$, where $F\colon \mathbb{N}^2 \to \mathsf{LO}$ is a functor and $0\text{-}\,\mathrm{supp}$, $1\text{-}\,\mathrm{supp}$ are natural transformations with the support condition (B.8).

Corollary B.5.18. *If F is a binary F-coded semidilator, then we can construct a semidilator $\mathfrak{f}(F)$. If F is an F-semidilator, then $\mathfrak{f}(F) = \mathfrak{f}(F \restriction \mathbb{N}^2)$ and $F \cong \mathfrak{a}(\mathfrak{f}(F \restriction \mathbb{N}^2))$.*

Also, calculating $\mathfrak{f}$ is primitive recursive in the parameter, so we have

Proposition B.5.19. *Suppose that F is a binary F-coded semidilator that is coded by a (primitive) recursive subset of $\mathbb{N}$. Then $\mathfrak{f}(F)$ is (primitive) recursive.*

We finish this section with the following theorem, stating that a 'slice' of a binary semidilator is a semidilator.

Proposition B.5.20. *Let D be a binary semibilator and X be a linear order. If we define a functor $E\colon \mathsf{LO} \to \mathsf{LO}$ by $E(Y) = D(X, Y)$, $E(f) = D(\mathsf{Id}_X, f)$, then E is naturally isomorphic to a semidilator. If D is a binary dilator and X is a well-order, then E is a dilator.*

Proof. Let us define a structure D_X as follows: D_X has field

$$\{(t, a) \mid t \in \text{field}(D) \wedge a \in [X]^{\text{arity}_0(t)}\},$$

with $\text{arity}^{D_X}(t, a) = \text{arity}_1^D(t)$, and for an arity diagram Д for (s, a), (t, b) for $s, t \in \text{field}(D)$, $a, b \in [X]^{<\omega}$,

$$D_X \vDash (s, a) <_{\text{Д}} (t, b) \iff D \vDash s <_{\text{Diag}_X(a,b),\text{Д}} t.$$

Then we can see that D_X is a semidilator. From now on, let us write $t(a)$ instead of (t, a). Then let us define the map $\mathsf{smn}^{1,1}_{D,X,Y}\colon D(X, Y) \to D_X(Y)$ by

$$\mathsf{smn}^{1,1}_{D,X,Y}(t(a; b)) = (t(a))(b).$$

Then we can easily see that $\mathsf{smn}^{1,1}_{D,X,Y}$ is an isomorphism natural in Y. □

By the previous proposition, we can identify $E = D(X, \cdot)$ with D_X. Note that $\mathsf{smn}^{1,1}_{D,X,Y}$ is not only natural in Y, but also natural in D and X. For future use, let us provide the formal definition of D_X:

Definition B.5.21. Let D be a binary semidilator and X a linear order. Let us define the semidilator $D_X = D(X, \cdot)$ as follows: Its terms have the form $t(a)$ for some $t \in \text{field}(D)$ and $a \in [X]^{\text{arity}_0(t)}$. The comparison rule is given by

$$D_X \vDash s(a) <_{\text{Д}} t(b) \iff D \vDash s <_{\text{Diag}_X(a,b),\text{Д}} t.$$

Remark B.5.22. It is often convenient to think of binary semidilators as functors taking a disjoint sum of two linear orders (which is a partial linear order) instead of two linear orders, which is possible since the map $(X, Y) \mapsto X \sqcup Y$ is a category isomorphism between LO^2 and the subcategory of the category of partial orders whose objects are a disjoint union of two linear orders and morphisms are the disjoint union of two increasing maps in LO. Under this perspective, we will frequently use the following notions:

- Instead of writing $\mathrm{arity}_0(t) = n_0$ and $\mathrm{arity}_1(t) = n_1$, we write $\mathrm{arity}(t) = n_0 \sqcup n_1$.
- We understand a binary F-semidilator F as a functor taking a disjoint union of two linear orders, and instead of using two support transformations $0\text{-}\,\mathrm{supp}^F$ and $1\text{-}\,\mathrm{supp}^F$, we use supp^F given by

$$\mathrm{supp}^F_{X,Y}(\sigma) = \{\underset{0}{x} \mid x \in 0\text{-}\,\mathrm{supp}^F_{X,Y}(\sigma)\} \cup \{\underset{1}{y} \mid y \in 1\text{-}\,\mathrm{supp}^F_{X,Y}(\sigma)\}.$$

- We also pretend $n_0 \sqcup n_1$ for $n_0, n_1 \in \mathbb{N}$ has a role of natural numbers in defining binary semidilators. For example, every finite object in LO^2 is uniquely isomorphic to an object of the category Nat^2 (which is a full subcategory of LO^2, whose objects are the disjoint sum of natural numbers), so we can talk about the isomorphism type $\|X\| \in \mathsf{Nat}^2$ of X with the isomorphism $\mathrm{en}_X \colon \|X\| \to X$. Also, we may think of the pair of arity diagrams Д, Ж in comparison rule $<_{Д,Ж}$ as a single 'arity diagram' over the category Nat^2 given by

$$Д \sqcup Ж = \begin{array}{ccc} Д(\bot) \sqcup Ж(\bot) & \longrightarrow & Д(0) \sqcup Ж(0) \\ \downarrow & & \downarrow{\scriptstyle Д(0,\top)\sqcup Ж(0,\top)} \\ Д(1) \sqcup Ж(1) & \xrightarrow[Д(1,\top)\sqcup Ж(1,\top)]{} & Д(\top) \sqcup Ж(\top) \end{array}$$

B.6 Examples of dilators

Example B.6.1. For a linear order X, a constant semidilator $\underline{X}(Y) = X$ is a semidilator. More precisely, $\underline{X}$ is a semidilator given by $\mathrm{field}(\underline{X}) = X$ and $\mathrm{arity}(x) = 0$ for every $x \in X$. The only arity diagram for two copies of 0 is the trivial diagram, and for the trivial diagram Д, we define

$$x <_{\text{Д}} y \iff X \vDash x < y.$$

If X is a well-order, then $\underline{X}$ is a dilator.

Example B.6.2. The usual operator for ordinals can be viewed as binary dilators. For example, the addition

$$X + Y = \{\mathsf{in}_0(x) \mid x \in X\} \cup \{\mathsf{in}_1(y) \mid y \in Y\}$$

with the order $\mathsf{in}_0(x) < \mathsf{in}_1(y)$ is a binary dilator. The addition maps two increasing maps $f \colon X \to X'$ and $g \colon Y \to Y'$ into a new map $f + g \colon X + Y \to X' + Y'$ defined by

$$(f + g)(\mathsf{in}_0(x)) = \mathsf{in}_0(f(x)),\ (f + g)(\mathsf{in}_1(y)) = \mathsf{in}_1(g(y)).$$

Also, the addition has the support transformations

- $0\text{-}\operatorname{supp}^+_{X,Y}(\mathsf{in}_0(x)) = \{x\}$, $0\text{-}\operatorname{supp}^+_{X,Y}(\mathsf{in}_1(y)) = \emptyset$.
- $1\text{-}\operatorname{supp}^+_{X,Y}(\mathsf{in}_0(x)) = \emptyset$, $1\text{-}\operatorname{supp}^+_{X,Y}(\mathsf{in}_1(y)) = \{y\}$.

Equivalently, the addition is a denotation system with two terms $\{\mathsf{in}_0, \mathsf{in}_1\}$ with

$$\mathrm{arity}_i(\mathsf{in}_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}$$

for $i, j \in \{0, 1\}$, with the comparison rules $\mathsf{in}_0 < \mathsf{in}_1$, $\mathsf{in}_0 <_{\mathrm{Diag}_2(\{0\},\{1\}),\emptyset} \mathsf{in}_0$, and $\mathsf{in}_1 <_{\mathrm{Diag}_2(\emptyset,\{0\},\{1\})} \mathsf{in}_1$.

We can also turn the multiplication

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

with the lexicographical order into a binary dilator. The detail is left to the reader.

For exponentials, consider

$$(1 + X)^Y = \exp(X, Y) = \{\langle (y_i, x_i) : i < n\rangle \in [Y \times X]^{<\omega} \mid \gamma \vDash y_0 > \cdots > y_{n-1}\}$$

equipped with the lexicographic order (so the pairs are compared by lexicographically, and the same for tuples). Function application is defined pointwise, and let us equip $0\text{-}\operatorname{supp}^{\exp}_{\alpha,\gamma}(\langle (x_i, y_i) \mid i < n\rangle) = \{y_i \mid i < n\}$, $1\text{-}\operatorname{supp}^{\exp}_{\alpha,\gamma} = \{x_i \mid i < n\}$.

When $\alpha = 1 + X$ is an ordinal (in particular when $\alpha = 2, \omega$), we express elements of α^Y by

$$\alpha^{y_0} \cdot x_0 + \alpha^{y_1} \cdot x_1 + \cdots \alpha^{y_{n-1}} \cdot x_{n-1}$$

instead of $\langle (y_0, x_0), (y_1, x_1), \cdots, (y_{n-1}, x_{n-1})\rangle$. (Zero corresponds to the empty sequence.)

Example B.6.3. Operators between two (semi)dilators give a new (semi)dilator. For example, for two (semi)dilators D and E, if we define

1. $(D + E)(X) = D(X) + E(X)$,
2. $(D + E)(f) = D(f) + E(f)$ for $f \colon X \to Y$,
3. $\operatorname{supp}^{D+E}_X(\sigma)$ is $\operatorname{supp}^D_X(\sigma)$ if $\sigma \in D(X)$ and $\operatorname{supp}^E_X(\sigma)$ otherwise.[1]

[1] Here is an abuse of notation: More precisely, $\operatorname{supp}^{D+E}_X(\sigma) = \operatorname{supp}^D_X(\sigma')$ if $\sigma = \mathsf{in}_0(\sigma')$ for some $\sigma' \in D(X)$, and vice versa.

then $D + E$ is a new (semi)dilator. In terms of denotation system, $D + E$ is the set of terms $\{\mathsf{in}_0(s) \mid s \in \text{field}\, D\} \cup \{\mathsf{in}_1(t) \mid t \in \text{field}\, E\}$ with the same arities and comparison rules from the previous semidilators, and $\mathsf{in}_0(s) <_{Д} \mathsf{in}_1(t)$ for every arity diagram Д between $\mathsf{in}_0(s)$ and $\mathsf{in}_1(t)$.

Example B.6.4. For a binary semidilator D, let us define the *diagonal* $\mathbf{\Delta}(D)$ of D by

$$\mathbf{\Delta}(D)(X) = D(X, X),\ \mathbf{\Delta}(D)(f) = D(f, f)$$

for a linear order X and an increasing $f \colon X \to Y$. $\mathbf{\Delta}(D)$ is a semidilator under the following support transformation:

$$\text{supp}_X^{\mathbf{\Delta}(D)}(\sigma) = 0\text{- supp}_X^D(\sigma) \cup 1\text{- supp}_X^D(\sigma).$$

We can turn $\mathbf{\Delta}$ into a functor from the category of binary semidilators to the category of semidilators by letting $\mathbf{\Delta}(\iota)_X = \iota_{X,X}$ for a binary semidilator morphism $\iota \colon D \to E$.

Example B.6.5. Let I be a linear order and $\{D_i \mid i \in I\}$ be a family of semidilators. We define the indexed sum $D = \sum_{i \in I} D_i$ as follows: D-terms are the form $\mathsf{in}_i(t)$ for $i \in I$ and $t \in D_i$. We take $\text{arity}^D(\mathsf{in}_i(t)) = \text{arity}^{D_i}(t)$, and for two D-terms $\mathsf{in}_{i_0}(t_0), \mathsf{in}_{i_1}(t_1) \in D$ and an arity diagram Д between these two, we define $D \vDash \mathsf{in}_{i_0}(t_0) <_{Д} \mathsf{in}_{i_1}(t_1)$ iff either

1. $I \vDash i_0 < i_1$, or
2. $i_0 = i_1$ and $D_{i_0} \vDash t_0 <_{Д} t_1$.

Note that some literature (especially, Girard's [26]) denotes $\mathsf{in}_i(t)$ by $\sum_{j<i} D_j + t$.

We can see that for a linear order X, $\left(\sum_{i \in I} D_i\right)(X) \cong \sum_{i \in I} D_i(X)$, so $\sum_{i \in I} D_i$ is a dilator if I is a well-order and each D_i is a dilator.

We say a semidilator D is *prime* if the size of field(D) is 1. The *order* of a prime semidilator is the arity of a unique term of the prime semidilator. The following example describes a prime dilator whose unique term has arity n:

Example B.6.6. Fix a permutation σ over n. Define the semidilator Prime_σ as follows: For a linear order X,

$$\mathsf{Prime}_\sigma(X) = \{\langle x_{\sigma(0)}, \cdots, x_{\sigma(n-1)}\rangle \mid X \vDash x_0 < \cdots < x_{n-1}\}$$

with the lexicographic order. Then define

$$\mathrm{supp}_X^{\mathsf{Prime}_\sigma}(\langle x_{\sigma(0)}, \cdots, x_{\sigma(n-1)}\rangle = \{x_0, \cdots, x_{n-1}\}.$$

We can show that Prime_σ is a dilator. To see Prime_σ is a prime dilator, observe that the trace of Prime_σ has the unique element $(n, \langle \sigma(0), \cdots, \sigma(n-1)\rangle)$. We call Prime_σ the *σ-lexicographical prime dilator of degree n.* We will see in Theorem C.2.4 that every prime dilator of order n is isomorphic to Prime_σ for some permutation σ over n.

Example B.6.7. For two semidilators D and E, a functor

$$(D \circ E)(X) = D(E(X)),\ (D \circ E)(f) = D(E(f))$$

is also a semidilator. We may construct a denotation system for $D \circ E$, but it is easier to handle $D \circ E$ as a Fruend-style semidilator: Then $D \circ E$ is naturally defined, and we can define $\mathrm{supp}_X^{D\circ E} \colon D(E(X)) \to [X]^{<\omega}$ by

$$\mathrm{supp}_X^{D\circ E}(\sigma) = \bigcup \mathrm{supp}_X^E[\mathrm{supp}_{E(X)}^D(\sigma)].$$

We can also define the denotation system for the composition: Naively, we may think of terms in $D \circ E$ take the form $t(t_0, \cdots, t_{m-1})$ for $t \in$ field(D), $t_0, \cdots, t_{m-1} \in$ field(E). This is not quite right since we also have to consider how to plug in parameters into each t_i; So for example, for a binary D-term t and unary E-terms t_0 and t_1, $t(t_0(x_0), t_1(x_0))$ and $t(t_0(x_0), t_1(x_1))$ would be different terms. We use an IU diagram to denote how

to plug in parameters into the innermost terms: For $t \in \text{field}(D)$ of arity m and $t_0, \cdots, t_{m-1} \in \text{field}(E)$ such that each t_i has arity m'_i, we say an IU diagram Д for m objects $a_0, \cdots, a_{m-1}$ is *adequate for* t, $t_0, \cdots, t_{m-1}$ if

1. $a_i = \text{arity}\, t_i$ for each $i < m$, and
2. For each $i < j$, we have $E \vDash t_i <_{\text{Д}(i,j)} t_j$.

Let us define the set of $(D \circ E)$-terms by the set of symbols of the form

$$t[\text{Д}](t_0, \cdots, t_{m-1}),$$

where $t \in \text{field}(D)$, $t_0, \cdots, t_{m-1} \in \text{field}(E)$, $\text{arity}^D(t) = m$, and Д is adequate for t, $t_0, \cdots, t_{m-1}$. Then there is a canonical bijection between the set of $(D \circ E)$-terms and $\text{Tr}(D \circ E)$:[2] First, note that elements of $\text{Tr}(D \circ E)$ is of the form $t(t_0(a_0), \cdots, t_{m-1}(a_{m-1}))$ for $t \in \text{field}(D)$, $t_i \in \text{field}(E)$, $\text{arity}^D(t) = m$, $a_i \in [\mathbb{N}]^{<\omega}$, $E(\mathbb{N}) \vDash t_i(a_i) < t_j(a_j)$ for $i < j$, and $a_0 \cup \cdots \cup a_{m-1} \in \mathbb{N}$. Now, let us set the bijection as follows: For an $(D \circ E)$-term $t[\text{Д}](t_0, \cdots, t_{m-1})$ with the IU diagram Д with objects $\{a_i \mid i \in \mathtt{L}_m\}$ and morphisms $\{e_{i,j} \mid i \leq j \in \mathtt{L}_m\}$, consider the map

$$t[\text{Д}](t_0, \cdots, t_{m-1}) \mapsto t(t_0(\text{ran}\, e_{0,0 \vee \cdots (m-1)}), \cdots, t_{m-1}(\text{ran}\, e_{m-1,0 \vee \cdots (m-1)})).$$

We can check that the inverse is given by

$$t(t_0(a_0), \cdots, t_{m-1}(a_{m-1})) \mapsto t[\text{Diag}_{\mathbb{N}}(a_0, \cdots, a_{m-1})](t_0, \cdots, t_{m-1}).$$

Hence we can 'pull back' the comparison rule defined over $\mathfrak{f}(\mathfrak{a}(D \circ E))$ to define the comparison rule between two $(D \circ E)$-terms; More precisely, suppose that we are given two $(D \circ E)$-terms $s[\text{Д}](s_0, \cdots, s_{m-1})$ and $t(\text{Ж})(t_0, \cdots, t_{n-1})$. We also assume that Д

[2] More precisely, $\text{Tr}(\mathfrak{a}(D \circ E))$.

has objects $\{a_i \mid i \in \mathtt{L}_m\}$ and morphisms $\{e_{ij} \mid i \leq j \in \mathtt{L}_m\}$, and **Ж** has objects $\{b_i \mid i \in \mathtt{L}_n\}$ and morphisms $\{f_{ij} \mid i \leq j \in \mathtt{L}_n\}$. For an arity diagram

$$\text{И} = \begin{array}{ccc} c_\cap & \longrightarrow & c_1 \\ \downarrow & & \downarrow{\scriptstyle g_1} \\ c_0 & \xrightarrow[g_0]{} & c_\cup \end{array}$$

with $c_0 = a_{0\vee\cdots\vee(m-1)}$ and $c_1 = b_{0\vee\cdots\vee(n-1)}$, we define $(D \circ E) \vDash s[\text{Д}](s_0, \cdots, s_{m-1}) <_\text{И} t(\text{Ж})(t_0, \cdots, t_{n-1})$ if and only if

$$\begin{aligned}(D \circ E)(\mathbb{N}) \vDash & s(s_0(\operatorname{ran} g_0 \circ e_{0,0\vee\cdots\vee(m-1)}), \cdots, s_{m-1}(\operatorname{ran} g_0 \circ e_{m-1,0\vee\cdots\vee(m-1)})) \\ & < t(t_0(\operatorname{ran} g_1 \circ f_{0,0\vee\cdots\vee(n-1)}), \cdots, t_{n-1}(\operatorname{ran} g_1 \circ f_{n-1,0\vee\cdots\vee(n-1)})).\end{aligned}$$

The following non-example describes a finite semidilator that is not a (pre)dilator:

Example B.6.8. Let opp be the *opposition semidilator*, a semidilator with an only unary term t given as follows: For the arity diagram $\text{Д} = \operatorname{Diag}_{\{0,1\}}(\{0\}, \{1\})$, we let $t <_{-\text{Д}} t$ holds. In terms of a Freund-style semidilator, opp is a semidilator reversing the order of a linear order with the obvious choice of support transformation. opp is a finite semidilator but not a dilator.

Some large cardinal notions exhibit a dilator structure:

Example B.6.9. This example requires a good understanding of sharps (e.g., [46, §9].) For a given set a, we can view $a^\sharp$ as a dilator as follows: We can think of $a^\sharp$ as an EM blueprint, so $a^\sharp$ is a theory of $L[a]$ with the first ω indiscernibles over the language $\{\in, \dot{a}, c_n\}_{n<\omega}$, in which c_n is interpreted as the nth indiscernible.[3]

Let us fix a set Sk of Skolem terms over the language $\{\in, \dot{a}\}$, and assume that the arity of $t \in \text{Sk}$ is k_t. For a linear order I, let us define a new model M_I by the set

[3]We assume that there are uncountably many $L[a]$-indiscernibles; Otherwise, the associated sharp predilator may not be a dilator.

$\{(t,a) \mid t \in \mathrm{Sk} \wedge a \in [I]^{k_n}\}$. Let us write $t(a)$ instead of (t,a), $c_{\{i_0,\cdots,i_{N-1}\}}$ instead of $c_{i_0},\cdots,c_{i_{N-1}}$, and define

$$M_I \vDash \phi(t_0(a_0),\cdots,t_{n-1}(a_{n-1})) \iff a^\sharp \vDash \phi(t_0(c_{\mathrm{en}_a^{-1}[a_0]}),\cdots,t_{n-1}(c_{\mathrm{en}_a^{-1}[a_{n-1}]}))$$

for an $\{\in,\dot{a}\}$-formula ϕ, where $a = \bigcup_{i<n} a_n$. Then M_I is a model satisfying the theory of $L[a]$ with the set of indiscernibles I. Then $\sharp_a(I) := \mathrm{Ord}^{M_I}$ is a predilator with the obvious choice of support transformation.

The existence of $a^\sharp$ includes the statement 'M_I is well-founded for a well-order I,' so $\sharp_a$ is also a dilator. If a is a real, then $\sharp_a$ is a countable dilator.

B.7 Supplement: Girard's dilator

In this section, we briefly review Girard's original definition of dilators and predilators that appear in various materials about dilators.

Definition B.7.1. A *G-semidilator* (G abbreviates Girard) is a functor from LO to LO preserving direct limits and pullbacks. A *G-dilator* is a functor from WO to WO preserving direct limits and pullbacks.

Girard pointed out in [26, Theorem 2.1.5] that every functor from Nat to LO preserving direct limits and pullbacks uniquely extends to a semidilator $\mathsf{LO} \to \mathsf{LO}$, so we may think of a G-dilator as a G-semidilator preserving well-orderedness. Also, the notions of G-semidilator and F-semidilator are equivalent in the following sense:

Proposition B.7.2. *Let F be an autofunctor over* LO*. Then the following are equivalent:*

1. *F preserves direct limits and pullbacks.*

2. *F admits a support transformation* $\mathrm{supp}^F \colon F \to [\cdot]^{<\omega}$ *with the support condition.*

Proof. We largely follow Freund's proof [21, Remark 2.2.2]. To prove from (2) to (1), let $F \colon \mathsf{LO} \to \mathsf{LO}$ be a functor with the support transformation. To show that F preserves a direct limit, suppose that $\langle X_i, f_{ij} \rangle_{i \leq j \in I}$ is a direct system of linear orders, and $X = \varinjlim \langle X_i, f_{ij} \rangle_{i \leq j \in I}$ be the direct limit with embeddings $f_i \colon X_i \to X$ with the universal property. By [26, Example 1.3.6], the universal property of the direct limit is equivalent to $X = \bigcup_{i \in I} \operatorname{ran} f_i$, so it suffices to show that $\bigcup_{i \in I} \operatorname{ran} F(f_i) = F(X)$.

$\bigcup_{i \in I} \operatorname{ran} F(f_i) \subseteq F(X)$ is immediate since $F(f_i)$ embeds $F(X)$. For the other inclusion, suppose that $\sigma \in F(X)$. Since $\mathrm{supp}^F_X(\sigma) \subseteq X = \bigcup_{i \in I} \operatorname{ran} f_i$ and $\mathrm{supp}^F_X(\sigma)$ is finite, there must be $i \in I$ such that $\mathrm{supp}^F_X(\sigma) \subseteq \operatorname{ran} f_i$. Hence by the support condition, $\sigma \in \operatorname{ran} F(f_i)$, as desired.

To show F preserves a pullback, suppose that we are given a diagram

$$\begin{array}{ccc} Y & \xrightarrow{h_1} & X_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_0 & \xrightarrow[f_0]{} & X \end{array} \tag{B.9}$$

Let $g = f_0 \circ h_0 = f_1 \circ h_1$. By [26, Theorem 1.5.5], (B.9) is a pullback diagram iff $\operatorname{ran} g = \operatorname{ran} f_0 \cap \operatorname{ran} f_1$. Hence, we claim that $\operatorname{ran} F(g) = \operatorname{ran} F(f_0) \cap \operatorname{ran} F(f_1)$ provided that (B.9) is a pullback diagram.

First, $\operatorname{ran} F(g) \subseteq \operatorname{ran} F(f_0)$ is clear since $F(g) = F(f_0) \circ F(h_0)$. Similarly, we have $\operatorname{ran} F(g) \subseteq \operatorname{ran} F(f_1)$ so $\operatorname{ran} F(g) \subseteq \operatorname{ran} F(f_0) \cap \operatorname{ran} F(f_1)$. Conversely, suppose that we are given an element of $\operatorname{ran} F(f_0) \cap \operatorname{ran} F(f_1)$ of the form $F(f_0)(\sigma_0)$ for some $\sigma_0 \in F(X_0)$ such that there is $\sigma_1 \in F(X_1)$ with $F(f_0)(\sigma_0) = F(f_1)(\sigma_1)$. Then we have $\mathrm{supp}^F_X(F(f_i)(\sigma_i)) = f_i[\mathrm{supp}^F_{X_i}(\sigma_i)]$ for $i = 0, 1$, so we get

$$f_0[\mathrm{supp}^F_{X_0}(\sigma_0)] = f_1[\mathrm{supp}^F_{X_1}(\sigma_1)] \subseteq \operatorname{ran} f_0 \cap \operatorname{ran} f_1 = \operatorname{ran} g.$$

Thus by the support condition, $F(f_0)(\sigma_0) \in \operatorname{ran} F(g)$.

For the remaining implication from (1) to (2), let F be an endofunctor over LO preserving direct limits and pullbacks. For a linear order X and $\sigma \in F(X)$, we want to define $\operatorname{supp}_X^F(\sigma)$ by the inclusion-least $a \in [X]^{<\omega}$ satisfying $\sigma \in \operatorname{ran} F(\iota_a)$ for the inclusion map $\iota_a \colon a \to X$. We claim that the least such a exists, which is clearly unique if it exists.

First, from the preservation property of F, we have that $F(X)$ is a direct limit of $\langle F(a), F(\iota_{a,b})\rangle_{a \subseteq b \in [X]^{<\omega}}$ for the inclusion map $\iota_{a,b} \colon a \to b$. Furthermore, the maps $F(\iota)$ witness the universal property of the direct limit. Hence if $\sigma \in F(X)$, then there is a finite $a \subseteq X$ such that $\sigma \in \operatorname{ran} F(\iota_a)$. Also, if $a, b \in [X]^{<\omega}$, $\iota_{a\cap b}$ is a pullback of ι_a and ι_b. Hence $\operatorname{ran} F(\iota_{a\cap b}) = \operatorname{ran} F(\iota_a) \cap \operatorname{ran} F(\iota_b)$, It allows us to define $\operatorname{supp}_X^F(\sigma)$ as desired.

We need to check that supp^F is a natural transformation and satisfies the support condition. For the naturality, we claim for an increasing $f \colon X \to Y$ and $\sigma \in F(X)$, the equality

$$\operatorname{supp}^F(F(f)(\sigma)) = f[\operatorname{supp}_X^F(\sigma)].$$

Let $b = f[\operatorname{supp}_X^F(\sigma)]$. We claim that b is the inclusion-minimal subset of Y such that $F(f)(\sigma) \in \operatorname{ran} F(\iota_b)$. For $a = \operatorname{supp}_X^F(\sigma)$, we have $\sigma \in \operatorname{ran} F(\iota_a)$, say $\sigma = F(\iota_a)(\sigma_0)$. Hence we have

$$F(f)(\sigma) = F(f) \circ F(\iota_a)(\sigma_0) = F(\iota_b) \circ F(f \restriction a)(\sigma_0) \in \operatorname{ran} F(\iota_b).$$

For the minimality, suppose that $F(f)(\sigma) \in \operatorname{ran} F(\iota_{b_0})$ for some $b_0 \subseteq b$. Let $a_0 = a \cap f^{-1}[b_0]$, then $f \restriction a_0$ is a bijection from a_0 to b_0. If $F(f)(\sigma) = F(\iota_{b_0})(\tau)$ for some $\tau \in F(\iota_{b_0})$, we have

$$F(f)(\sigma) = F(\iota_{b_0})(\tau) = F(\iota_{b_0} \circ (f \restriction a_0) \circ (f \restriction a_0)^{-1})(\tau)$$
$$= F(f)((\iota_{a_0}) \circ (f \restriction a_0)^{-1})(\tau)),$$

so $\sigma \in \operatorname{ran} \iota_{a_0}$. Then by the inclusion-minimality of a, $a = a_0$, so we have $b = b_0$.

Lastly, for the support condition, suppose that $\tau \in F(Y)$ be such that $\operatorname{supp}^F_Y(\tau) \subseteq \operatorname{ran} f$. By the definition of supp^F, $\operatorname{supp}^F_Y(\tau) \subseteq \operatorname{ran} f$ implies $\tau \in \operatorname{ran} F(\iota_b)$ for some finite $b \subseteq \operatorname{ran} f$ and the inclusion map $\iota_b \colon b \to Y$. If we factor ι_b into two inclusion maps $\iota'_b \colon b \to \operatorname{ran} f$ and $\iota \colon \operatorname{ran} f \to Y$, we have $\tau \in \operatorname{ran} F(\iota)$, so there is $\tau' \in F(\operatorname{ran} f)$ such that $\tau = F(\iota)(\tau')$.

An increasing map $f \colon X \to Y$ also factors through an isomorphism $f' \colon X \to \operatorname{ran} f$ and an inclusion map $\iota \colon \operatorname{ran} f \to Y$, and $F(f') \colon F(X) \to F(\operatorname{ran} f)$ is also an isomorphism. Now take $\sigma = F(f')^{-1}(\tau')$, then it satisfies

$$F(f)(\sigma) = F(f) \circ F(f')^{-1}(\tau') = F(\iota)(\tau') = \tau,$$

proving the support condition. □

Girard's dilator is easy to define, unlike denotation systems or support transformation-based ones. However, Girard's definition is the most impractical one (at least, for me) to prove something. Freund-style dilators with support transformations or denotation systems are more intuitive and practical. Despite that, Girard's notion of dilator has a systematic advantage in that its definition is easy to generalize, allowing us to define general ptykes far more easily than other formulations. Furthermore, the idea of Girard's definition appears in various places in the theory of dilators and ptykes.

APPENDIX C

STRUCTURES OF DILATORS

> Φοβεῖταί τις μεταβολήν; τί γὰρ δύναται χωρὶς μεταβολῆς γενέσθαι;[1]
>
> Marcus Aurelius, *Meditations*, 7.18.

In the previous chapter, we introduced various equivalent definitions of semidilators and dilators. In this chapter, we examine the properties of dilators and how to characterize them. Every content in this chapter is due to Girard [26, 29].

C.1 Perfect decomposition theorem

We can factor non-zero natural numbers into a product of prime numbers. Similarly, we can 'factor' semidilators into a sum of smaller semidilators with no further 'factor.' The way we factor semidilators is the sum, and a semidilator with no factor is what we will call a *perfect* semidilator:

Definition C.1.1. A semidilator D is *perfect* if it cannot be decomposed into a sum of two non-zero semidilators; That is, there are no $D_0, D_1 \ncong \underline{0}$ such that $D \cong D_0 + D_1$.

Note that Girard in [29] used the term 'connected dilator' to mean a perfect dilator. The next theorem says we can decompose every semidilator into an ordered sum of perfect semidilators:

Theorem C.1.2 (Perfect decomposition theorem). *For every semidilator D, there is a*

[1] *Is any man afraid of change? Why, what can take place without change?* (Translated by George Long.)

linear order $\mathrm{lh}(D)$ *and an indexed family of perfect semidilators* $\{D_i \mid i \in \mathrm{lh}(D)\}$ *with an isomorphism* $\epsilon^D \colon D \cong \sum_{i \in \mathrm{lh}(D)} D_i$. *Moreover,*

1. *If* $\iota \colon D \to E$ *is an embedding, then there is an increasing map* $\mathrm{lh}(\iota) \colon \mathrm{lh}(D) \to \mathrm{lh}(E)$ *and embeddings* $\iota_i \colon D_i \to E_{\mathrm{lh}(\iota)(i)}$ *such that if we define the map* $\hat{\iota} \colon \sum_{i \in \mathrm{lh}(D)} D_i \to \sum_{i \in \mathrm{lh}(E)} E_i$ *by*

$$\hat{\iota}(\mathsf{in}_i(t)) = \mathsf{in}_{\mathrm{lh}(\iota)(i)}(\iota_i(t)),$$

   then $\hat{\iota}$ *is an embedding and the following diagram commutes:*

$$\begin{array}{ccc} D & \xrightarrow{\epsilon^D} & \sum_{i \in \mathrm{lh}(D)} D_i \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \hat{\iota}} \\ E & \xrightarrow[\epsilon^E]{} & \sum_{i \in \mathrm{lh}(E)} E_i \end{array} \tag{C.1}$$

   Furthermore, such $\mathrm{lh}(\iota)$ *and* ι_i *satisfying* (C.1) *are unique.*

2. *The linear order* $\mathrm{lh}(D)$ *and the indexed family* $\{D_i \mid i \in \mathrm{lh}(D)\}$ *are unique up to isomorphism.*

3. *If* D *is a predilator, then each* D_i *is a predilator.*

4. *If* D *is a dilator, then* $\mathrm{lh}(D)$ *is a well-order and each* D_i *is a dilator.*

Proof. Let D be a semidilator. Let us define a binary relation $\underline{\ll}^D$ over $\mathrm{field}(D)$ as follows:

$$s \mathrel{\underline{\ll}^D} t \iff \exists \text{Д}(s \leq_{\text{Д}} t).$$

Informally, the strict relation $s \ll^D t \iff s \mathrel{\underline{\ll}^D} t \wedge t \not\mathrel{\underline{\ll}^D} s$ means '$s(a)$ is larger than $t(b)$ no matter what a and b are.' Then we can see that $\underline{\ll}^D$ is a preorder. Then let us consider the equivalence relation $\asymp^D$ induced by the preorder $\underline{\ll}^D$, that is,

$$s \asymp^D t \iff s \mathrel{\underline{\ll}^D} t \wedge t \mathrel{\underline{\ll}^D} s.$$

Then define $\mathrm{lh}(D) := (\mathrm{field}(D), \lessapprox^D)/\asymp^D$, which is a linear order. For each $i \in \mathrm{lh}(D)$, let D_i be a subsemidilator of D whose terms are $t \in D$ belonging to the equivalence class i.

Lemma C.1.3. *For each $i \in \mathrm{lh}(D)$, D_i is perfect.*

Proof. Suppose not, assume that $D_i \cong E_0 + E_1$ for some non-zero semidilators E_0 and E_1. Let $f\colon E_0 + E_1 \to D_i$ be an isomorphism, and take two $t_0 \in f[E_0]$, $t_1 \in f[E_1]$. Then $t_0 \neq t_1$. Since D_i is an $\asymp^D$-equivalence class, there is an arity diagram

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n_0 \\ \big\downarrow & & \big\downarrow{\scriptstyle e_0} \\ n_1 & \xrightarrow[e_1]{} & n_\cup \end{array} \tag{C.2}$$

such that $t_1 <_{\text{Д}} t_0$. Hence $D(n_\cup) \vDash t_1(\mathrm{ran}\, e_1) < t_0(\mathrm{ran}\, e_0)$, and $(E_0 + E_1)(n_\cup) \vDash f^{-1}(t_1)(\mathrm{ran}\, e_1) < f^{-1}(t_0)(\mathrm{ran}\, e_0)$, which is impossible since $f^{-1}(t_0)(\mathrm{ran}\, e_0) \in E_0(n_\cup)$, $f^{-1}(t_1)(\mathrm{ran}\, e_1) \in E_1(n_\cup)$, and every element of $E_0(n_\cup)$ is less than that of $E_1(n_\cup)$. □

Also, observe that if $i, j \in \mathrm{lh}(D)$, $s \in D_i$, $t \in D_j$, and $i < j$, then $s <_{\text{Д}} t$ holds for every arity diagram Д for s and t. Hence we can show

Lemma C.1.4. *The map $\epsilon^D(t) = \mathsf{in}_{[t]_{\asymp^D}}(t)$ is an isomorphism from D to $\sum_{i \in \mathrm{lh}(D)} D_i$.*

Now let us decompose an embedding $\iota\colon D \to E$. We define $\mathrm{lh}(\iota)\colon \mathrm{lh}(D) \to \mathrm{lh}(E)$ by $\mathrm{lh}(\iota)([t]_{\asymp^D}) = [\iota(t)]_{\asymp^E}$. Clearly $s \lessapprox^D t$ implies $\iota(s) \lessapprox^E \iota(t)$, so we have the following:

Lemma C.1.5. $\mathrm{lh}(\iota)$ *is well-defined and increasing.*

If $s \in D_i$, then $\iota(s) \in E_{\mathrm{lh}(\iota)(i)}$. Hence $\iota_i := \iota \restriction D_i$ is an embedding from D_i to $E_{\mathrm{lh}(\iota)(i)}$. Hence if we define $\widehat{\iota}(\mathsf{in}_i(t)) = \mathsf{in}_{f(i)}(\iota_i(t))$, then we can easily prove $\widehat{\iota}$ is

an embedding from $\sum_{i\in\mathrm{lh}(D)} D_i$ to $\sum_{i\in\mathrm{lh}(E)} E_i$, and the commutativity of (C.1) easily follows. The uniqueness of $\mathrm{lh}(\iota)$ and $\{\iota_i \mid i \in \mathrm{lh}(D)\}$ satisfying (C.1) follows from that ϵ^D and ϵ^E are isomorphisms.

Now let us prove the uniqueness of $\mathrm{lh}(D)$ and $\{D_i \mid i \in \mathrm{lh}(D)\}$. Suppose we are given a linear order J, a family of non-zero perfect semidilators $\{D'_j \mid j \in J\}$, and an isomorphism $\epsilon' \colon \sum_{j\in J} D'_j \to D$. To define an increasing map $J \to \mathrm{lh}(D)$, let us observe that a semidilator embedding preserves $\ll$ and $\asymp$, and every term in a perfect semidilator is $\cong$-equivalent:

Lemma C.1.6. *Let E be a non-zero perfect semidilator. If $s, t \in E$, then $s \asymp^E t$.*

Proof. Suppose that there are two non-$\asymp^E$-equivalent E-terms, so $|\mathrm{lh}(E)| \geq 2$. Fix $i_0, i_1 \in \mathrm{lh}(E)$ such that $\mathrm{lh}(E) \vDash i_0 < i_1$, and define two subsemidilators $E_0, E_1 \subseteq E$ by

- $E_0 = \{s \in E \mid \mathrm{lh}(E) \vDash [s]_{\asymp^E} < i_1\}$,
- $E_1 = \{s \in E \mid \mathrm{lh}(E) \vDash [s]_{\asymp^E} \geq i_1\}$.

Then both E_0 and E_1 are not empty, and every E-term is in either E_0 or E_1. If we define $f \colon E \to E_0 + E_1$ by

$$f(t) = \begin{cases} \mathsf{in}_0(t) & t \in E_0, \\ \mathsf{in}_1(t) & t \in E_1, \end{cases}$$

then f is a bijection. We can also show that both f and f^{-1} is a semidilator embedding, so $f \colon E \cong E_0 + E_1$, contradicting that E is perfect. □

Hence $\epsilon^D \circ \epsilon' \colon \sum_{j\in J} D'_j \to \sum_{i\in\mathrm{lh}(D)} D_i$ maps a summand D'_j of $\sum_{j\in J} D'_j$ to another unique summand D_i of $\sum_{i\in\mathrm{lh}(D)} D_i$. Hence we can define a map $h \colon J \to \mathrm{lh}(D)$

given by

$$h(j) = i \iff \epsilon^D \circ \epsilon' \text{ maps } D'_j \text{ to } D_i.$$

Then h is increasing: If $j_0 \leq j_1$, $t_0 \in D'_{j_0}$, $t_1 \in D'_{j_1}$, then $\mathsf{in}_{j_0}(t_0) \ll^{\sum_{j\in J} D'_j} \mathsf{in}_{j_1}(t_1)$, so $(\epsilon^D \circ \epsilon')(\mathsf{in}_{j_0}(t_0)) \ll^D (\epsilon^D \circ \epsilon')(\mathsf{in}_{j_1}(t_1))$. Hence we have $h(j_0) \leq h(j_1)$. Since $\epsilon^D \circ \epsilon'$ is an isomorphism, h is onto, so $h \colon J \cong \mathrm{lh}(D)$.

By definition of h, $\epsilon^D \circ \epsilon' \restriction D'_j \colon D'_j \to D_{h(j)}$. But we can see that the inverse of $\epsilon^D \circ \epsilon'$ also preserves $\asymp$-equivalence classes, and each summand of the sum $\sum_{j\in J} D'_j$ and $\sum_{i\in \mathrm{lh}(D)} D_i$ form an $\asymp$-equivalence class. Hence $(\epsilon^D \circ \epsilon')^{-1}$ maps $D_{h(j)}$ to D'_j, so $D'_j \cong D_{h(j)}$.

We defined D_i as a subsemidilator of D, so if D is a (pre)dilator, then so is D_i. Now we claim that $\mathrm{lh}(D)$ is a well-order when D is a dilator. Suppose not, if $\mathrm{lh}(D)$ has an infinite descending sequence

$$\mathrm{lh}(D) \vDash [t_0]_{\asymp^D} > [t_1]_{\asymp^D} > [t_2]_{\asymp^D} > \cdots.$$

Then we have

$$D(\omega) \vDash t_0(0, 1, \cdots, n_0 - 1) > t_1(n_0, \cdots, n_0 + n_1 - 1) > \cdots,$$

a contradiction. It finishes the proof of the perfect decomposition theorem. □

Remark C.1.7. Suppose that a semidilator D admits the perfect decomposition $\epsilon^D \colon D \to \sum_{i\in \mathrm{lh}(D)} D_i$. The isomorphism from D to $\sum_{i\in \mathrm{lh}(D)} D_i$ may not be unique even when D is a predilator; A silly example would be $D = \underline{\mathbb{Q}}$. Hence, the statement on Theorem C.1.2 depends on the particular choice of ϵ^D although the choice is the most natural one. In particular, the decomposition of a semidilator embedding may not work well with a different isomorphism from D to the sum.

If D is a dilator, however, the isomorphism between D and $\sum_{i\in \mathrm{lh}(D)} D_i$ becomes unique. Hence, we do not need to specify ϵ^D in the statement of Theorem C.1.2 when we only consider dilators.

Remark C.1.8. The isomorphism ϵ^D, the linear order $\mathrm{lh}(D)$ and the indexed family of perfect semidilators $\{D_i \mid i \in \mathrm{lh}(D)\}$ are 'effectively definable' from D in the following sense: If D is a countable semidilator, they are primitive recursive in D. In general, they are rudimentary in D.

Remark C.1.9. In the statement of the perfect decomposition theorem, we stated that every semidilator is isomorphic to the 'external sum' $\sum_{i\in \mathrm{lh}(D)} D_i$. By examining the proof, we can see that the proof indicates we can decompose D into the 'internal sum' of perfect dilators: If each D_i is an $\asymp^D$-equivalence class, then $D = \bigcup_{i\in \mathrm{lh}(D)} D_i$ and every D_i-term is 'less than' every D_j-term when $i < j$. D is isomorphic to the external sum, and *equal to* the internal sum. Using the internal sum decomposition for semidilators in place of the external sum is useful when we define, for example, the separation functor for predilators of type Ω. (See [38, 26, 29] for the separation functor.)

The perfect decomposition theorem has a focal role in classifying dilators into the following types:

Definition C.1.10. A semidilator D is

1. *of type* 0 if $D \cong \underline{0}$.
2. *of type* 1 if $D \cong D' + \underline{1}$.
3. *of type* ω if $D \cong \sum_{i\in I} D_i$ for some linear order I of limit ordertype.
4. *of type* Ω if $D \cong D' + D''$ for some perfect $D'' \ncong \underline{1}$.

If we think of a dilator D as a class ordinal (i.e., $D(\mathrm{Ord})$), then each type roughly corresponds to zero, successor ordinal, limit ordinal of cofinality $<$ Ord, and that of

cofinality = Ord. The following proposition gives a criterion for a perfect dilator, which is useful to find dilators of type Ω:

Proposition C.1.11. *If D is a non-zero predilator, then the following are equivalent:*

1. *D is perfect.*
2. *For every $t \in \operatorname{field} D$, $\{t(b) \mid b \in [\omega]^{\operatorname{arity} t}\}$ is cofinal in $D(\omega)$.*

Proof. D is perfect if and only if D has only one $\asymp^D$-equivalence class, which is also equivalent to the fact that for every two $s, t \in \operatorname{field}(D)$ there is Д such that $s \leq_{Д} t$. We will claim that it is equivalent to the second statement.

For one direction, let $a \in [\omega]^{\operatorname{arity} s}$. We want to find $b \in [\omega]^{\operatorname{arity} t}$ such that $\operatorname{Diag}_\omega(a, b) = Д$, where

$$Д = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{C.3}$$

One of the problems is that $\min \operatorname{ran} e_0 > \min \operatorname{ran} e_1$ is possible, which becomes problematic when $\min a = 0$, so $\min b$ cannot take any value over ω. Another problem is that, for example, $e_0(0) < e_1(0) < e_0(1)$ is possible, which becomes problematic when $a = \{0, 1, \cdots\}$, so $b(0)$ cannot take any value.

To get around all of these issues, we replace a with

$$a' = \{a(i) + n_1 \cdot (1 + i) \mid i < n_0\}.$$

(Recall that $|a| = n_0 = \operatorname{arity} s$ and $|b| = n_1 = \operatorname{arity} t$.) Then $D(\omega) \vDash t(a) < t(a')$ since D is a predilator, and we can find b such that $\operatorname{Diag}_\omega(a', b) = Д$. Hence $D(\omega) \vDash s(a) < s(a') < t(b)$.

Conversely, suppose that we have $s, t \in \text{field}(D)$ such that $t <_{Д} s$ for every arity diagram Д for t and s. In particular, Д can be an arity diagram (C.3) with $\max e_0 < \min e_1$, so we have $D(\omega) \vDash t(a) < s(0, 1, \cdots, \text{arity } s - 1)$ for any choice of $a \in [\omega]^{\text{arity } t}$. □

Example C.1.12. For a well-order α, the dilator $D(X) = X + \underline{\alpha}$ is either of type 1 if α is a successor ordinal, or type ω if α is a limit ordinal. But the dilator $E(X) = X + X$ or $F(X) = X \cdot X$ is of type Ω: Clearly, the identity functor is perfect, so E is of type Ω. For F, observe that elements of the form $\langle n+1, n\rangle$, $\langle n, n\rangle$, and $\langle n, n+1\rangle$ are cofinal in ω^2, so Proposition C.1.11 implies F is perfect.

Proposition C.1.13. *Every semidilator is either of type* 0*,* 1*,* ω*, or* Ω*.*

Proof. By the perfect decomposition theorem, we can decompose a given dilator D into $\sum_{i \in I} D_i$. If I is empty, then D is of type 0. If I has no maximum, then D is of type ω. If I has a maximum, then the last summand determines the type of D: If $D \cong \underline{1}$, then D is of type 1. Otherwise, D is of type Ω. □

We can view ordinals as objects generated from 0 by the successor operator and the limit process. Similarly, we can view dilators as those generated from $\underline{0}$ and generated by the successor operator (type 1), the limit process (type ω), or some other process that should be related to type Ω.

C.2 Prime dilators

In this section, we analyze dilators of size 1, called *prime dilators*:

Definition C.2.1. A dilator D is *prime* if $|\text{field } D| = 1$. A *degree* of a prime dilator is the arity of its unique term.

We introduced an example of a prime dilator in Example B.6.6. Let us reiterate its definition for the reader's convenience:

Definition C.2.2. Let σ be a permutation over $n = \{0, 1, \cdots, n-1\}$. For a linear order X, let us define

$$\mathsf{Prime}_\sigma(X) = \{\langle x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}\rangle : x_0 < x_1 < \cdots < x_{n-1} \in X\}$$

equipped with $<_{\mathsf{KB}}$ (or the lexicographic order). For $f : X \to Y$, define

$$\mathsf{Prime}_\sigma(f)(\langle x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}\rangle) = \langle f(x_{\sigma(0)}), f(x_{\sigma(1)}), \cdots, f(x_{\sigma(n-1)})\rangle$$

and

$$\mathrm{supp}_X^{\mathsf{Prime}_\sigma}(\langle x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}\rangle) = \{x_0, x_1, \cdots, x_{n-1}\}.$$

Proposition C.2.3. Prime_σ *is a prime dilator.*

Proof. It is easy to see that Prime_σ is a semidilator. Also, if X is a well-order, then $\mathsf{Prime}_\sigma(X)$ is a well-order.

To see Prime_σ is a prime dilator, let us compute $\mathrm{Tr}(\mathsf{Prime}_\sigma)$. We can see that $\mathsf{Prime}_\sigma(k) \cong 0$ for $k < n$, and $\mathsf{Prime}_\sigma(n) = \{\langle \sigma(0), \cdots, \sigma(n-1)\rangle\}$. No elements of $\mathsf{Prime}_\sigma(k)$ for $k > n$ have support k, so $\mathrm{Tr}(\mathsf{Prime}_\sigma) = \{(n, \langle \sigma(0), \cdots, \sigma(n-1)\rangle)\}$. □

In this remaining part of the section, we will claim that Prime_σ is the only possible prime dilator. That is, every prime dilator is isomorphic to some Prime_σ. The following proof is a modification of Freund's argument [23, Proposition 2.7] in our setting:

Theorem C.2.4. *Every prime dilator of degree n is isomorphic to Prime_σ for some permutation σ over n.*

Proof. The main trouble is how to find the permutation σ. That means we must first decide which component to compare from a given prime dilator.

Here is a trick: Let us consider the following sequences:

$$\begin{aligned}
c_0 &= \textcircled{1} \quad 2 \quad 4 \quad 6 \quad 8 \quad \cdots \\
c_1 &= 0 \quad \textcircled{3} \quad 4 \quad 6 \quad 8 \quad \cdots \\
c_2 &= 0 \quad 2 \quad \textcircled{5} \quad 6 \quad 8 \quad \cdots \\
c_3 &= 0 \quad 2 \quad 4 \quad \textcircled{7} \quad 8 \quad \cdots \\
c_4 &= 0 \quad 2 \quad 4 \quad 6 \quad \textcircled{9} \quad \cdots
\end{aligned}$$

We stated c_i as an infinite sequence, but let us chop off to length n if the dilator D has degree n, so we have n elements $c_0, \cdots, c_{n-1}$ that are members of $(2n)^n$. We can see that in each ith column, c_i has the largest value among the others.

Let t be the unique D-term. Now compare the order of

$$t(c_0), t(c_1), \cdots, t(c_{n-1}).$$

over $D(2n)$. If $t(c_i)$ is the largest, then the i-th component has the highest priority in the comparison. Similarly, if $t(c_j)$ is the second largest under this order, then the j-th component has the second highest priority. It gives the following permutation: Let us define $\sigma : n \to n$ in a way that

$$D(2n) \vDash t(c_{\sigma(0)}) > t(c_{\sigma(1)}) > \cdots > t(c_{\sigma(n-1)}).$$

so we have

$$i < j \iff D(2n) \vDash t(c_{\sigma(i)}) > t(c_{\sigma(j)}).$$

It remains to show that $D \cong \mathsf{Prime}_\sigma$ holds. To see this, let us prove the following: For a given linear order X and $a, b \in [X]^n$,

$$\begin{aligned}
D(X) \vDash t(a) < t(b) \iff \exists j < n \big[&\mathrm{en}_a(\sigma(j)) < \mathrm{en}_b(\sigma(j)) \\
&\wedge \forall i < j [\mathrm{en}_a(\sigma(i)) = \mathrm{en}_b(\sigma(i))] \big]. \quad \text{(C.4)}
\end{aligned}$$

To make the argument easier, we will replace X with $2{\cdot}X$, and assume every element of a, b has the form $2 \cdot x$, which can be done by considering the increasing map $f \colon X \to 2 \cdot X$, $f(x) = 2 \cdot x$ and replacing X, a, b with their images under f.

Let us observe that proving the backward implication of (C.4) suffices by the linearity of the orders. We prove the backward implication by induction on the number of indices k such that $\mathrm{en}_a(\sigma(k)) > \mathrm{en}_b(\sigma(k))$. If the number is 0, then $\mathrm{en}_a(i) \leq \mathrm{en}_b(i)$ for every $i < n$, so $D(X) \vDash t(a) \leq t(b)$ since D is a predilator. Now let us assume the inductive hypothesis, and let $k < n$ be the least index such that $\mathrm{en}_a(k) > \mathrm{en}_b(k)$, and let $k = \sigma(l)$.

If j is the natural number in the right-hand side of (C.4), then $j < l$. Now let $\overline{a}$ be the set obtained from a by replacing $\mathrm{en}_a(\sigma(j))$ and $\mathrm{en}_a(k) = \mathrm{en}_a(\sigma(l))$ by $\mathrm{en}_a(\sigma(j)) + 1$, $\mathrm{en}_b(k)$ respectively. Then the map $a \to \overline{a}$ given by $\mathrm{en}_a(i) \mapsto \mathrm{en}_{\overline{a}}(i)$ is order-preserving since

1. If $k' < k$, then $\mathrm{en}_a(k') \leq \mathrm{en}_b(k') < \mathrm{en}_b(k) < \mathrm{en}_a(k)$ by the minimality of k.
2. If $j' > \sigma(j)$, then $\mathrm{en}_a(\sigma(j)) + 1 < \mathrm{en}_a(j'), \mathrm{en}_b(j')$.

Hence we have

$$\mathrm{en}_{\overline{a}}(\sigma(i)) = \begin{cases} \mathrm{en}_a(\sigma(j)) + 1 & i = j, \\ \mathrm{en}_b(\sigma(l)) & i = l, \\ \mathrm{en}_a(\sigma(i)) & \text{otherwise.} \end{cases}$$

Now we claim that $D(X) \vDash t(a) < t(\overline{a}) < t(b)$. We can see that in $t(\overline{a}) < t(b)$, the number of i with $\overline{a}(i) > b(i)$ is reduced by 1, and we still have (C.4) holds for a replaced by $\overline{a}$. Hence, the inductive hypothesis tells $D(X) \vDash t(\overline{a}) < t(b)$ holds.

Hence, it remains to prove $D(X) \vDash t(a) < t(\overline{a})$. To see this, let us consider the set

$$Y = \{\mathrm{en}_a(i), \mathrm{en}_a(i) + 1 \mid i < n,\ i \neq k\} \cup \{\mathrm{en}_b(k), \mathrm{en}_a(k)\}.$$

Then $|Y| = 2n$. If $g \colon 2n \to Y$ is the increasing enumeration, then we can see that

- $g[c_{\sigma(j)}] = \{\mathrm{en}_a(i) \mid i < n \wedge i \neq k, \sigma(j)\} \cup \{\mathrm{en}_a(\sigma(j)) + 1, \mathrm{en}_b(\sigma(l))\} = \mathrm{en}_{\overline{a}}$,
- $g[c_{\sigma(l)}] = \{\mathrm{en}_a(i) \mid i < n \wedge i \neq k\} \cup \{\mathrm{en}_a(\sigma(l))\} = a$.

Hence $D(X) \vDash t(a) < t(\overline{a})$ is equivalent to $D(2n) \vDash t(c_{\sigma(l)}) < t(c_{\sigma(j)})$, which holds by the definition of σ. □

For each predilator D and $t \in \mathrm{field}(D)$, $D \restriction \{t\}$ is a prime dilator, which is isomorphic to Prime_σ for some permutation σ over arity t. The permutation σ gauges which component (indexed from the smallest to the largest) has a higher priority in comparison than others:

Definition C.2.5. Let D be a predilator and $t \in \mathrm{field}(D)$. The *priority permutation* Σ_t^D *for* t is the unique permutation σ over arity t such that $D \restriction \{t\} \cong \mathsf{Prime}_\sigma$.

From the proof of (C.4) in Theorem C.2.4, we have the following:

Theorem C.2.6. *Let D be a predilator and $t \in \mathrm{field}(D)$. For a linear order X and $a, b \in [X]^{\mathrm{arity}\, t}$, we have*

$$D(X) \vDash t(a) < t(b) \iff \exists j < \mathrm{arity}\, t \big[\mathrm{en}_a(\Sigma_t^D(j)) < \mathrm{en}_b(\Sigma_t^D(j)) \wedge \forall i < j \big[\mathrm{en}_a(\Sigma_t^D(i)) = \mathrm{en}_b(\Sigma_t^D(i))\big]\big]. \quad \square$$

C.3 Abstract construction of predilators

The previous section characterizes each term of a predilator. However, most predilators have more than a single term, so characterizing predilators requires characterizing the term comparison. Again, we follow notions and proofs in [23, §2].

Definition C.3.1. Let D be a predilator and $s, t \in \text{field}(D)$. Let $\hat{p}_{s,t} = \hat{p}^D_{s,t} \leq \min(\text{arity}\, s, \text{arity}\, t)$ be the largest natural number such that

$$\forall i, j < \hat{p}_{s,t}[\Sigma^D_s(i) < \Sigma^D_s(j) \iff \Sigma^D_t(i) < \Sigma^D_t(j)].$$

For $p \leq \hat{p}_{s,t}$ and an arity diagram Д for s, t of the form

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{C.5}$$

we say Д is *p-congruent (relative to s and t)* if $e_0(\Sigma^D_s(i)) = e_1(\Sigma^D_t(i))$ holds for every $i < p$,

Then we say $p \leq \hat{p}_{s,t}$ is *secure (relative to s and t)* if $D \vDash s <_\text{Д} t$ holds for some p-congruent Д, then $D \vDash s <_\text{Д} t$ holds for every p-congruent Д. In other words, the validity of $D \vDash s <_\text{Д} t$ does not depend on the choice of a p-congruent Д.

Put differently, we say $\text{Д} = \text{Diag}_X(a, b)$ is p-congruent relative to s and t when $\text{en}_a(\Sigma^D_s(i)) = \text{en}_b(\Sigma^D_t(i))$ for every $i < p$. p is secure relative to s and t means the comparison between s and t depends only on the first p parameters enumerated under their comparison priority. We claim that a secure number exists since $\hat{p}_{s,t}$ is secure:

Lemma C.3.2. *$\hat{p}_{s,t}$ is secure.*

Proof. Note that arity $s = n_0$ and arity $t = n_1$. If $\hat{p}_{s,t} = n_0 \leq n_1$, then $e_0 = e_1 \circ \iota$ holds for the map $\iota \colon n_0 \to n_1$ defined by $\iota(\Sigma^D_s(i)) = \Sigma^D_t(i)$ for $i < n_0$. It is order-preserving by the defining property of $\hat{p}_{s,t}$. Hence

$$\begin{aligned} D \vDash s <_\text{Д} t &\iff D(n_\cup) \vDash D(e_0)(s(0, \cdots, n_0 - 1)) < D(e_1)(t(0, \cdots, n_1 - 1)) \\ &\iff D(n_1) \vDash D(\iota)(s(0, \cdots, n_0 - 1)) < t(0, \cdots, n_1 - 1), \end{aligned}$$

and the last formula does not depend on Д as long as Д is $\hat{p}_{s,t}$-congruent. Hence $\hat{p}_{s,t}$ is secure. The case $\hat{p}_{s,t} = n_1 \leq n_0$ follows from a symmetric argument.

Thus let us consider the case when $\hat{p}_{s,t} < \min(n_0, n_1)$. By the maximality of $\hat{p}_{s,t}$, there is $j < \hat{p}_{s,t}$ such that

$$[\Sigma_s^D(j) < \Sigma_s^D(\hat{p}_{s,t}) \text{ and } \Sigma_t^D(j) > \Sigma_t^D(\hat{p}_{s,t})] \\ \text{or} \quad [\Sigma_s^D(j) > \Sigma_s^D(\hat{p}_{s,t}) \text{ and } \Sigma_t^D(j) < \Sigma_t^D(\hat{p}_{s,t})].$$

Without loss of generality, suppose that the first disjunct holds. Note in this case $s \neq t$ since $\hat{p}_{t,t} = \text{arity}\, t$. Also, by the assumption on $j, \hat{p}_{s,t}$ and the $\hat{p}_{s,t}$-congruence of Д, we have

$$e_1(\Sigma_t^D(\hat{p}_{s,t})) < e_1(\Sigma_t^D(j)) = e_0(\Sigma_s^D(j)) < e_0(\Sigma_s^D(\hat{p}_{s,t})). \tag{C.6}$$

Now suppose that $D \vDash s <_{\text{Д}} t$, and let

$$\text{Д}' = \begin{array}{ccc} n'_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle e'_1} \\ n_0 & \xrightarrow[e'_0]{} & n'_\cup \end{array}$$

be another $\hat{p}_{s,t}$-congruent arity diagram relative to s and t. Then we have that (C.6) holds when e_0 and e_1 are replaced by e'_0 and e'_1 respectively; i.e., $e'_1(\Sigma_t^D(\hat{p}_{s,t})) < e'_0(\Sigma_s^D(\hat{p}_{s,t}))$.

Now, let us prove the following form of the amalgamation lemma:

Lemma C.3.3. *We can find a finite linear order Y and embeddings $h\colon n_\cup \to Y$, $h'\colon n'_\cup \to Y$ such that the diagram*

$$\begin{array}{ccc} \hat{p}_{s,t} & \xrightarrow{e'_0 \circ \Sigma_s^D} & n'_\cup \\ {\scriptstyle e_0 \circ \Sigma_s^D}\downarrow & & \downarrow{\scriptstyle h'} \\ n_\cup & \xrightarrow[h]{} & Y \end{array}$$

commutes and

$$Y \vDash h(e_1 \circ \Sigma_t^D(\hat{p}_{s,t})) < h'(e_1' \circ \Sigma_t^D(\hat{p}_{s,t})) < h'(e_0' \circ \Sigma_s^D(\hat{p}_{s,t})) < h(e_0 \circ \Sigma_s^D(\hat{p}_{s,t})). \tag{C.7}$$

(Note that $e_0 \circ \Sigma_s^D(i) = e_1 \circ \Sigma_t^D(i)$ for $i < \hat{p}_{s,t}$ since Д is $\hat{p}_{s,t}$-congruent, and similar for e_0' and e_1'.)

Proof. First, let us observe that the following are the same, respectively:

1. The position of $e_1(\Sigma_t^D(\hat{p}_{s,t}))$ and $e_0(\Sigma_s^D(\hat{p}_{s,t}))$ over the set
$$H = \{e_0 \circ \Sigma_s^D(i) \mid i < \hat{p}_{s,t}\} \cup \{e_0(\Sigma_s^D(\hat{p}_{s,t})), e_1(\Sigma_t^D(\hat{p}_{s,t}))\} \subseteq n_{\cup},$$
2. The position of $e_1'(\Sigma_t^D(\hat{p}_{s,t}))$ and $e_0'(\Sigma_s^D(\hat{p}_{s,t}))$ over the set
$$H' = \{e_0' \circ \Sigma_s^D(i) \mid i < \hat{p}_{s,t}\} \cup \{e_0'(\Sigma_s^D(\hat{p}_{s,t})), e_1'(\Sigma_t^D(\hat{p}_{s,t}))\} \subseteq n_{\cup}'.$$

It follows from

$$\Sigma_t^D(\hat{p}_{s,t}) < \Sigma_t^D(j) < \Sigma_s^D(j) < \Sigma_s^D(\hat{p}_{s,t})$$

and the following equivalences: For $i < \hat{p}_{s,t}$,

1. $e_0 \circ \Sigma_s^D(i) < e_1 \circ \Sigma_t^D(\hat{p}_{s,t}) \iff \Sigma_t^D(i) < \Sigma_t^D(\hat{p}_{s,t}) \iff e_0' \circ \Sigma_s^D(i) < e_1' \circ \Sigma_t^D(\hat{p}_{s,t})$,
2. $e_0 \circ \Sigma_s^D(\hat{p}_{s,t}) < e_1 \circ \Sigma_t^D(i) \iff \Sigma_s^D(\hat{p}_{s,t}) < \Sigma_s^D(i) \iff e_0' \circ \Sigma_s^D(\hat{p}_{s,t}) < e_1' \circ \Sigma_t^D(i)$.

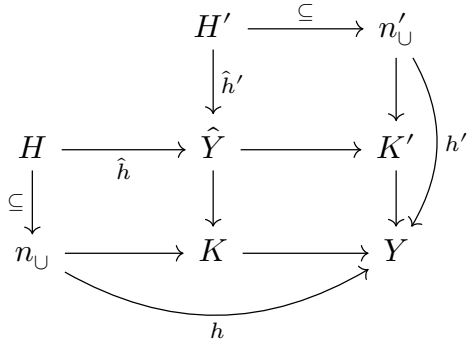
Hence we can find a finite linear order $\hat{Y}$ of size $\hat{p}_{s,t}+4$ and maps $\hat{h} \colon H \to \hat{Y}$, $\hat{h}' \colon H' \to \hat{Y}$ such that the diagram

$$\begin{array}{ccc} \hat{p}_{s,t} & \xrightarrow{e_0' \circ \Sigma_s^D} & H' \\ {\scriptstyle e_0 \circ \Sigma_s^D}\downarrow & & \downarrow{\scriptstyle \hat{h}'} \\ H & \xrightarrow[\hat{h}]{} & \hat{Y} \end{array}$$

commutes and

$$\hat{Y} \vDash \hat{h}(e_1 \circ \Sigma_t^D(\hat{p}_{s,t})) < \hat{h}'(e_1' \circ \Sigma_t^D(\hat{p}_{s,t})) < \hat{h}'(e_0' \circ \Sigma_s^D(\hat{p}_{s,t})) < \hat{h}(e_0 \circ \Sigma_s^D(\hat{p}_{s,t})).$$

To complete the proof, let us apply Theorem A.1.8 three times to get the desired amalgamation: From the diagram



We apply Theorem A.1.8 to $\hat{h}$ and the inclusion map $H \subseteq n_\cup$ to get K, and to $\hat{h}'$ and the inclusion map $H' \subseteq n_\cup'$ to get K'. Then we apply Theorem A.1.8 once again to K and K' to get Y. If we define h and h' by the appropriate composition displayed in the diagram, then Y is the desired order. The finiteness follows from the additional statement in Theorem A.1.8. □

Going back to the main proof, let us observe that

$$D(Y) \vDash s(\operatorname{ran} h' \circ e_0') < s(\operatorname{ran} h \circ e_0)$$

since $h' \circ e_0'(\Sigma_s^D(i)) = h \circ e_0(\Sigma_s^D(i))$ for $i < \hat{p}_{s,t}$ and $h' \circ e_0'(\Sigma_s^D(\hat{p}_{s,t})) < h \circ e_0(\Sigma_s^D(\hat{p}_{s,t}))$, so we can apply Theorem C.2.6. Similarly, we have $D(Y) \vDash t(\operatorname{ran} h \circ e_1) < t(\operatorname{ran} h' \circ e_1')$. Hence we have

$$\begin{aligned}
& D \vDash s <_{\text{Д}} t \\
\iff\ & D(n_\cup) \vDash s(\operatorname{ran} e_0) < t(\operatorname{ran} e_1) \iff D(Y) \vDash s(\operatorname{ran} h \circ e_0) < t(\operatorname{ran} h \circ e_1) \\
\implies\ & D(Y) \vDash s(\operatorname{ran} h' \circ e_0') < t(\operatorname{ran} h' \circ e_1') \iff D(n_\cup') \vDash s(\operatorname{ran} e_0') < t(\operatorname{ran} e_1') \\
\iff\ & D \vDash s <_{\text{Д}'} t.
\end{aligned}$$

Since Д and Д′ are arbitrary $\hat{p}_{s,t}$-congruent arity diagrams, we have the desired result.

□

Hence, the following notion is well-defined:

Definition C.3.4. For a predilator D, let $\mathbf{p}^D(s,t)$ be the least secure number relative to s and t. Then we define $\varepsilon^D_{s,t} = +1$ if $D \vDash s <_{Д} t$ holds for every (equivalently, some) $\mathbf{p}^D(s,t)$-congruent arity diagram Д for s and t, and $\varepsilon^D_{s,t} = -1$ otherwise.

Hence if $\varepsilon^D_{s,t} = +1$ and $a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for every $i < \mathbf{p}^D(s,t)$, then we have $D(X) \vDash s(a) < t(b)$ no matter what the other values of a and b are. We may understand $\mathbf{p}^D(s,t)$ as the 'index bound of the comparison', as we do not consider indices beyond $\mathbf{p}^D(s,t)$ (modulo permutation) to compare $s(a)$ and $t(b)$. Also, if $\varepsilon^D(s,t) = -1$ and if Д is a $\mathbf{p}^D(s,t)$-congruent arity diagram, then $D \vDash t <_{-Д} s$.

It turns out that Σ^D, $\mathbf{p}^D$, and ε^D completely characterize the comparison rule of D:

Theorem C.3.5. *Let D be a predilator and X a linear order. For $s, t \in \mathrm{field}(D)$ and $a, b \subseteq X$ such that $|a| = \mathrm{arity}\, s$, $|b| = \mathrm{arity}\, t$, we have $D(X) \vDash s(a) < t(b)$ if and only if*

1. *$a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for every $i < \mathbf{p}^D(s,t)$ and $\varepsilon^D_{s,t} = +1$, or*
2. *There is $j < \mathbf{p}^D(s,t)$ such that $a(\Sigma^D_s(j)) < b(\Sigma^D_t(j))$ and $a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for every $i < j$.*

Proof. Observe that $\varepsilon^D_{s,t} = -\varepsilon^D_{t,s}$ and $\mathbf{p}^D(s,t) = \mathbf{p}^D(t,s)$. Hence, if the latter side of 'if and only if' fails, then the first disjunct holds with s and t interchanged. Henc,e it suffices to prove the backward implication.

Suppose that the first disjunct holds; i.e., $a(\Sigma_s^D(i)) = b(\Sigma_t^D(i))$ for every $i < \mathbf{p}^D(s,t)$ and $\varepsilon_{s,t}^D = +1$. In this case, $\mathrm{Diag}_X(a,b)$ is $\mathbf{p}^D(s,t)$-congruent, so $\varepsilon_{s,t}^D = +1$ implies $D \vDash s <_{\mathrm{Diag}_X(a,b)} t$, so $D(X) \vDash s(a) < t(b)$.

Now suppose that the second disjunct holds. Then the witnessing j is not secure, so there is a j-congruent arity diagram

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{C.8}$$

for s and t (so arity $s = |a| = n_0$ and arity $t = |b| = n_1$) such that $D \vDash s <_{\text{Д}} t$.

Lemma C.3.6. *We can find a finite linear order Y and embeddings $h \colon a \cup b \to Y$, $h' \colon n_\cup \to Y$ such that the diagram*

$$\begin{array}{ccc} j & \xrightarrow{\mathrm{en}_a \circ \Sigma_s^D} & a \cup b \\ {\scriptstyle e_0 \circ \Sigma_s^D}\downarrow & & \downarrow{\scriptstyle h} \\ n_\cup & \xrightarrow[h']{} & Y \end{array}$$

commutes and

$$Y \vDash h'(e_1(\Sigma_t^D(j))) < h(\mathrm{en}_b(\Sigma_t^D(j))) \text{ and } h(\mathrm{en}_a(\Sigma_s^D(j))) < h'(e_0(\Sigma_s^D(j))).$$

Proof. First, let us observe that the following are the same, respectively:

1. The position of $\{\mathrm{en}_b(\Sigma_t^D(j)), \mathrm{en}_a(\Sigma_s^D(j))\}$ over the set

$$H = \{\mathrm{en}_a(\Sigma_s^D(i)) \mid i < j\} \cup \{\mathrm{en}_a(\Sigma_s^D(j)), \mathrm{en}_b(\Sigma_t^D(j))\} \subseteq a \cup b,$$

2. The position of $\{e_1(\Sigma_t^D(j)), e_0(\Sigma_s^D(j))\}$ over the set

$$H' = \{e_0 \circ \Sigma_s^D(i) \mid i < j\} \cup \{e_0(\Sigma_s^D(j)), e_1(\Sigma_t^D(j))\} \subseteq n_\cup.$$

Its proof is similar to the corresponding part of the proof Lemma C.3.3 with one significant difference: In this case, we may have, for example,

$$\mathrm{en}_a(\Sigma_s^D(j)) < \mathrm{en}_b(\Sigma_t^D(j)) \quad \text{but} \quad e_0(\Sigma_s^D(j)) > e_1(\Sigma_t^D(j)).$$

However, in this case, $\mathrm{en}_a(\Sigma_s^D(j))$ and $\mathrm{en}_b(\Sigma_t^D(j))$ are adjacent to the each other in H, and similarly for $e_0(\Sigma_s^D(j))$ and $e_1(\Sigma_t^D(j))$ in H'. Hence, it is not a problem to carry over the next argument.

Hence we can find a finite linear order $\hat{Y}$ of size $j+4$ and maps $\hat{h} : H \to \hat{Y}, \hat{h}' : H' \to \hat{Y}$ such that the diagram

$$\begin{array}{ccc} j & \xrightarrow{a \circ \Sigma_s^D} & H \\ {\scriptstyle e_0 \circ \Sigma_s^D}\downarrow & & \downarrow{\scriptstyle \hat{h}} \\ H' & \xrightarrow[\hat{h}']{} & \hat{Y} \end{array}$$

commutes and

$$\hat{Y} \vDash h'(e_1(\Sigma_t^D(j))) < h(\mathrm{en}_b(\Sigma_t^D(j))) \wedge h(\mathrm{en}_a(\Sigma_s^D(j))) < h'(e_0(\Sigma_s^D(j))).$$

Then apply Theorem A.1.8 to get a desired Y and h, h'. □

Granted to Lemma C.3.6, we can apply Theorem C.2.6 as in the proof of Lemma C.3.2 to have

$$D(Y) \vDash D(h)(s(a)) < D(h')(s(\operatorname{ran} e_0)) < D(h')(t(\operatorname{ran} e_1)) < D(h)(t(b)).$$

Hence we have $D(a \cup b) \vDash s(a) < t(b)$, so $D(X) \vDash s(a) < t(b)$. □

Then let us analyze constraints on $\mathbf{p}^D$ and ε^D. The following lemma states $2^{-\mathbf{p}^D}$ shows a ultrametric-like behavior:

Lemma C.3.7. *For $s, t, u \in \mathrm{field}(D)$, we have $\mathbf{p}^D(s,u) \geq \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$.*

Proof. Let $p < \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$. We claim that p is not secure relative to s and u, which implies $p < \mathbf{p}^D(s,u)$.

From $p < \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$, we have for every $i, j \leq p$,

$$\Sigma_s^D(i) < \Sigma_s^D(j) \iff \Sigma_t^D(i) < \Sigma_t^D(j) \iff \Sigma_u^D(i) < \Sigma_u^D(j).$$

Hence we can find a finite linear order Y with embeddings $f, f' \colon \operatorname{arity} s \to Y$, $g, g' \colon \operatorname{arity} t \to Y$, $h, h' \colon \operatorname{arity} u \to Y$ such that

1. For every $i \leq p$,

$$f(\Sigma_s^D(i)) = f'(\Sigma_s^D(i)) = g(\Sigma_t^D(i)) = g'(\Sigma_t^D(i)) = h(\Sigma_u^D(i)) = h'(\Sigma_u^D(i)).$$

2. $f(\Sigma_s^D(p)) < g(\Sigma_t^D(p)) < h(\Sigma_u^D(p))$.
3. $f'(\Sigma_s^D(p)) > g'(\Sigma_t^D(p)) > h'(\Sigma_u^D(p))$.

Then by Theorem C.3.5, we have

1. $D(Y) \vDash s(\operatorname{ran} f) < t(\operatorname{ran} g) < u(\operatorname{ran} h)$, and
2. $D(Y) \vDash s(\operatorname{ran} f') > t(\operatorname{ran} g') > u(\operatorname{ran} h')$.

Then $\operatorname{Diag}_Y(\operatorname{ran} f, \operatorname{ran} h)$ and $\operatorname{Diag}_Y(\operatorname{ran} f', \operatorname{ran} h')$ are p-congruent arity diagrams but they do not agree on comparing s and u. □

For this reason, we often call $\mathbf{p}^D(s,t)$ the *distance between s and t*.

The next lemma has a crucial role in the abstract characterization of predilators.

Lemma C.3.8. *For $s, t, u \in \operatorname{field}(D)$, suppose that $\varepsilon_{s,t}^D = \varepsilon_{t,u}^D = +1$. Then $\varepsilon_{s,u}^D = +1$ and $\mathbf{p}^D(s,u) = \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$.*

Proof. One direction of the inequality was shown in Lemma C.3.7. Hence we prove from the assumption that $\varepsilon^D_{s,u} = +1$ and $\mathbf{p}^D(s,u) \leq \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$ hold.

Let $p = \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$. For an p-congruent arity diagram Д for s and u, we claim that $D \vDash s <_{Д} u$, proving both the inequality and $\varepsilon^D_{s,u} = +1$. For technical convenience, we prove the equivalent inequality $D(\omega \cdot (1 + n_\cup)) \vDash s(a) < u(c)$, where

- $a = \{\omega \cdot (1 + e_0(0)), \cdots, \omega \cdot (1 + e_0(n_0 - 1))\}$, and
- $c = \{\omega \cdot (1 + e_1(0)), \cdots, \omega \cdot (1 + e_1(n_1 - 1))\}$.

1. If $\mathbf{p}^D(s,t) = \mathbf{p}^D(t,u)$, let us find $b \subseteq \omega \cdot (1 + n_\cup)$ of size arity t such that $a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for $i < \mathbf{p}^D(s,t)$. Then Theorem C.3.5 implies $D(\omega \cdot (1 + n_\cup)) \vDash s(a) < t(b) < u(c)$.
2. If $\mathbf{p}^D(s,t) < \mathbf{p}^D(t,u)$, let us find $b \subseteq \omega \cdot (1 + n_\cup)$ of size arity t such that $a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for $i < \mathbf{p}^D(s,t)$ and $b(\Sigma^D_t(\mathbf{p}^D(s,t))) < c(\Sigma^D_u(\mathbf{p}^D(s,t)))$. Then again by Theorem C.3.5, we have $D(\omega \cdot (1 + n_\cup)) \vDash s(a) < t(b) < u(c)$.
3. If $\mathbf{p}^D(s,t) > \mathbf{p}^D(t,u)$, let us find $b \subseteq \omega \cdot (1 + n_\cup)$ of size arity t such that $a(\Sigma^D_s(i)) = b(\Sigma^D_t(i))$ for $i < \mathbf{p}^D(t,u)$ and $a(\Sigma^D_s(\mathbf{p}^D(t,u))) < b(\Sigma^D_t(\mathbf{p}^D(t,u)))$. Then Theorem C.3.5 implies $D(\omega \cdot (1 + n_\cup)) \vDash s(a) < t(b) < u(c)$.

Hence, in all possible cases, we get the inequality. □

Hence we can define a linear order $<^D$ over field(D) from ε^D as follows:

Definition C.3.9. Let D be a predilator. Let us define $<^D$ over field(D) by $s <^D t$ iff $s \neq t$ and $\varepsilon^D_{s,t} = +1$.

Lemma C.3.8 implies $<^D$ is a linear order. However, $<^D$ may not be well-founded even when D is a dilator. Also, we have

Lemma C.3.10. *Let D be a predilator and $s, t, u \in \mathrm{field}(D)$. If $s <^D t <^D u$, then $\mathbf{p}^D(s,u) = \min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u))$.* □

Hence, we have the following:

Theorem C.3.11. *Every predilator is characterized by the following data:*

1. *The domain set X with a linear order $<^X$.*
2. *A function $\mathbf{p} \colon X \times X \to \mathbb{N}$ such that*
   - *$\mathbf{p}(x,y) = \mathbf{p}(y,x)$ for $x, y \in X$.*
   - *$\mathbf{p}(x,z) = \min(\mathbf{p}(x,y), \mathbf{p}(y,z))$ for $x, y, z \in X$ such that $x \leq^X y \leq^X z$.*
3. *For each $x \in X$, a permutation Σ_x over $\mathbf{p}(x,x)$ such that*

$$\forall i, j < \mathbf{p}(x,y)[\Sigma_x(i) < \Sigma_x(j) \iff \Sigma_y(i) < \Sigma_y(j)].$$

Proof. For a given predilator D, $(\mathrm{field}(D), <^D, \mathbf{p}^D, \Sigma^D)$ satisfies the conditions. Conversely, we can construct a predilator from the given data as follows: Define a predilator D by $\mathrm{field}(D) = X$, and for $t \in \mathrm{field}(D)$, $\mathrm{arity}\, t = \mathbf{p}(t,t)$. For $s, t \in \mathrm{field}(D)$ and an arity diagram Д for s and t of the form (C.8), let us define $s <_Д t$ iff either

1. $s \leq^X t$ and $e_0(\Sigma_s(i)) = e_1(\Sigma_t(i))$ for $i < \mathbf{p}(s,t)$, or
2. There is $i < \mathbf{p}(s,t)$ such that $e_0(\Sigma_s(i)) < e_1(\Sigma_t(i))$ and $e_0(\Sigma_s(j)) = e_1(\Sigma_t(j))$ for every $j < i$.

It is possible to prove that D is a semidilator, and D being a predilator follows from the definition. Furthermore, Theorem C.3.5 implies $<^D$ and $<^X$ coincide, $\mathbf{p} = \mathbf{p}^D$, and $\Sigma = \Sigma^D$. □

Note that we can use a priority *order* instead of priority permutation to formulate the abstract construction: For $s \in D$, let us define a linear order $\pi\!\!\pi^D(s) = (\text{arity } s, <_{\pi\!\!\pi^D(s)})$ over arity s by

$$i <_{\pi\!\!\pi^D(x)} j \iff \Sigma^D_s(i) < \Sigma^D_s(j).$$

We can conversely turn a linear order $\prec$ over m into a permutation over m by letting

$$\sigma(i) = j \iff i \text{ is the } j\text{th largest element under } \prec.$$

Hence, we can characterize the abstract construction as follows:

Proposition C.3.12. *Every predilator is characterized by the following data:*

1. *The domain set X with a linear order $<^X$.*
2. *A function $\mathbf{p} \colon X \times X \to \mathbb{N}$ such that*
   - $\mathbf{p}(x, y) = \mathbf{p}(y, x)$ *for* $x, y \in X$.
   - $\mathbf{p}(x, z) = \min(\mathbf{p}(x, y), \mathbf{p}(y, z))$ *for* $x, y, z \in X$ *such that* $x \leq^X y \leq^X z$.
3. *For each $x \in X$, a linear order $\pi\!\!\pi(x) = (\mathbf{p}(x, x), <_{\pi\!\!\pi(x)})$ over $\mathbf{p}(x, x)$ such that*

$$\forall i, j < \mathbf{p}(x, y)[i <_{\pi\!\!\pi(x)} j \iff i <_{\pi\!\!\pi(y)} j].$$ □

The abstract construction of predilators becomes important in the theory of *dendrograms*, which we will explain in the next chapter.

We end this section with the following lemma. This lemma is placed here because I do not find any better place to put it.

Lemma C.3.13. *Let D be a perfect predilator. Then $\mathbf{p}(s,t) > 0$ for every $s,t \in$ field(D).*

Proof. If $\mathbf{p}^D(s,t) = 0$, then we have $D \vDash s <_{\text{Д}} t$ for every 0-congruent arity diagram Д, but every arity diagram is 0-congruent. Hence $\mathbf{p}^D(s,t) = 0$ means s and t belong to different $\asymp^D$-components, which is impossible when D is perfect. □

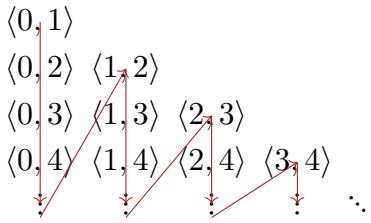
C.4 Flowers

We may view dilators as a uniform procedure taking an ordinal and returning another ordinal. The way a dilator produces a new ordinal sometimes shows unexpected behavior.

Example C.4.1. Consider $\mathsf{Prime}_{\mathsf{Id}_2}$, a prime dilator with a unique term t of arity 2, and it satisfies the following for a linear order X:

$$\mathsf{Prime}_{\mathsf{Id}_2}(X) \vDash t(a_0,a_1) < t(b_0,b_1) \iff X \vDash (a_0 < b_0) \vee [(a_0 = b_0) \wedge (a_1 < b_1)].$$

Then $\mathsf{Prime}_{\mathsf{Id}_2}(\omega)$ compares pairs as follows:



Hence even though $\mathsf{Prime}_{\mathsf{Id}_2}(n) \cong n(n-1)/2$ for each natural number n, $\mathsf{Prime}_{\mathsf{Id}_2}(\omega) \cong \omega^2$. In some sense, $\mathsf{Prime}_{\mathsf{Id}_2}$ is not 'continuous.'

What is 'wrong' with the previous example? We can see that $\mathsf{Prime}_{\mathsf{Id}^2}$ 'inserts' new elements in the middle. That is, $\mathsf{Prime}_{\mathsf{Id}^2}(n)$ is not an initial segment of $\mathsf{Prime}_{\mathsf{Id}^2}(n+1)$. Thus, we name a dilator that does not insert new elements in the middle:

Definition C.4.2. A semidilator D is a *semiflower* if whenever X, Y are linear orders such that $X \subseteq Y$ is an initial segment of Y, then $D(X)$ is an initial segment of $D(Y)$. A semiflower is a *flower* if it is also a dilator.

We think of semiflowers as semidilators that do not insert new elements in the middle. We will see that every semiflower $D(x)$ has the form $a + \sum_{y<x} E(y)$ for some linear order a and a semidilator E. The following definition gives a way to define a new semidilator $\int D$ from D corresponding to $\sum_{y<x} D(y)$:

Definition C.4.3. For a semidilator D, let us define $\int D$ as follows: The set of $\int D$-terms is $\{t^{\int} \mid t \in \text{field}(D)\}$ with $\text{arity}^{\int D} t^{\int} = \text{arity}^{D} t + 1$. For an arity diagram Д of the form

$$\text{Д} = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup \end{array} \tag{C.9}$$

let us define the comparison rule $t_0^{\int} <_{\text{Д}} t_1^{\int}$ if and only if

1. Either $\max e_0 < \max e_1$, or
2. If $\max e_0 = \max e_1$, and if $\text{Д}^- = \text{Diag}_{n_\cup}(\text{ran}\, e_0 \setminus \{\max e_0\}, \text{ran}\, e_1 \setminus \{\max e_1\})$, then $D \vDash t_0 <_{\text{Д}^-} t_1$.

The idea of the comparison rule becomes evident in the proof of the following lemma:

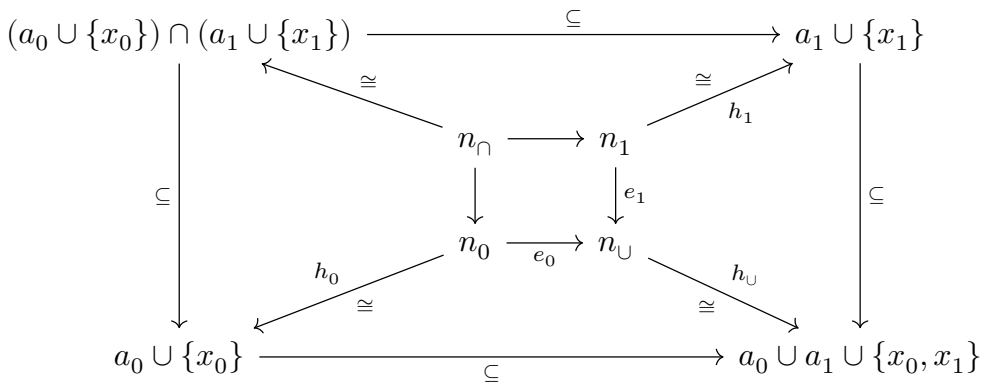
Lemma C.4.4. *For a linear order X and a semidilator D, $(\int D)(X)$ is isomorphic to $\sum_{x \in X} D(X \upharpoonright x)$, where $X \upharpoonright x = \{y \in X \mid y < x\}$.*

Proof. Let us define the map $f \colon \sum_{x \in X} D(X \upharpoonright x) \to (\int D)(X)$ by

$$f((x, t(a))) = t^{\int}(a \cup \{x\}).$$

Here $a \subseteq X \restriction x$ and $|a| = \operatorname{arity} t$. Then clearly f is well-defined and onto, so it remains to show that f is increasing.

Let $(x_0, t_0(a_0)), (x_1, t_1(a_1)) \in \sum_{x \in X} D(X \restriction x)$, and $Д = \mathrm{Diag}(a_0 \cup \{x_0\}, a_1 \cup \{x_1\})$ has the form (C.9). In particular, we have that the following diagram commutes:



$(x_0, t_0(a_0)) < (x_1, t_1(a_1))$ holds when either $x_0 < x_1$, or ($x_0 = x_1$ and $t_0(a_0) < t_1(a_1)$.) In the first case, observe that $a_0 \cup a_1 \cup \{x_0, x_1\} \vDash x_0 < x_1$ implies $n_\cup \vDash h_\cup^{-1}(x_0) < h_\cup^{-1}(x_1)$. Then by commutativity, $h_\cup^{-1}(x_0) = \max e_0$ and $h_\cup^{-1}(x_1) = \max e_1$, so $\max e_0 < \max e_1$.

In the second case, $D(X) \vDash t_0(a_0) < t_1(a_1)$ if and only if $D \vDash t_0 <_{\mathrm{Diag}_X(a_0, a_1)} t_1$. We can see that $h_\cup$ witnesses

$$\mathrm{Diag}_X(\operatorname{ran} e_0 \setminus \{\max e_0\}, \operatorname{ran} e_1 \setminus \{\max e_1\}) = \mathrm{Diag}_X(a_0, a_1),$$

as desired. □

Proposition C.4.5. *For a semidilator D, $\int D$ is a semiflower. For a semidilator embedding $f \colon D \to E$, if we define $\int f \colon \int D \to \int E$ by $(\int f)(t) = f(t)^\int$, then $\int f$ is also a semidilator morphism.*

Proof. Let $X \subseteq Y$ be linear orders such that Y end-extends X. For $s^\int(a), t^\int(b) \in (\int D)(Y)$ for some $s, t \in \mathrm{field}(D)$ and $a, b \subseteq Y$, $s^\int(a) \leq t^\int(b)$ implies $\max a \leq \max b$.

In particular, if $\max b \in X$, then $\max a \in X$, and we can see that $\max a \in X$ implies $s^{\int}(a) \in (\int D)(X)$. $\int f$ being a semidilator morphism follows from the definition of the comparison rule of $\int D$ and $\int E$. □

Hence $\int D$ is always a semiflower. To prove the converse, stating that every semiflower is of the form $a + \int D$, we need to analyze the structure of a semiflower:

Lemma C.4.6. *Let D be a semiflower. Then for every D-terms s, t,*

1. *If* $0 = \operatorname{arity} s < \operatorname{arity} t$ *and an arity diagram* Д *for* s, t*, then* $s <_{Д} t$.
2. *If* $\operatorname{arity} s, \operatorname{arity} > 0$ *and if* Д *is an arity diagram for* s, t *of the form* (C.9) *satisfying* $\max e_0 < \max e_1$*, then* $s <_{Д} t$.

Proof. 1. Suppose that the given claim fails and $n = \operatorname{arity} t$. Then $t <_{-Д} s$ holds, so $D(n) \vDash t(0, \cdots, n-1) < s$. It means $D(0)$ is not an initial segment of $D(n)$ even though 0 is an initial segment of n.

2. Suppose similarly that the given claim fails. If $n_0 = \operatorname{arity} s$ and $n_1 = \operatorname{arity} t$, then $t <_{-Д} s$ holds. We have $D(n_{\cup}) \vDash t(\operatorname{ran} e_1) < s(\operatorname{ran} e_0)$ and $t(\operatorname{ran} e_1) \notin D(\max e_0)$. Hence $D(n_{\cup})$ is not an initial segment of $D(\max e_0)$ even though $\max e_0 < n_{\cup}$. □

Lemma C.4.7. *Let D be a semiflower. Let us define a new structure ∂D as follows: The field of ∂D is $\{t^{\partial} \mid t \in \operatorname{field}(D) \wedge \operatorname{arity}^{D}(t) \geq 1\}$, and $\operatorname{arity}^{\partial D}(t^{\partial}) = \operatorname{arity}^{D}(t) - 1$. For each arity diagram* Д *for $t_0^{\partial}, t_1^{\partial}$ of the form* (C.9)*, let us define*

$$Д^{+} = \operatorname{Diag}_{n_{\cup}+1}(\operatorname{ran} e_0 \cup \{n_{\cup}\}, \operatorname{ran} e_1 \cup \{n_{\cup}\}). \tag{C.10}$$

Then define $t_0^{\partial} <_{Д} t_1^{\partial}$ iff $D \vDash t_0 <_{Д^{+}} t_1$.

Then ∂D is a semidilator. Furthermore, if D and E are semiflowers and $f : D \to E$, the map $\partial f : \partial D \to \partial E$ given by $(\partial f)(t^{\partial}) = (f(t))^{\partial}$ is a semidilator embedding.

Proof. Irreflexivity is easy to check. For linearity, observe that if Д is not trivial then $Д^+$ is also not trivial, and $-(Д^+) = (-Д)^+$.

Now, let us verify the transitivity. Let $\mathbf{Д} = \langle n_i, f_{ij} \mid i \leq j \in \mathtt{L}_3 \rangle$ be an IU diagram for $t_0^\partial, t_1^\partial, t_2^\partial$ with $n_i = \operatorname{arity} t_i^\partial$. If we define

$$\mathbf{Д}^+ = \operatorname{Diag}_{n_{0\vee1\vee2}}(\operatorname{ran} f_{0,0\vee1\vee2} \cup \{n_{0\vee1\vee2}\}, \operatorname{ran} f_{1,0\vee1\vee2} \cup \{n_{0\vee1\vee2}\}, \operatorname{ran} f_{2,0\vee1\vee2} \cup \{n_{0\vee1\vee2}\}),$$

then $\mathbf{Д}^+(i,j) = \mathbf{Д}(i,j)^+$ for every $i, j \in \mathtt{L}_3$. From $t_0^\partial <_{\mathbf{Д}(0,1)} t_1^\partial <_{\mathbf{Д}(1,2)} t_2^\partial$ we have $t_0 <_{\mathbf{Д}^+(0,1)} t_1 <_{\mathbf{Д}^+(1,2)} t_2$, and the transitivity of D implies $t_0 <_{\mathbf{Д}^+(0,2)} t_2$. Hence we get $t_0^\partial <_{\mathbf{Д}(0,2)} t_2^\partial$. The 'furthermore' part follows from the definition of the comparison rule of ∂D and ∂E. □

Theorem C.4.8. *Let D be a semiflower. If $\mathsf{Init}(D)$ is the linear order given by the nullary D-terms, then $D \cong \mathsf{Init}(D) + \int(\partial D)$.*

Proof. Let us define $\iota \colon D \to \mathsf{Init}(D) + \int(\partial D)$ by

$$\iota(t) = \begin{cases} (0, t) & \text{if } \operatorname{arity}^D(t) = 0, \\ (1, (t^\partial)^\int) & \text{if } \operatorname{arity}^D(t) > 0. \end{cases}$$

Then ι is a bijection. To prove ι is an isomorphism, it suffices to prove that ι is a semidilator embedding.

Let $s, t \in \operatorname{field}(D)$ and Д is an arity diagram for s, t. Suppose that $D \vDash s <_Д t$. If both of s and t are nullary, then Д is trivial and clearly $\iota(s) <_Д \iota(t)$. By Lemma C.4.6, it is impossible that $\operatorname{arity} s > 0$ and $\operatorname{arity} t = 0$. If $\operatorname{arity} s = 0 < \operatorname{arity} t$, then $\iota(s) = (0, s) <_Д (1, (t^\partial)^\int) = \iota(t)$ by definition of semidilator sum.

Hence, we remain in the case $\operatorname{arity} s, \operatorname{arity} t > 0$. Suppose that Д has the form (C.9). Then by Lemma C.4.6, $\max e_0 > \max e_1$ is impossible. If $\max e_0 < \max e_1$, then the

definition of $\int$ implies $\int(\partial D) \vDash (s^{\partial})^{\int} <_{Д} (t^{\partial})^{\int}$, so $\iota(s) <_{Д} \iota(t)$. If $\max e_0 = \max e_1$, and if we define

$$Д^{-} = \mathrm{Diag}_{n_{\cup}}(\mathrm{ran}\, e_0 \setminus \{\max e_0\}, \mathrm{ran}\, e_1 \setminus \{\max e_1\}),$$

then we can prove $Д = (Д^{-})^{+}$. Hence $D \vDash s <_{Д} t$ implies $\partial D \vDash s^{\partial} <_{Д^{-}} t^{\partial}$, so $\int(\partial D) \vDash (s^{\partial})^{\int} <_{Д} (t^{\partial})^{\int}$. Hence we get $\iota(s) <_{Д} \iota(t)$. □

Conversely, the 'differentiation' of the 'integration' of a dilator gives the previous dilator back:

Proposition C.4.9. *Let D be a semidilator. Then $D \cong \partial(\int D)$.*

Proof. The obvious choice for the isomorphism is $t \mapsto (t^{\int})^{\partial}$. It is clearly a bijection preserving the arity, so it suffices to show that the map preserves the comparison relation.

Suppose that $s, t \in \mathrm{field}(D)$ and $Д$ be an arity diagram between s and t. Consider $Д^{+}$ defined in (C.10), then we have $\int D \vDash s^{\int} <_{Д^{+}} t^{\int}$. Then by the definition of differentiation, we get $\partial(\int D) \vDash (s^{\int})^{\partial} <_{Д} (t^{\int})^{\partial}$. □

From the proof of Theorem C.4.8, we have the following description of a semidilator:

Corollary C.4.10. *A semidilator D is a semiflower iff for every $s, t \in \mathrm{field}(D)$ and an arity diagram $Д$ for s, t of the form (C.9), we have*

1. *If* arity $s = 0 <$ arity t, *then* $s <_{Д} t$.
2. *If* arity s, arity $t > 0$ *and* $\max e_0 < \max e_1$, *then* $s <_{Д} t$.

Proof. One direction holds by Lemma C.4.6. For the other direction, observe that the proof of Theorem C.4.8 still carries over for D, so $D \cong \mathsf{Init}(D) + \int(\partial D)$. Hence D is isomorphic to a semiflower. □

Hence, for a countable semidilator D, the assertion 'D is a semiflower' is Π^0_1. Before finishing this section, let us provide the abstract construction characterization of preflowers:

Proposition C.4.11. *Let D be a predilator. Then D is a preflower if and only if the following holds: For every $s, t \in \mathrm{field}(D)$,*

1. *If* $\mathrm{arity}(s) = 0 < \mathrm{arity}\, t$, *then* $s <^D t$.
2. *If* $\mathrm{arity}\, s, \mathrm{arity}\, t > 0$, *then* $\mathbf{p}^D(s,t) \geq 1$ *and* $\Sigma^D_s(0) = \mathrm{arity}\, s - 1$, $\Sigma^D_t(0) = \mathrm{arity}\, t - 1$.

Proof. Suppose that D is a preflower. Then the first condition is immediate. For the second condition, suppose that $\mathrm{arity}\, s, \mathrm{arity}\, t > 0$. Then $\mathbf{p}^D(s,t) > 0$, otherwise either $s <_{\text{Д}} t$ holds for *every* arity diagram Д or $t <_{\text{Д}} s$ for every Д. For $\Sigma^D_s(0) = \mathrm{arity}\, s - 1$, observe that the largest parameter governs the comparison, i.e., For $a, b \in [X]^{\mathrm{arity}\, s}$, $D(X) \vDash s(a) < s(b)$ if $\max a < \max b$ and all other components of a and b have no role in this inequality. The converse follows from Corollary C.4.10. □

We finish this section by introducing bilators, which are closely related to dilators of type Ω:

Definition C.4.12. A *semibilator* is a binary semidilator D satisfying the following:

1. For every linear order X, $D(X, \cdot)$ is a semiflower.
2. $D(X, Y)$ is not constant in Y; Formally, it is not true that for every linear order X, $D_X = D(X, \cdot)$ is isomorphic to a constant semidilator.

A semibilator D is a *bilator* if D is both a semibilator and a binary dilator.

We will see in the next section that bilators have a role of 'fundamental sequence' for class ordinals. It is convenient to see the first argument of a bilator as a 'scale' (like ω_1^{CK}, ω_1, or Ord) and the second argument as the index for the 'fundamental sequence.' The fundamental sequence should be continuous, which is why we required a bilator to be a flower in the second argument.

We can characterize bilators in terms of the denotation system as follows:

Proposition C.4.13. *A binary semidilator D is a semibilator iff the following holds:*

1. *For every $s, t \in \mathrm{field}(D)$, and a pair of arity diagrams $(Д, Ж)$ for s, t, and*

$$Д = \begin{array}{ccc} m_\cap & \longrightarrow & m_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ m_0 & \xrightarrow[e_0]{} & m_\cup \end{array}, \quad Ж = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ n_0 & \xrightarrow[f_0]{} & n_\cup \end{array} \tag{C.11}$$

   (a) *If* $\mathrm{arity}_1(s) = 0 < \mathrm{arity}_1(t)$, *then* $s <_{Д,Ж} t$.

   (b) *If* $\mathrm{arity}_1(s), \mathrm{arity}_1(t) > 0$ *and* $\max f_0 < \max f_1$, *then* $s <_{Д,Ж} t$.

2. *Not every $t \in \mathrm{field}(D)$ satisfy* $\mathrm{arity}_1(t) = 0$.

Proof. Suppose that D is a semibilator. For two given $s, t \in \mathrm{field}(D)$, observe that

$$D \vDash s <_{Д,Ж} t \iff D(m_\cup, n_\cup) \vDash s(\mathrm{ran}\, e_0; \mathrm{ran}\, f_0) < t(\mathrm{ran}\, e_1; \mathrm{ran}\, f_1).$$

Hence the first clause holds from that $D(m_\cup, \cdot)$ is a semiflower and Corollary C.4.10. Also, if every $t \in \mathrm{field}(D)$ satisfies $\mathrm{arity}_1(t) = 0$, then D is constant in the second argument. It proves the forward direction.

Then let us prove the backward direction. D is a semiflower in the second argument since the first clause implies D_X is a semiflower for each linear order X by Corollary C.4.10. For D being non-constant in the second argument, let $t \in \mathrm{field}(D)$ be a term such that $\mathrm{arity}_1(t) \neq 0$. Then $D(\mathrm{arity}_0(t), \cdot)$ has a term $t(\mathrm{arity}_0\, t)$ whose arity is $\mathrm{arity}_1(t) > 0$, so is not a constant semidilator. □

C.5 Dilators of type Ω

We may view dilators of type Ω as class ordinals of large cofinality, like Ord + Ord, Ord · Ord, or Ord^{Ord}, all of which can be understood as well-orders of the form $D(\text{Ord})$ for some dilator D of type Ω. We want to extract a fundamental sequence for ordinals, or equivalently, a 'fundamental sequence' for dilators of type Ω. A fundamental sequence is a continuous sequence of ordinals that corresponds to a flower. That is, if E is an 'associated bilator' for D, then the critical sequence for $D(\text{Ord})$ should be $\langle E(\text{Ord}, \alpha) \mid \alpha \in \text{Ord} \rangle$.

Then what is the 'associated bilator?' If E is a bilator, then we can see that $D(X) = E(X, X)$ is a dilator of type Ω. However, not every dilator of type Ω has the form $E(X, X)$ for some bilator E:

Example C.5.1. Let us consider $\mathsf{Prime}_{\mathsf{Id}_2}$, which is perfect but not a flower. There can be no bilator E such that $E(X, X)$ is naturally isomorphic to $\mathsf{Prime}_{\mathsf{Id}_2}(X)$: If there is such E, then observe that the number of E-terms is less than or equal to that of $\mathsf{Prime}_{\mathsf{Id}_2}$. Hence E has a unique term t, and $\text{arity}_0(t) + \text{arity}_1(t) = 2$. Since E is a bilator, $\text{arity}_1(t) \geq 1$, but $\text{arity}_1(t) \neq 2$, otherwise $\mathsf{Prime}_{\mathsf{Id}_2}$ would be a flower that is not. Hence $\text{arity}_0(t) = \text{arity}_1(t) = 1$, and there are two possibilities for t: t compares the first component first, then compares the second component, or vice versa. Each case gives $E(X, Y) \cong X \times Y$ or $E(Y, X) \cong Y \times X$, but in both cases, the dilator $X \mapsto E(X, X)$ has three terms.

Does it mean our idea of 'a bilator as a fundamental sequence' is wrong? We can salvage this idea with a slight modification: For a bilator E, let $\partial_1 E$ be a binary dilator obtained from E by taking the derivative operator to the second coordinate. Then we have

$$E(X, X) \cong \mathsf{Init}(\lambda Y.E(X, Y)) + \sum_{x \in X} (\partial_1 E)(X, X \restriction x).$$

If X is 'sufficiently large,' then X and $X_{>x} = \{y \in X \mid y > x\}$ are isomorphic, so we may expect

$$E(X,X) \cong \mathsf{Init}(\lambda Y.E(X,Y)) + \sum_{x \in X} (\partial_1 E)(X_{>x}, X \restriction x).$$

Hence for a given dilator D of type Ω, we instead find a bilator E satisfying

$$D(X) \cong \mathsf{Init}(\lambda Y.E(X,Y)) + \sum_{x \in X} (\partial_1 E)(X_{>x}, X \restriction x).$$

Based on this idea, let us define the unification functor as follows:

Definition C.5.2. Let E be a prebilator. Let us define $\mathrm{Un}(E)$ as follows: For a linear order X, define

$$\mathrm{Un}(E)(X) = \{t(a;b) \in E(X,X) \mid |a| \cdot |b| \neq 0 \to \max b < \min a\},$$

and for an increasing $f \colon X \to Y$, $\mathrm{Un}(E)(f) = E(f,f)$. Also, we define the support function by

$$\mathrm{supp}_X^{\mathrm{Un}(E)}(\sigma) = 0\text{-}\,\mathrm{supp}_{X,X}^E(\sigma) \cup 1\text{-}\,\mathrm{supp}_{X,X}^E(\sigma).$$

Clearly, $\mathrm{Un}(E)$ is a predilator for a binary predilator E. We can also provide the denotation system for $\mathrm{Un}(E)$ as follows:

Proposition C.5.3. *Let E be a prebilator. Then $\mathrm{Un}(E)$ has the following denotation system: Its set of terms is $\{t^{\mathrm{Un}} \mid t \in \mathrm{field}(E)\}$, and $\mathrm{arity}(t^{\mathrm{Un}}) = \mathrm{arity}_0(t) + \mathrm{arity}_1(t)$. For two $s,t \in \mathrm{field}(E)$ and an arity diagram $Д$ of the form (C.9) for s^{Un} and t^{Un}, $\mathrm{Un}(E) \vDash s^{\mathrm{Un}} <_Д t^{\mathrm{Un}}$ is given as follows: If we define*

- $Д_1 = \mathrm{Diag}_{n_\cup}(e_0[\mathrm{arity}_1(s)], e_1[\mathrm{arity}_1(t)])$, *and*
- $Д_0 = \mathrm{Diag}_{n_\cup}(e_0[\mathrm{arity}(s^{\mathrm{Un}}) \setminus \mathrm{arity}_1(s)], e_1[\mathrm{arity}(t^{\mathrm{Un}}) \setminus \mathrm{arity}_1(t)])$,

then $E \vDash s <_{Д_0, Д_1} t$.

Proof. We have defined $\mathrm{Un}(E)$ as a semidilator with a support transformation, and we have the functor $\mathfrak{f}$ that turns a semidilator with a support transformation into a denotation system. (See Proposition B.4.5) We prove that $\mathfrak{f}(\mathrm{Un}(E))$ satisfies all the aforementioned descriptions for a denotation system. First, it is clear that

$$\begin{aligned}&\mathrm{Tr}(\mathrm{Un}(E))\\&= \{t(\mathrm{arity}_0(t), \cdots, \mathrm{arity}_0(t) + \mathrm{arity}_1(t) - 1; 0, 1, \cdots, \mathrm{arity}_0(t) - 1) \mid t \in \mathrm{field}(E)\}.\end{aligned}$$

From now on, let us define

$$t^{\mathrm{Un}} = (\mathrm{arity}_0(t), \mathrm{arity}_1(t), t(\mathrm{arity}_0(t), \cdots, \mathrm{arity}_0(t)+\mathrm{arity}_1(t)-1; 0, 1, \cdots, \mathrm{arity}_0(t)-1)).$$

Now let Д be an arity diagram for s^{Un} and t^{Un} of the form (C.9). We also define Д_0 and Д_1 as in the statement of the proposition. We claim the following two are equivalent:

1. $E \vDash s <_{\text{Д}_0,\text{Д}_1} t$, and
2. $\mathrm{Un}(E)(n_\cup) \vDash \mathrm{Un}(E)(e_0)(s^{\mathrm{Un}}) < \mathrm{Un}(E)(e_1)(t^{\mathrm{Un}})$ (Here we abuse the notation; so s^{Un} means its second component and the same for t^{Un}.)

First, let us observe that the second item is equivalent to

$$\begin{aligned}\mathrm{Un}(E)(n_\cup) \vDash s(e_0[\mathrm{arity}(s^{\mathrm{Un}}) \setminus \mathrm{arity}_0(s)]; e_0[\mathrm{arity}_0(s)])&\\< t(e_1[\mathrm{arity}(t^{\mathrm{Un}}) \setminus \mathrm{arity}_0(t)]; e_1[\mathrm{arity}_0(t)]).&\end{aligned}$$

We defined $\mathrm{Un}(E)(n_\cup)$ as a suborder of $E(n_\cup, n_\cup)$, so the above comparsion also holds over $E(n_\cup, n_\cup)$. Then the definition of a binary semidilator application yields $E \vDash s <_{\text{Д}_0,\text{Д}_1} t$. The converse follows from the same argument. □

Remark C.5.4. The current definition of $\mathrm{Un}(E)$ has a subtle aspect that E is a denotation system, but $\mathrm{Un}(E)$ is a Freund-styled predilator. Thus our definition of Un takes

a denotation system and returns an F-predilator. One way to resolve this issue is to define $\mathrm{Un}(E)$ as a denotation system, by taking Proposition C.5.3 as a definition, so Un takes a denotation system and returns another denotation system. We may also define Un for Freund-styled binary predilators, whose details are left to the reader. We will conflate denotation systems with functors with support transformation as described in Remark B.4.12.

The following proposition says the unification of a prebilator yields a predilator of type Ω:

Proposition C.5.5. *If E is a prebilator, then $\mathrm{Un}(E)$ is a predilator of type Ω. We also have $\mathrm{Un}(E)$ is perfect if $\mathrm{arity}_1(t) > 0$ for every $t \in \mathrm{field}(E)$.*

Proof. Let us first consider the case when $\mathrm{arity}_1(t) > 0$ for every $t \in \mathrm{field}(E)$. Take any E-term t. If we prove the set

$$\{t(a;b) \mid a \in [\omega]^{\mathrm{arity}_0(t)} \wedge b \in [\omega]^{\mathrm{arity}_1(t)} \wedge \max b < \min a\}$$

is cofinal in $\mathrm{Un}(E)(\omega)$, then $\mathrm{Un}(E)$ is perfect by Proposition C.1.11. To see this, let $s \in \mathrm{field}(E)$ and $a', b' \subseteq \omega$. Since $E(\omega, \cdot)$ is a flower, we have $E(\omega,\omega) \vDash s(a';b') < t(a;b)$ as long as we have $\max b' < \max b$.

Then let us consider the case when we have a term $s \in \mathrm{field}(E)$ with $\mathrm{arity}_1(s) = 0$. Proposition C.4.13 implies if $\mathrm{arity}_1(t) > 0$, then for every linear order X and $a, a', b' \subseteq X$, $\mathrm{Un}(D)(X) \vDash s(a;) < t(a';b')$. In the view of Proposition C.5.3, we have $\mathrm{Un}(E) \vDash s^{\mathrm{Un}} <_{Д} t^{\mathrm{Un}}$ for every choice of arity diagram Д for s^{Un} and t^{Un}. Hence s^{Un} lies from a different $\asymp^{\mathrm{Un}(E)}$-component from that of t^{Un}. If we let $E_0 = \{t \in \mathrm{Un}(E) \mid \mathrm{arity}_1(t) = 0\}$ and $E_1 = \{t \in \mathrm{Un}(E) \mid \mathrm{arity}_1(t) > 0\}$, then the previous consideration shows E_1 is perfect and every perfect component of E_0 is

$\ll^{\mathrm{Un}(E)}$-less than E_1 (following terminologies in the proof of Theorem C.1.2.) Hence $\mathrm{Un}(E)$ is of type Ω. □

Remark C.5.6. By definition, $\mathrm{Un}(E)(X) \subseteq E(X,X)$ for every prebilator E. This inclusion is functorial in the following sense: If we define the diagonal functor $\mathbf{\Delta}(E)$ by $\mathbf{\Delta}(E)(X) = E(X,X)$, then the map $\iota_E \colon \mathrm{Un}(E) \to \mathbf{\Delta}(E)$ given by

$$\iota_{E,X}(t(a;b)) = t(a;b)$$

is a natural embedding. There is no reason to restrict E to prebilators, and ι_E is defined for every binary semidilator E. Indeed, both $\mathbf{\Delta}$ and Un are *ptykes of type* $o \times o \to o$, and ι is a ptyx embedding.

We also need a way to extract a bilator from a dilator of type Ω, which will be called the *separation functor*. We first define the *pre-separation functor*, which will have a crucial role in defining the separation functor as follows:

Definition C.5.7. Let D be a predilator. For a linear order X and Y, let us define

$$\mathrm{PreSep}(D)(X,Y) = \{t(a) \in D(Y+X) \mid \{a(0), \cdots, a(\Sigma_t^D(0))\} \subseteq \{\underset{0}{y} \mid y \in Y\} \\ \wedge \{a(\Sigma_t^D(0)+1), \cdots, a(\mathrm{arity}\, t - 1)\} \subseteq \{\underset{1}{x} \mid x \in X\}\}.$$

For $f \colon X \to X'$ and $g \colon Y \to Y'$, we define

$$\mathrm{PreSep}(D)(f,g)(t(a)) := D(g+f)(t(a)) = t((g+f)[a]).$$

Then define $0\text{-}\mathrm{supp}^{\mathrm{PreSep}(D)}_{X,Y}$ and $1\text{-}\mathrm{supp}^{\mathrm{PreSep}(D)}_{X,Y}$ by

1. $0\text{-}\mathrm{supp}^{\mathrm{PreSep}(D)}_{X,Y}(t(a)) = \{x \in X \mid \underset{1}{x} \in a\}$,
2. $1\text{-}\mathrm{supp}^{\mathrm{PreSep}(D)}_{X,Y}(t(a)) = \{y \in Y \mid \underset{0}{y} \in a\}$.

That is, elements of $\mathrm{PreSep}(D)(X,Y) \subseteq D(Y+X)$ takes the form

$$t(\underset{0}{y_0}, \cdots \underset{0}{y_i}, \underset{1}{x_0}, \cdots \underset{1}{x_{\text{arity } t-i-1}})$$

for some $t \in D$, $y_0, \cdots, y_i \in Y$ and $x_0, \cdots, x_{\text{arity } t-i-1} \in X$, where $i = \Sigma_t^D(0)$ is the argument position with the highest comparison priority of t.

Proposition C.5.8. *If D is a predilator, then* $\mathrm{PreSep}(D)$ *is also a predilator. Furthermore, if $D \ncong \underline{1}$ is perfect, then* $\mathrm{PreSep}(D)$ *is a prebilator.*

Proof. For a notational convenience, let us write

1. $a|_{\Sigma_t^D(0)} = \{a(0), \cdots, a(\Sigma_t^D(0))\}$, and
2. $a|^{\Sigma_t^D(0)} = a \setminus a|_{\Sigma_t^D(0)}$.

For $f \colon X \to X'$, $g \colon Y \to Y'$, $t \in \mathrm{field}(D)$, and $a \subseteq Y + X$ such that $t(a) \in \mathrm{PreSep}(D)(X,Y)$, we have

$$\begin{aligned}\mathrm{PreSep}(D)(f,g)(t(a)) &= t((g+f)[a]) \\ &= t(g[a|_{\Sigma_t^D(0)}] \cup (\mathsf{Id}_Y + f)[a|^{\Sigma_t^D(0)}]) \in \mathrm{PreSep}(D)(\operatorname{ran} f, \operatorname{ran} g).\end{aligned}$$

It shows $\mathrm{PreSep}(D)(f,g) \colon \mathrm{PreSep}(D)(X,Y) \to \mathrm{PreSep}(D)(X',Y')$. To see $\mathrm{PreSep}(D)$ is a binary predilator, we need to check the support condition. Suppose that we have $f \colon X \to X'$, $g \colon Y \to Y'$, and $\sigma \in \mathrm{PreSep}(D)(X',Y')$ such that $0\text{-}\operatorname{supp}^{\mathrm{PreSep}(D)}_{X',Y'}(\sigma) \subseteq \operatorname{ran} f$ and $1\text{-}\operatorname{supp}^{\mathrm{PreSep}(D)}_{X',Y'}(\sigma) \subseteq \operatorname{ran} g$. It means

$$\begin{aligned}\operatorname{supp}^D_{Y'+X'}(\sigma) = \{\underset{1}{x} \mid x \in 0\text{-}\operatorname{supp}^{\mathrm{PreSep}(D)}_{X',Y'}(\sigma)\} \cup \{\underset{0}{y} \mid y \in 1\text{-}\operatorname{supp}^{\mathrm{PreSep}(D)}_{X',Y'}(\sigma)\} \\ \subseteq \operatorname{ran}(g+f),\end{aligned}$$

so σ (as an element of $D(Y+X)$) is in $\operatorname{ran} D(g+f) = \mathrm{PreSep}(D)(f,g)$. It shows the support condition.

To see the furthermore part, observe that if $D \ncong \underline{1}$ is perfect, then every D-term is non-nullary. Hence for a fixed X, $\mathrm{PreSep}(D)(X,0) \cong 0$ and $\mathrm{PreSep}(D)(X,n) \ncong 0$ for some $n \in \mathbb{N}$. It shows $\mathrm{PreSep}(D)$ is not constant in the second argument. To see $\mathrm{PreSep}(D)(X,\cdot)$ is a preflower, observe again that the previous observation also shows no $\mathrm{PreSep}(D)$-term t satisfies $\mathrm{arity}_1(t) = 0$, so it suffices to verify the second condition of Corollary C.4.10. Observe that every element of $\mathrm{PreSep}(D)(X,Y)$ takes the form

$$\sigma = s\Big(a_0, \cdots, a_{\Sigma_s^D(0)}, \underbrace{b_0, \cdots, b_{\mathrm{arity}\, s - \Sigma_s^D(0) - 1}}_{1}\Big).$$

for some $s \in \mathrm{field}(D)$. Since D is perfect, s must be non-nullary. Now let $Y' \subseteq Y$ be an initial segment of Y. For

$$\tau = t\Big(a'_0, \cdots, a'_{\Sigma_t^D(0)}, \underbrace{b'_0, \cdots, b'_{\mathrm{arity}\, s - \Sigma_s^D(0) - 1}}_{1}\Big),$$

if $D(Y + X) \vDash \sigma < \tau$ and $\tau \in \mathrm{PreSep}(X, Y')$, then either $\mathbf{p}^D(s,t) = 0$ and $\varepsilon^D_{s,t} = +1$, or $a_{\Sigma_s^D(0)} \leq a'_{\Sigma_t^D(0)}$. We can exclude the first case by Lemma C.3.13, so we have the second case that implies $\sigma \in \mathrm{PreSep}(X, Y')$. Hence $\mathrm{PreSep}(X, Y')$ is also an initial segment of $\mathrm{PreSep}(X, Y)$. □

We can also calculate the denotation system for $\mathrm{PreSep}(D)$:

Proposition C.5.9. *For a predilator D,* $\mathrm{PreSep}(D)$ *has the following denotation system: Every* $\mathrm{PreSep}(D)$*-term has the form t^{PreSep} for some $t \in \mathrm{field}(D)$, with* $\mathrm{arity}_0(t^{\mathrm{PreSep}}) = \mathrm{arity}\, t - \Sigma_t^D(0) - 1$, $\mathrm{arity}_1(t^{\mathrm{PreSep}}) = \Sigma_t^D(0) + 1$. *For a pair of arity diagrams* Д *and* Ж *of the form* (C.11)*, we have $s^{\mathrm{PreSep}} <_{Д,Ж} t^{\mathrm{PreSep}}$ if and only if when we define*

$$Д + Ж = \begin{array}{ccc} n_\cap + m_\cap & \longrightarrow & n_1 + m_1 \\ \downarrow & & \downarrow{\scriptstyle f_1 + e_1} \\ n_0 + m_0 & \xrightarrow[f_0 + e_0]{} & n_\cup + m_\cup \end{array} \tag{C.12}$$

then $D \vDash s <_{Д+Ж} t$.

Proof. The details are left to the reader. □

Definition C.5.10. Let D be a predilator of type Ω. We define $\mathrm{Sep}(D)$ as follows: Consider the internal sum decomposition (cf. Remark C.1.9) $\{D_i \mid i \in \mathrm{lh}(D)\}$ of a predilator D. Since D is of type Ω, $\mathrm{lh}(D)$ has the maximum, say i_0. Let D' be a sub-semidilator of D of field $\bigcup_{i<i_0} D_i$, and let us define

$$\mathrm{Sep}(D)(X,Y) = D'(X) + \mathrm{PreSep}(D_{i_0})(X,Y).$$

Arguments X and Y indicate how to interpret arities of the terms; That is, we think of every D'-term of $\mathrm{Sep}(D)$ as terms with $\mathrm{arity}_1(t) = 0$.

We can see that $\mathrm{Sep}(D)$ is a semibilator and satisfies

$$\mathrm{Sep}(D)(X,Y) \cong \left(\sum_{i<i_0} D_i\right)(X) + \mathrm{PreSep}(D_{i_0})(X,Y).$$

Also, Proposition C.5.9 gives the following denotation system for $\mathrm{Sep}(D)$. Let us say a term $t \in \mathrm{field}(D)$ is a *top term* if t belongs to the largest indexed perfect decomposition of D; Put differently, t is a top term if $s \ll^D t$ for every $s \in \mathrm{field}(D)$.

Proposition C.5.11. *Suppose that D is a predilator of type Ω. Then $\mathrm{Sep}(D)$ has the following denotation system: If D' is the set of top terms of D, then every $\mathrm{Sep}(D)$-term has the form t^{Sep} for some $t \in \mathrm{field}(D)$. We also define*

1. $\mathrm{arity}_0^{\mathrm{Sep}(D)}(t^{\mathrm{Sep}}) = \mathrm{arity}^D(t)$, $\mathrm{arity}_1^{\mathrm{Sep}(D)}(t^{\mathrm{Sep}}) = 0$ *for* $t \in \mathrm{field}(D \setminus D')$,
2. $\mathrm{arity}_i^{\mathrm{Sep}(D)}(t^{\mathrm{Sep}}) = \mathrm{arity}_i^{\mathrm{PreSep}(D')}(t^{\mathrm{PreSep}})$ *for* $t \in \mathrm{field}(D')$.

We compare terms as follows: For a pair Д *and* Ж *of arity diagrams for s and t,*

1. *If* $s,t \in \mathrm{field}(D \setminus D')$, $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{\text{Д,Ж}} t^{\mathrm{Sep}}$ *iff* $D \vDash s <_{\text{Д}} t$. *(In this case, every object in* Ж *is* 0*.)*

2. $s, t \in \mathrm{field}(D')$, $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{\text{Д,Ж}} t^{\mathrm{Sep}}$ *iff* $\mathrm{PreSep}(D') \vDash s^{\mathrm{PreSep}} <_{\text{Д,Ж}} t^{\mathrm{PreSep}}$.

3. *If* $s \in \mathrm{field}(D \setminus D')$ *and* $t \in \mathrm{field}(D')$, *then* $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{\text{Д,Ж}} t^{\mathrm{Sep}}$ *holds.*

Proof. The details are left to the reader. □

Then we can see that Sep and Un are inverses of each other in some sense. To state what the 'some sense' means, let us define

Definition C.5.12. The *category of prebilators* is the induced subcategory of binary predilators whose objects are predilators. The *category of predilators of type* Ω is the induced subcategory of predilators whose objects are predilators of type Ω. We write each category by PreBil and $\Omega\mathsf{PreDil}$ respectively.

We can turn Sep and Un as functors: For prebilators E, F and $\iota \colon E \to F$, we define $\mathrm{Un}(\iota) \colon \mathrm{Un}(E) \to \mathrm{Un}(F)$ by

$$\mathrm{Un}(\iota)(s^{\mathrm{Un}}) = \iota(s)^{\mathrm{Un}}.$$

Then we can see that $\mathrm{Un}(\iota)$ is a predilator morphism. Also, for predilators D, E of type Ω and $\iota \colon D \to E$, we define $\mathrm{Sep}(\iota) \colon \mathrm{Sep}(D) \to \mathrm{Sep}(E)$ by

$$\mathrm{Sep}(\iota)(s^{\mathrm{Sep}}) = \iota(s)^{\mathrm{Sep}}.$$

It is routine to see that $\mathrm{Sep}(\iota)$ is a binary predilator morphism. The following theorem says Un and Sep are not only functors, but also category equivalences:

Theorem C.5.13. $\mathrm{Un} \circ \mathrm{Sep}$ *is naturally isomorphic to* $\mathsf{Id}_{\Omega\mathsf{PreDil}}$. $\mathrm{Sep} \circ \mathrm{Un}$ *is naturally isomorphic to* $\mathsf{Id}_{\mathsf{PreBil}}$.

Proof. For the first claim, it suffices to prove that the map $t \mapsto (t^{\mathrm{Sep}})^{\mathrm{Un}}$ is a predilator isomorphism from D to $\mathrm{Un}(\mathrm{Sep}(D))$ if D is a predilator of type Ω. This map is clearly

bijective and natural, so it suffices to prove that the map preserves the comparison relation. Suppose that we have $D \vDash s <_{Д} t$, where Д is an arity diagram for s and t of the form (C.9). Let us divide the cases:

1. If none of s and t are top terms: Then we have $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{Д,\mathrm{Diag}_0(0,0)} t^{\mathrm{Sep}}$, so $\mathrm{Un}(\mathrm{Sep}(D)) \vDash (s^{\mathrm{Sep}})^{\mathrm{Un}} <_{Д} (t^{\mathrm{Sep}})^{\mathrm{Un}}$.
2. If both s and t are top terms: If we define

   - $Д_1 = \mathrm{Diag}_{n_\cup}(e_0[\Sigma_s^D(0)+1], e_1[\Sigma_t^D(0)+1])$, and
   - $Д_0 = \mathrm{Diag}_{n_\cup}(\mathrm{ran}\, e_0 \setminus e_0[\Sigma_s^D(0)+1], \mathrm{ran}\, e_1 \setminus e_1[\Sigma_t^D(0)+1])$,

   then we have $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{Д_0,Д_1} t^{\mathrm{Sep}}$. Then observe that $Д_0 + Д_1$ and Д are the same arity diagram, so we have $\mathrm{Un}(\mathrm{Sep}(D)) \vDash (s^{\mathrm{Sep}})^{\mathrm{Un}} <_{Д} (t^{\mathrm{Sep}})^{\mathrm{Un}}$.
3. If s is not a top term but t is: Then for *every* Д′ and Ж′, we have $\mathrm{Sep}(D) \vDash s^{\mathrm{Sep}} <_{Д',Ж'} t^{\mathrm{Sep}}$. If we take Д′ and Ж′ by $Д_0$ and $Д_1$ we defined in the previous case, then we have $\mathrm{Un}(\mathrm{Sep}(D)) \vDash (s^{\mathrm{Sep}})^{\mathrm{Un}} <_{Д} (t^{\mathrm{Sep}})^{\mathrm{Un}}$.

For the next claim, it suffices to show that the map $t \mapsto (t^{\mathrm{Un}})^{\mathrm{Sep}}$ preserves the comparison relation. Suppose that we have $E \vDash s <_{Д,Ж} t$ for a prebilator E, $s, t \in \mathrm{field}(E)$ and a pair of arity diagrams Д and Ж of the form (C.11). If we define $Д + Ж$ as we defined in (C.12), we have $\mathrm{Un}(E) \vDash s^{\mathrm{Un}} <_{Д+Ж} t^{\mathrm{Un}}$.

Now let us classify $\mathrm{Un}(E)$-terms. Observe that E is a prebilator, hence

1. If $\mathrm{arity}_1(s) = 0 < \mathrm{arity}_1(t)$, then $E(\mathbb{N}) \vDash s(a;) < t(b,c)$ for every $a, b, c \in [\mathbb{N}]^{<\omega}$. It shows $E \vDash s <_{Д,Ж} t$ for every pair of arity diagrams Д, Ж for s, t.
2. If $\mathrm{arity}_1(s), \mathrm{arity}_1(t) > 0$, then $E(\mathbb{N}) \vDash s(a;b) < t(c;d)$ when $\max b < \max d$. In particular, for every arity diagram Д we can find Ж such that $E \vDash s <_{Д,Ж} t$.

From this, we can deduce the following:

1. If $\text{arity}_1(s) = 0 < \text{arity}_1(t)$, then $s^{\text{Un}} \ll^{\text{Un}(D)} t^{\text{Un}}$.
2. If $\text{arity}_1(s), \text{arity}_1(t)$, then $s^{\text{Un}} \underline{\ll}^{\text{Un}(D)} t^{\text{Un}}$.

Hence the set of top terms of $\text{Un}(D)$ is precisely the set of terms of the form t^{Un} with $\text{arity}_1(t) > 0$.

Going back to the main argument, suppose that $E \models s <_{\text{Д,Ж}} t$ so we have $\text{Un}(E) \models s^{\text{Un}} <_{\text{Д+Ж}} t^{\text{Un}}$.

1. If $\text{arity}_1(s) = \text{arity}_1(t) = 0$, then s^{Un} and t^{Un} are not top terms of $\text{Un}(E)$ and Ж is trivial. Hence $\text{Sep}(\text{Un}(E)) \models (s^{\text{Un}})^{\text{Sep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{Sep}}$.
2. If $\text{arity}_1(s) = 0 < \text{arity}_1(t)$, then s^{Un} is not a top term but t^{Un} is. Hence $\text{Sep}(\text{Un}(E)) \models (s^{\text{Un}})^{\text{Sep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{Sep}}$.
3. The case $\text{arity}_1(s) > 0 = \text{arity}_1(t)$ cannot happen.
4. If $\text{arity}_1(s), \text{arity}_1(t) > 0$, then both s^{Un} and t^{Un} are top terms. Then $\text{Sep}(\text{Un}(E)) \models (s^{\text{Un}})^{\text{Sep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{Sep}}$ iff $\text{PreSep}(\text{Un}(E)') \models (s^{\text{Un}})^{\text{PreSep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{PreSep}}$, where $\text{Un}(E)'$ is a subpredilator of $\text{Un}(E)$-top terms. But $\text{PreSep}(\text{Un}(E)') \models (s^{\text{Un}})^{\text{PreSep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{PreSep}}$ iff $\text{Un}(E)' \models s^{\text{Un}} <_{\text{Д+Ж}} t^{\text{Un}}$ by Proposition C.5.9, which follows from $\text{Un}(E) \models s^{\text{Un}} <_{\text{Д+Ж}} t^{\text{Un}}$ we already have.

Hence in each cases, we have $\text{Sep}(\text{Un}(E)) \models (s^{\text{Un}})^{\text{Sep}} <_{\text{Д,Ж}} (t^{\text{Un}})^{\text{Sep}}$. □

The next theorem shows that the way we defined the unification functor 'coincides' with what we explained at the beginning of this section. (Take $F = \text{Sep}(D)$ for the comparison.)

Theorem C.5.14. *Let F be a prebilator. Then we can find an isomorphism natural in X such that*

$$\phi_{F,X} \colon \mathrm{Un}(F)(X) \cong \hat{F}(X) := F(X,0) + \sum_{x \in X} \partial_1 F(X_{>x}, X \restriction x)$$

given by

$$\phi_{F,X}(t(a;b)) = \begin{cases} \underline{\dfrac{t(a;)}{0}} & \text{if } |b| = 0, \\ \underline{\dfrac{\mathsf{in}_{\max b}(t^{\partial_1}(a; b \setminus \{\max b\}))}{1}} & \text{if } |b| > 0. \end{cases}$$

Furthermore, for $f \colon X \to Y$, if we define

$$\hat{F}(f) \colon F(X,0) + \sum_{x \in X} \partial_1 F(X_{>x}, X \restriction x) \to F(Y,0) + \sum_{y \in Y} \partial_1 F(Y_{>y}, Y \restriction y)$$

by

1. $\hat{F}(f)(\underline{\dfrac{t(a;)}{0}}) = \underline{\dfrac{t(f[a];)}{0}}$, *and*
2. $\hat{F}(f)\Big(\underline{\dfrac{\mathsf{in}_x(t^{\partial_1}(a;b))}{1}}\Big) = \underline{\dfrac{\mathsf{in}_{f(x)}(t^{\partial_1}(f[a];f[b]))}{1}}$,

then $\hat{F}(f) \circ \phi_{F,X} = \phi_{F,Y} \circ \mathrm{Un}(F)(f)$. Moreover, if F and G are prebilators and $\iota \colon F \to G$ is a binary predilator morphism, if we define $\hat{\iota}_X \colon \hat{F}(X) \to \hat{G}(X)$ by

1. $\hat{\iota}(\underline{\dfrac{t(a;)}{0}}) = \underline{\dfrac{\iota(t)(a;)}{0}}$, *and*
2. $\hat{\iota}\Big(\underline{\dfrac{\mathsf{in}_x(t^{\partial_1}(a;b))}{1}}\Big) = \underline{\dfrac{\mathsf{in}_x(\iota(t)^{\partial_1}(a;b))}{1}}$,

then $\hat{\iota}_X \circ \phi_{F,X} = \phi_{G,X} \circ \iota_X$. (In particular, $\phi_{F,X}$ is also natural in F.)

Proof. We first claim that $\phi_{F,X}$ is monotone: Suppose that we have $\mathrm{Un}(F)(X) \vDash s(a;b) < t(a';b')$. The case when $|b| = 0$ or $|b'| = 0$ is easy. If $|b|, |b'| > 0$, then F being a prebilator implies $\max b < \max b'$, or ($\max b = \max b'$ and $s^{\partial_1}(a; b \setminus \{\max b\}) <$

$t^{\partial_1}(a'; b' \setminus \{\max b'\})$). $\phi_{F,X}$ being onto is easy to check. The remaining part of the proof follows from the direct calculation, so we omit the details. □

The following theorem shows Un is close to the diagonal functor $\mathbf{\Delta}$ for a 'sufficiently large' argument:

Theorem C.5.15. *Let F be a binary predilator. Then we have an isomorphism $\phi_F \colon \mathbf{\Delta}(F) \circ \omega^{\underline{1}+\mathsf{Id}} \cong \mathrm{Un}(F) \circ \omega^{\underline{1}+\mathsf{Id}}$. Furthermore, for binary predilators F, G and a binary predilator morphism $\iota \colon F \to G$, we have*

$$(\mathrm{Un}(\iota) \circ \omega^{\underline{1}+\mathsf{Id}}) \circ \phi_F = \phi_G \circ (\mathbf{\Delta}(\iota) \circ \omega^{\underline{1}+\mathsf{Id}}). \tag{C.13}$$

Proof. We claim that there is a support-preserving natural isomorphism

$$\phi_{F,X} \colon F(\omega^{1+X}, \omega^{1+X}) = \mathbf{\Delta}(F)(\omega^{1+X}) \cong \mathrm{Un}(F)(\omega^{1+X}).$$

For notational convenience, let us write $\mathbf{0} < \mathbf{1}$ for the first two nullary terms of $X \mapsto \omega^{1+X}$. For an F-term t, and $x_0, \cdots, x_{m-1}, y_0, \cdots, y_{n-1} \in \omega^{1+X}$, let us define

$$\phi_{F,X}(t(x_0, \cdots, x_{m-1}; y_0, \cdots, y_{n-1})) = \begin{cases} t(x_0, \cdots, x_{m-1};) & n = 0, \\ t(y_{n-1} + \mathbf{1} + x_0, \cdots, y_{n-1} + \mathbf{1} + x_{m-1}; y_0, \cdots, y_{n-1}) & n > 0. \end{cases}$$

Then $\phi_{F,X}$ is increasing: Suppose that we have

$$F(\omega^{1+X}, \omega^{1+X}) \vDash t(x_0, \cdots, x_{m-1}; y_0, \cdots, y_{n-1}) < t'(x'_0, \cdots, x'_{m'-1}; y'_0, \cdots, y'_{n'-1}).$$

The case $n = n' = 0$ is clear. The case $n = 0 < n'$ follows from the fact that the map $x' \mapsto y_{n'-1} + \mathbf{1} + x'$ is increasing and F is a binary predilator. The case $n' = 0 < n$ cannot happen since F is a prebilator, so F is a preflower in the second component. If $n, n' > 0$, then we have $y_{n-1} \leq y'_{n'-1}$. If $y_{n-1} < y'_{n'-1}$, then the desired inequality

follows from that F is a prebilator. If $y_{n-1} = y'_{n'-1}$, then the desired inequality follows from that $x \mapsto y_{n-1} + \mathbf{1} + x$ is increasing. To see $\phi_{F,X}$ is onto, observe that for $x > y$ in ω^{1+X}, there is a unique $z \in \omega^{1+X}$ such that $y = x + \mathbf{1} + z$.

We can easily see that $\phi_{F,X}$ is natural in X, so that if $f \colon X \to Y$ is increasing, then $\mathrm{Un}(F)(\omega^{\underline{1}+f}) \circ \phi_{F,X} = \phi_{F,Y} \circ F(\omega^{\underline{1}+f}, \omega^{\underline{1}+f})$. It remains to show that $\phi_{F,X}$ is support preserving, i.e.,

$$\mathrm{supp}_X^{\mathrm{Un}(F)\circ\omega^{\underline{1}+\mathsf{Id}}}(\phi_{F,X}(t(x_0, \cdots, x_{m-1}; y_0, \cdots, y_{n-1}))) \\ = \mathrm{supp}_X^{\mathbf{\Delta}(F)\circ\omega^{\underline{1}+\mathsf{Id}}}(t(x_0, \cdots, x_{m-1}; y_0, \cdots, y_{n-1})). \quad \text{(C.14)}$$

(See Example B.6.4 for the definition of the support transformation of $\mathbf{\Delta}(D)$.) We can see that

$$\mathrm{supp}_X^{\mathbf{\Delta}(F)\circ\omega^{\underline{1}+\mathsf{Id}}}(t(x_0, \cdots, x_{m-1}; y_0, \cdots, y_{n-1})) \\ = \bigcup_{i<m} \mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(x_i) \cup \bigcup_{j<n} \mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(y_j).$$

We have (C.14) when $n = 0$. If $n > 0$, let us observe that

$$\mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(x_i) \subseteq \mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(y_{n-1} + \mathbf{1} + x_i) \subseteq \mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(y_{n-1}) \cup \mathrm{supp}_X^{\omega^{\underline{1}+\mathsf{Id}}}(x_i),$$

so we have (C.14). To see (C.13), we need to check for $\iota \colon F \to G$ and a linear order X, we have

$$\mathrm{Un}(\iota)_{\omega^{1+X}} \circ \phi_{F,X} = \phi_{G.X} \circ \mathbf{\Delta}(\iota)_{\omega^{1+X}},$$

which follows from a direct calculation. □

C.6 Supplement: Abstract construction of semidilators

In this section, we provide the abstract construction of semidilators. Semidilators have an extra structure we will call the set of *positively compared indices*:

Proposition C.6.1. *Let $t \in \mathrm{field}(D)$ with $\mathrm{arity}\, t = n$. Then we can find $\wp_t^D \subseteq n$ with the following holds: For a linear order X and $a, b \in [X]^n$, if $\mathrm{en}_a(i) \leq \mathrm{en}_b(i)$ for all $i \in \wp_t^D$ and $\mathrm{en}_a(i) \geq \mathrm{en}_b(i)$ for all $i \notin \wp_t^D$, then $D(X) \vDash t(a) \leq t(b)$.*

Proof. Fix $t \in \mathrm{field}(D)$ of arity n. For $i < n$, let us define $c_i^- = \{0, 1, \cdots, i-1, i, i+2, \cdots, n\}$ and $c_i^+ = \{0, 1, \cdots, i-1, i+1, i+2, \cdots, n\}$. Then set $\wp_t^D = \{i < n \mid D(n+1) \vDash t(c_i^-) < t(c_i^+)\}$.

Now let X be a linear order and $a, b \in [X]^n$. We prove the desired statement by induction on the number of i such that $\mathrm{en}_a(i) \neq \mathrm{en}_b(i)$: Assume the induction hypothesis, and suppose that we are given a and b. Let $m < n$ be the largest number such that $\mathrm{en}_a(m) \neq \mathrm{en}_b(m)$. We have two cases:

1. Suppose that $m \in \wp_t^D$ and $\mathrm{en}_a(m) < \mathrm{en}_b(m)$. Let us consider b' of size n satisfying
$$\mathrm{en}_{b'}(i) = \begin{cases} \mathrm{en}_b(i) & i \neq m, \\ \mathrm{en}_a(m) & i = m. \end{cases}$$
Then $D(X) \vDash t(a) < t(b')$ by the induction hypothesis. Now let us check $D(X) \vDash t(b') < t(b)$: Let us take $c = b \cup \{\mathrm{en}_a(m)\}$. If $\mathrm{en}_a(m) < \mathrm{en}_b(m)$, then $\mathrm{en}_a(m) = \mathrm{en}_c(m) < \mathrm{en}_c(m+1) = \mathrm{en}_b(m)$. By the assumption, $m \in \wp_t^D$, so
$$D(X) \vDash t(\mathrm{en}_c(0), \cdots, \mathrm{en}_c(m-1), \mathrm{en}_c(m), \cdots) < t(\mathrm{en}_c(0), \cdots, \mathrm{en}_c(m-1), \mathrm{en}_c(m+1), \cdots)$$
and we have $D(X) \vDash t(b') < t(b)$. Similarly, $\mathrm{en}_a(m) > \mathrm{en}_b(m)$ implies $D(X) \vDash t(b') < t(b)$.
2. Suppose that $m \notin \wp_t^D$ and $\mathrm{en}_a(m) > \mathrm{en}_b(m)$. Then the previous argument works

by considering a' of size n satisfying

$$\mathrm{en}_{a'}(i) = \begin{cases} \mathrm{en}_a(i) & i \neq m, \\ \mathrm{en}_b(m) & i = m. \end{cases}$$

instead; We can prove $D(X) \vDash t(a) < t(a')$, and $D(X) \vDash t(a') < t(b)$ follows from the induction hypothesis. □

Semidilators also have priority permutations:

Theorem C.6.2. *For a semidilator D and $t \in \mathrm{field}(D)$ of arity n, there is a priority permutation Σ_t^D satisfying the following: For $a, b \in [X]^n$ with $\mathrm{en}_a(\Sigma_t^D(i)) < \mathrm{en}_b(\Sigma_t^D(i))$ and $\mathrm{en}_a(\Sigma_t^D(j)) = \mathrm{en}_b(\Sigma_t^D(j))$ for every $j < i$, then*

- *If $\Sigma_t^D(i) \in \wp_t^D$, then $D(X) \vDash t(a) < t(b)$.*
- *If $\Sigma_t^D(i) \notin \wp_t^D$, then $D(X) \vDash t(a) > t(b)$.*

Proof. Its proof is more or less similar to that of Theorem C.2.4, but instead of considering $c_0, \cdots, c_{n-1}$ over $2n$, let us consider the following c_i over

$$\hat{n} = \{0^- < 0 < 0^+ < 1^- < 1 < 1^+ < \cdots < (n-1)^- < (n-1) < (n-1)^+\}$$

as follows:

$$c_i = \begin{cases} \{0, 1, \cdots, (i-1), i^+, \cdots, (n-1)\} & \text{if } i \in \wp_t^D, \\ \{0, 1, \cdots, (i-1), i^-, \cdots, (n-1)\} & \text{if } i \notin \wp_t^D. \end{cases}$$

Then define $\sigma = \Sigma_t^D$ in a way that

$$D(\hat{n}) \vDash t(c_{\sigma(0)}) > t(c_{\sigma(1)}) > \cdots > t(c_{\sigma(n-1)}).$$

Now suppose that $a, b \in [X]^n$, $a(\sigma(j)) = b(\sigma(j))$ for every $j < i$ and $a(\sigma(i)) < b(\sigma(i))$. We prove $D(X) \vDash t(a) < t(b)$ by induction on the number of k such that

$$\mathrm{en}_a(k) \neq \mathrm{en}_b(k) \wedge \neg[\mathrm{en}_a(k) < \mathrm{en}_b(k) \leftrightarrow k \in \wp_t^D]. \tag{C.15}$$

The case when the number of such k is 0 is clear by the definition of $\wp_t^D$. Otherwise, pick the least k satisfying (C.15), and let $k = \sigma(l)$. Note that $l > i$, so $D(\hat{n}) \vDash t(c_{\sigma(l)}) < t(c_{\sigma(i)})$.

For a technical convenience, let us replace X with $\hat{X} = \{x^- < x < x^+ \mid x \in X\}$ with $x^+ < y^-$ when $x, y \in X$, $x < y$. Then we have the following cases:

1. $\sigma(i) \in \wp_t^D$: If $\sigma(l) \in \wp_t^D$, let us consider $\overline{a}$ satisfying

$$\mathrm{en}_{\overline{a}}(\sigma(j)) = \begin{cases} \mathrm{en}_a(\sigma(i))^+ & j = i, \\ \mathrm{en}_b(\sigma(l)) & j = l, \\ \mathrm{en}_a(\sigma(j)) & \text{otherwise.} \end{cases}$$

Then the proof of Theorem C.2.4 works as follows: We can see that $D(X) \vDash t(\overline{a}) < t(b)$ by induction hypothesis. For $D(X) \vDash t(a) < t(\overline{a})$, consider the map $f \colon c_{\sigma(l)} \cup c_{\sigma(i)} \to a \cup \overline{a}$ given by

$$f(x) = \begin{cases} \mathrm{en}_a(\sigma(i)) & x = \sigma(i), \\ \mathrm{en}_a(\sigma(i))^+ & x = \sigma(i)^+, \\ \mathrm{en}_b(\sigma(l)) & x = \sigma(l), \\ \mathrm{en}_a(\sigma(l)) & x = \sigma(l)^+, \\ \mathrm{en}_a(x) & \text{otherwise.} \end{cases}$$

We can see that f is order-preserving, and the most tricky case is when $x = \sigma(l)$; In this case, the minimality of $k = \sigma(l)$ implies for $k' < k$, $f(k') = a(k') \leq b(k') < b(k)$. Furthermore, $f[c_{\sigma(l)}] = a$, $f[c_{\sigma(i)}] = \overline{a}$. Thus $D(c_{\sigma(i)} \cup c_{\sigma(l)}) \vDash t(c_{\sigma(l)}) < t(c_{\sigma(i)})$ implies $D(X) \vDash t(a) < t(\overline{a})$.

If $\sigma(l) \notin \wp_t^D$, let us consider $\overline{b}$ satisfying

$$\mathrm{en}_{\overline{b}}(\sigma(j)) = \begin{cases} \mathrm{en}_b(\sigma(i))^- & j = i, \\ \mathrm{en}_a(\sigma(l)) & j = l, \\ \mathrm{en}_b(\sigma(j)) & \text{otherwise.} \end{cases}$$

In this case, $\sigma(l)^- \in c_l$ and $a(\sigma(l)) < b(\sigma(l))$. The induction hypothesis shows $D(X) \vDash t(a) < t(\overline{b})$. We can also show $D(X) \vDash t(\overline{b}) < t(b)$ with the help of the following f:

$$f(x) = \begin{cases} \mathrm{en}_b(\sigma(i))^- & x = \sigma(i)^-, \\ \mathrm{en}_b(\sigma(i)) & x = \sigma(i), \\ \mathrm{en}_a(\sigma(l)) & x = \sigma(l)^-, \\ \mathrm{en}_b(\sigma(l)) & x = \sigma(l), \\ \mathrm{en}_b(x) & \text{otherwise.} \end{cases}$$

2. $\sigma(i) \notin \wp_t^D$: If $\sigma(l) \in \wp_t^D$, we can prove $D(X) \vDash t(a) > t(\overline{b}) > t(b)$. If If $\sigma(l) \notin \wp_t^D$, we can prove $D(X) \vDash t(a) > t(\overline{a}) > t(b)$. The detail is left to the reader. □

Now, let us find the remaining structures of semidilators.

Definition C.6.3. Let D be a semidilator and $s, t \in \mathrm{field}(D)$. Let $\hat{p}_{s,t} = \hat{p}_{s,t}^D \leq \min(\mathrm{arity}\, s, \mathrm{arity}\, t)$ be the largest natural number such that

1. $\forall i < \hat{p}_{s,t}[i \in \wp_t^D \iff i \in \wp_t^D]$.
2. $\forall i, j < \hat{p}_{s,t}[\Sigma_s^D(i) < \Sigma_s^D(j) \iff \Sigma_t^D(i) < \Sigma_t^D(j)]$.

Then define the secureness of $p < \hat{p}_{s,t}$ as we did in Definition C.3.1.

Lemma C.6.4. *$\hat{p}_{s,t}$ is secure.*

Proof. Suppose that arity $s = n_0$, arity $t = n_1$. The case $\hat{p}_{s,t} = n_0$ or $\hat{p}_{s,t} = n_1$ follows from the argument in the proof of Lemma C.3.2. Thus let us consider the case $\hat{p}_{s,t} < \min(n_0, n_1)$. Then we have one of the following:

1. $\hat{p}_{s,t} \in \wp_s^D \nleftrightarrow \hat{p}_{s,t} \in \wp_t^D$, or
2. There is $j < \hat{p}_{s,t}$ such that

$$[\Sigma_s^D(j) < \Sigma_s^D(\hat{p}_{s,t}) \text{ and } \Sigma_t^D(j) > \Sigma_t^D(\hat{p}_{s,t})] \\ \text{or} \quad [\Sigma_s^D(j) > \Sigma_s^D(\hat{p}_{s,t}) \text{ and } \Sigma_t^D(j) < \Sigma_t^D(\hat{p}_{s,t})].$$

Suppose that the first case holds. Without loss of generality, assume that $\hat{p}_{s,t} \in \wp_s^D$ and $\hat{p}_{s,t} \notin \wp_t^D$. Now let

$$Д = \begin{array}{ccc} n_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ n_0 & \xrightarrow[e_0]{} & n_\cup, \end{array} \qquad Д' = \begin{array}{ccc} n'_\cap & \longrightarrow & n_1 \\ \downarrow & & \downarrow{\scriptstyle e'_1} \\ n_0 & \xrightarrow[e'_0]{} & n'_\cup \end{array}$$

be two $\hat{p}_{s,t}$-congruent arity diagrams. Then we can find a finite linear order Y and embeddings $h : n_\cup \to Y$, $h' : n'_\cup \to Y$ such that

$$\begin{array}{ccc} \hat{p}_{s,t} & \xrightarrow{e'_0 \circ \Sigma_s^D} & n'_\cup \\ {\scriptstyle e_0 \circ \Sigma_s^D}\downarrow & & \downarrow{\scriptstyle h'} \\ n_\cup & \xrightarrow[h]{} & Y \end{array}$$

commutes and

$$Y \vDash h'(e'_0 \circ \Sigma_s^D(\hat{p})) < h(e_0 \circ \Sigma_s^D(\hat{p})) \wedge h'(e'_1 \circ \Sigma_t^D(\hat{p})) < h(e_1 \circ \Sigma_t^D(\hat{p})).$$

With the help of h, h', we can prove $D(n_\cup) \vDash s(\operatorname{ran} e_0) < t(\operatorname{ran} e_1)$ implies $D(n'_\cup) \vDash s(\operatorname{ran} e'_0) < t(\operatorname{ran} e'_1)$.

Now suppose that $\hat{p}_{s,t} \in \wp_s^D \leftrightarrow \hat{p}_{s,t} \in \wp_t^D$ but the second case holds. If $\hat{p}_{s,t}, \in \wp_s^D \cap \wp_t^D$, the same proof of Lemma C.3.2 works. Otherwise, a similar proof works, but we use a variant of Lemma C.3.3 with (C.7) replaced with

$$Y \vDash h'(e'_1 \circ \Sigma_t^D(\hat{p}_{s,t})) < h(e_1 \circ \Sigma_t^D(\hat{p}_{s,t})) < h(e_0 \circ \Sigma_s^D(\hat{p}_{s,t})) < h'(e'_0 \circ \Sigma_s^D(\hat{p}_{s,t})). \ \square$$

Hence, we can define $\mathbf{p}^D(s,t)$ by the least secure number relative to s and t, and also $\varepsilon_{s,t}^D$ in a usual manner. Then we have the following:

Theorem C.6.5.

1. *For every* $i < \mathbf{p}^D(s,t)$, $i \in \wp_s^D \iff i \in \wp_t^D$.
2. *For every* $i, j < \mathbf{p}^D(s,t)$, $\Sigma_s^D(i) < \Sigma_s^D(j)$ *iff* $\Sigma_t^D(i) < \Sigma_t^D(j)$.
3. *For* $a, b \in [X]^{<\omega}$, *if* $\mathrm{en}_a(\Sigma_s^D(i)) = \mathrm{en}_b(\Sigma_t^D(i))$ *for every* $i < \mathbf{p}^D(s,t)$, *then* $D(X) \vDash s(a) < t(b)$ *iff* $\varepsilon_{s,t}^D = +1$.
4. *If there is* $i < \mathbf{p}^D(s,t)$ *such that* $\mathrm{en}_a(\Sigma_s^D(i)) < \mathrm{en}_b(\Sigma_t^D(i))$ *and* $\mathrm{en}_a(\Sigma_s^D(j)) = \mathrm{en}_b(\Sigma_t^D(j))$ *for every* $j < i$, *then*

$$\Sigma_s^D(i) \in \wp_s^D \iff D(X) \vDash s(a) < t(b).$$

Proof. The first three are clear by definition. For the last statement, we can apply the same proof of Theorem C.3.5. $\square$

Lastly, the proofs of Lemma C.3.7 and that of Lemma C.3.7 also work for semidilators with slight modifications. Hence we have the following:

Theorem C.6.6. *Every semidilator is characterized by the following data:*

1. *The domain set* X *with a linear order* $<^X$.

2. *The arity function* $\text{arity} : X \to \mathbb{N}$.

3. *A function* $\mathbf{p} : X \times X \to \mathbb{N}$ *such that* $\mathbf{p}(x, x) = \text{arity}(x)$ *and*

   - $\mathbf{p}(x, y) = \mathbf{p}(y, x)$ *for* $x, y \in X$.
   - $\mathbf{p}(x, z) = \min(\mathbf{p}(x, y), \mathbf{p}(y, z))$ *for* $x, y, z \in X$ *such that* $x \leq^X y \leq^X z$.

4. *For each* $x \in X$, *a permutation* Σ_x *over* $\mathbf{p}(x, x)$ *such that*

$$\forall i, j < \mathbf{p}(x, y)[\Sigma_x(i) < \Sigma_x(j) \iff \Sigma_y(i) < \Sigma_y(j)]. \qquad \square$$

5. *For each* $x \in X$, *the set* $\wp_x \subseteq \text{arity}(x)$ *satisfying the following: For two* $x, y \in X$, $\wp_x \cap \mathbf{p}(x, y) = \wp_y \cap \mathbf{p}(x, y)$.

APPENDIX D

DENDROIDS AND DENDROGRAMS

> Ask veit ek standa, heitir Yggdrasill hár baðmr, ausinn hvíta auri; þaðan koma döggvar þærs í dala falla; stendr æ yfir grœnn Urðar brunni.[1]
>
> Völuspá: The Vala's Prophecy 19

Dendroids were introduced to visualize dilators as tree structures. Girard only defined dendroids, but in this chapter, we also define a notion of predendroid corresponding to predilators. Roughly, a (pre)dendroid is a functorial tree representation of a (pre)dilator á la Giarard or Freund. However, a (pre)dendroid is a combination of a (pre)dilator and a parameter linear order, and we often need to separate them, like in Chapter 4, on which the iteration of the measure family needs to happen over a single tree structure for a finite dilator. The notion of *(Pre)dendrogram* provides the tree structure for a (pre)dilator without relying on parameter linear orders, which are discovered independently by Aguilera -Freund-Pakhomov [1] under the name *cell decomposition*, and me. Predendrogram provides a convenient way to construct (pre)dilators by hand, which has a pivotal role in developing the theory of 2-ptykes. Many contents of this chapter appeared in [40, 41] and an unpublished work joint with Aguilera about a measurable dilator [7] in verbatim, and the main idea of Section D.2 is due to Freund [23]. I found that the theory of dendrograms coheres better with the theory of ptykes, so they should be defined first. Hence, we do not follow the historical order in which dendroids were discovered first.

[1] *I know an ash standing Yggdrasil hight, a lofty tree, laved with limpid water: thence come the dews into the dales that fall; ever stands it green over Urd's fountain.* (Translated by Benjamin Thorpe.)

D.1 Predendrograms

Definition D.1.1. A *predendrogram* is a structure $(C, \lessdot, \multimap, \pi)$ satisfying the following:

1. $(C, \multimap)$ is a forest with the immediate successor relation $\multimap$. Moreover, if $\multimap^*$ is the transitive closure of $\multimap$, then for each $x \in C$, $\{y \in C \mid y \multimap^* x\}$ is finite and well-ordered by $\multimap^*$. We call the size of $\{y \in C \mid y \multimap^* x\}$ the *length of* x and denote it by $\mathrm{lh}(x)$.
2. For every $x \in C$, either x is a terminal point (i.e., x has no immediate successor) or there is y such that $x \multimap^* y$ and y is a terminal point.
3. The *local comparison order* $\lessdot$ is a partial order over C. Moreover, the following sets are linearly ordered by $\lessdot$:
   - The set of *roots* of C, i.e. elements with no immediate predecessor.
   - The set immediate successors of x for each $x \in C$.
4. The *priority order* π is a function of domain C. For each $x \in C$, $\pi(x) = (\mathrm{lh}(x), <_{\pi(x)})$ is a linear order over $\mathrm{lh}(x)$. Moreover,
   - $x \multimap y$ implies $\pi(x) \subseteq \pi(y)$.
   - If x and y have the same immediate predecessor, then $\pi(x) = \pi(y)$.

We denote the set of terminal points of C by $\mathrm{term}(C)$. We also define $\mathrm{pred}(x)$ as the $\multimap$-increasing enumeration of $\{y \in C \mid y \multimap^* x\} \cup \{x\}$. $\mathrm{pred}(x)$ is a sequence of length $\mathrm{lh}(x) + 1$. If $\langle x_0, \cdots, x_{m-1}, x_m \rangle$ is the sequence of predecessors of x and $l \leq m$, we write $x \upharpoonright l$ for x_l.

We can visualize predendrograms as an ordered directed forest. For every x, y in the predendrogram such that $x \multimap y$, we draw an edge from x to y. We label each edge for

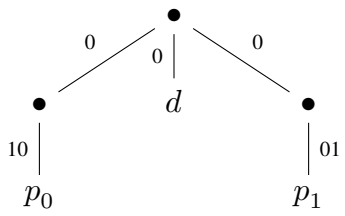


Figure D.1: An example of a dendrogram C

$x \multimap y$ the enumeration of $\mathrm{lh}\, y$ under the priority order $\pi(y)$. When $x \lessdot y$, we put y at the right-hand side of x.

Example D.1.2. The following tree visualises an example of a predendrogram with three terminal nodes p_0, d, p_1 such that

1. There is one tree in the forest (so $p_0 \upharpoonright 0 = d \upharpoonright 0 = p_1 \upharpoonright 0$.)
2. p_0 and p_1 have length 2, d has length 0.
3. $p_0 \upharpoonright 1 \lessdot d \upharpoonright 1 \lessdot p_1 \upharpoonright 1$.
4. $\pi(p_0) = \{1 < 0\}$, $\pi(p_1) = \{0 < 1\}$.

We will see later that this predendrogram represents a dilator $D(X) = X^2$.

A predendrogram encodes the abstract construction of a predilator in a forest form. To motivate the definition of a predendrogram, let us introduce a way to turn a predendrogram into a predilator:

Definition D.1.3. Let C be a predendrogram and X be a linear order. We define $\mathrm{Fun}(C)(X)$ to be the set of tuples of the form

$$\langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m \rangle$$

such that $x_m \in \mathrm{term}(C)$, $\langle x_0, x_1, \cdots, x_m \rangle = \mathrm{pred}(x_m)$ and $\xi_0, \cdots, \xi_{m-1} \in X$ be such that

$$\forall i, j < m \big[i <_{\pi(x_m)} j \iff X \vDash \xi_i < \xi_j \big] \tag{D.1}$$

We call x_i the *term parts* of the tuple, and ξ_i the *parameter parts* of the tuple. We equip $\mathrm{Fun}(C)(X)$ with the Kleene-Brouwer order. For an increasing map $f \colon X \to Y$, we define $\mathrm{Fun}(C)(f) \colon \mathrm{Fun}(C)(X) \to \mathrm{Fun}(C)(Y)$ by

$$\mathrm{Fun}(C)(f)(\langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m \rangle) = \langle x_0, f(\xi_0), \cdots, x_{m-1}, f(\xi_{m-1}), x_m \rangle.$$

We also define $\mathrm{supp}_X(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle) = \{\xi_0, \cdots, \xi_{m-1}\}$. We denote $\mathrm{Fun}(C)$ by the *functorization of* C. We also say $\xi_0, \cdots, \xi_{m-1}$ *are* $\pi(x_m)$*-coherent* if (D.1) holds.

Proposition D.1.4. *For a predendrogram* C, $\mathrm{Fun}(C)$ *is an F-predilator.*

Proof. It is easy to see that $\mathrm{Fun}(C)$ is an F-semidilator. To see $\mathrm{Fun}(C)$ is an F-predilator, we can apply Proposition B.3.2. □

Like predilators, predendrograms admit subobjects and morphisms.

Proposition D.1.5. *Let* C *be a predendrogram and* $C' \subseteq C$ *be such that* $\mathrm{term}(C') \subseteq \mathrm{term}(C)$. *Then the inherited substructure* C' *is also a predendrogram. We call* C' *a* sub-predendrogram of C*, and denote it by* $C' \Subset C$. □

We visualize $\mathrm{Fun}(C)(A)$ for a predendrogram C and a linear order A as follows. We do it recursively by the length of the nodes.

1. In the zeroth stage, we copy the edges in the zeroth level and their succeeding components A times, and label them by elements of A. For example, if $A = 3 = \{0, 1, 2\}$, then we copy the zeroth-level edges and their succeeding components

three times, and label them by 0, 1, and 2. We ordered the new copies of labels under the size of edge labels.

2. In the first stage, we do not touch the terminal points. For non-terminal points, we will copy their succeeding edges again, and label them by a_0a_1 for $a_0, a_1 \in A$ in a way that the preceding edge has label a_0, and the order of a_0a_1 respects the priority order of the succeeding point of the edge. If there is no such a_1, then we delete the whole edge and its succeeding component. We order the new copies under the size of a_1. For example, we start from a node $x \multimap y$ with $A = 3$, the previous edge is labeled by 2 and $\pi(y) = \{1 < 0\}$. Then the possible labels the current edge can have are 20 and 21, so we copy the edge and its succeeding component twice, and label the edges by 20 and 21, respectively. The edge with label 21 comes to the right of the edge with label 20. If everything is the same but the previous edge is labeled by 0, then there is no label the succeeding edge takes. Hence, we eliminate the edge.

3. We repeat a similar progress: In Stage n, suppose that we have a non-terminal point x of length n, and $x \multimap y$. We assume the preceding edge of x is labeled by $a_0a_1 \cdots a_{n-1}$. We copy the edge $x \multimap y$ again, and label them by $a_0a_1 \cdots a_{n-1}a_n$. a_n is chosen in a way that $i \mapsto a_i$ is an order-preserving map from $(\mathrm{lh}\, y, \pi(y))$ to A. We again remove the edge $x \multimap y$ and its succeeding component if we cannot choose such a_n. We order the duplicated labels and their succeeding components under the size of a_n.

4. Remove all the non-terminal nodes without terminal nodes after finishing the duplication.

Example D.1.6. Let C be the predendrogram represented in Example D.1.2. We can visualize $\mathrm{Fun}(C)(3)$ as follows:

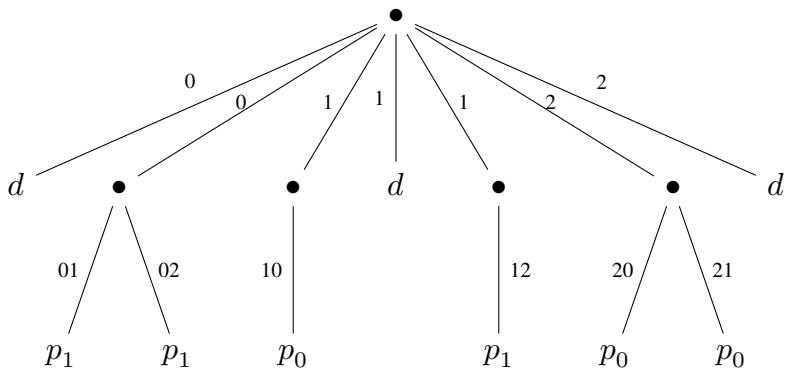


Figure D.2: A visualization of $\mathrm{Fun}(C)(3)$

Definition D.1.7. Let C and C' be two predendrograms. We say $h\colon C \to C'$ is a *predendrogram morphism* if it repects to $\lessdot$, $\multimap$, π, and the terminal points. We write $h\colon C \to C'$ if h is a predendrogram morphism.

The following proposition establishes the basic properties of a predendrogram morphism. Its proof is easy and tedious, so we leave its proof to the reader:

Proposition D.1.8. *Let $h\colon C \to C'$ be a predendrogram morphism.*

1. *If $x, y \in C$ satisfies $x \multimap^* y$, then $h(x) = h(y) \restriction \mathrm{lh}\, x$. In particular, h is determined by $h \restriction \mathrm{term}(C)$.*
2. *h is one-to-one.*
3. *If $h\colon C \to (C')$ is onto, then h^{-1} is a predendrogram morphism from C' to C.* □

We can turn Fun a functor from the category of predendrograms to the category of F-predilators:

Proposition D.1.9. *Let $h\colon C \to C'$ be a predendrogram morphism and X a linear order. Define $\mathrm{Fun}(h)_X\colon \mathrm{Fun}(C)(X) \to \mathrm{Fun}(C')(X)$ as follows: For*

- $x \in \text{term}(C)$, $y = h(x) \in \text{term}(C')$,
- $\text{pred}(x) = \langle x_0, \cdots, x_m \rangle$ *and* $\text{pred}(y) = \langle y_0, \cdots, y_m \rangle$. *(In particular,* $h(x_i) = y_i$ *for every* $i \leq m$*.)*
- $\xi_0, \cdots, \xi_{m-1} \in X$ *are* $\varpi(x)$*-coherent,*

let us define

$$\text{Fun}(h)_X(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle) = \langle y_0, \xi_0, \cdots, \xi_{m-1}, y_m \rangle.$$

Then $\text{Fun}(h) \colon \text{Fun}(C) \to \text{Fun}(C')$ *is a natural transformation.*

Proof. We first check that $\text{Fun}(h)_X$ is order-preserving. Suppose that we have

$$\text{Fun}(C)(X) \vDash \langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle <_{\mathsf{KB}} \langle x'_0, \xi'_0, \cdots, \xi'_{m'-1}, x'_{m'} \rangle.$$

We have the following two possible cases:

1. $x_i = x'_i$, $\xi_i = \xi'_i$ for every $i < k$ and $x_k < x'_k$.
2. $x_i = x'_i$ for every $i \leq k$, $\xi_i = \xi'_i$ for every $i < k$, and $\xi_k < \xi'_k$.

Suppose that $y = h(x_m)$, $y' = h(x'_m)$. $\text{pred}(y) = \langle y_0, \cdots, y_m \rangle$, and $\text{pred}(y') = \langle y'_0, \cdots, y'_{m'} \rangle$. In both cases we previously considered, we have $y_i = y'_i$ and $\xi_i = \xi'_i$ for $i < k$. In the first case, we have $y_k < y'_k$, so

$$\text{Fun}(C)(X) \vDash \langle y_0, \xi_0, \cdots, \xi_{m-1}, y_m \rangle <_{\mathsf{KB}} \langle y'_0, \xi'_0, \cdots, \xi'_{m'-1}, y'_{m'} \rangle.$$

In the second case, we additionally have $y_k = y'_k$ and $\xi_k < \xi'_k$, so we have the desired inequality. Now we need to check for an increasing map $f \colon X \to Y$,

$$\text{Fun}(h)_Y \circ \text{Fun}(C)(f) = \text{Fun}(C')(f) \circ \text{Fun}(h)_X,$$

which follows from the direct calculation. □

Conversely, we can extract a predendrogram from a predilator, called the *cell decomposition*. We first define the field of the cell decomposition:

Definition D.1.10. For a predilator D and $n \in \mathbb{N}$, define $\equiv_n^D$ over $\mathrm{field}(D)$ by

$$s \equiv_n^D t \iff \mathbf{p}^D(s,t) > n \vee s = t.$$

For technical convenience, we let $s \equiv_{-1}^D t$ for every $s, t \in \mathrm{field}(D)$.

Then $\equiv_n^D$ respects $<^D$ in the following sense, where $<^D$ is a linear order over D given by the abstract construction of predilators:

Lemma D.1.11. *$\equiv_n^D$-equivalence classes form an interval on $(D, <^D)$. As a consequence, $(D, <^D)/\equiv_n^D$ forms a linear order.*

Proof. For $s, t, u \in \mathrm{field}(D)$, suppose that $s <^D t <^D u$ and $\mathbf{p}^D(s,u) > n$. Then $\min(\mathbf{p}^D(s,t), \mathbf{p}^D(t,u)) = \mathbf{p}^D(s,u) > n$, so s, t, u are all $\equiv_n^F$-equivalent. □

Proposition D.1.12. *For a predilator D, let us define its cell decomposition* $\mathrm{Cell}(D)$ *as follows:*

1. $\mathrm{Cell}(D) = \{[t]_{\equiv_n^D} \mid t \in \mathrm{field}\, D \wedge n \leq \mathrm{arity}\, t\}$.
2. $\mathrm{Cell}(D) \vDash [s]_{\equiv_m^D} \multimap [t]_{\equiv_n^D}$ *iff* $n = m + 1$ *and* $s \equiv_m^D t$.
3. $\mathrm{Cell}(D) \vDash [s]_{\equiv_m^D} \lessdot [t]_{\equiv_n^D}$ *iff* $m = n$, $s \equiv_{m-1}^D t$, *and* $s <^D t$.
4. $\pi^{\mathrm{Cell}(D)}([s]_{\equiv_m^D}) = \pi^D(s) \cap (m \times m)$.

Then $\mathrm{Cell}(D)$ *is a predendrogram.*

Proof. We first claim that $\lessdot$ and π are well-defined. Suppose that $s \equiv_m^D s'$, $t \equiv_m^D t'$, and s, s', t, t' are all $\equiv_{m-1}^D$-equivalent. If $s <^D t$, then Lemma D.1.11 implies that

every element of $[s]_{\equiv_m^D} = [s']_{\equiv_m^D}$ as an interval over $(D, <^D)$ is less than that of every element of $[t]_{\equiv_m^D} = [t']_{\equiv_m^D}$. Hence $s' <^D t'$. This shows $\lessdot$ is well-defined. For the well-definedness of $\pi([s]_{\equiv_m^D})$, we argue that $s \equiv_m^D s'$ implies $\pi^D(s) \cap (m \times m) = \pi^D(s') \cap (m \times m)$: The assumption $s \equiv_m^D s'$ implies either $m < \mathbf{p}^D(s, s')$ or $s = s'$. The latter case is trivial, so let us focus on the former. For $i, j < m < \mathbf{p}^D(s, s')$, we have that $\Sigma_s^D(i) < \Sigma_s^D(j)$ iff $\Sigma_{s'}^D(i) < \Sigma_{s'}^D(j)$. Then we get the desired equality by the definition of $\pi^D(s)$.

The first three conditions of Definition D.1.1 are clear. For the last condition, observe that the well-definedness proof of $\pi([s]_{\equiv_m^D})$ with $m + 1 \leq \mathbf{p}^D(s, t)$ also shows that

$$s \equiv_m^D t \implies \pi^D(s) \cap (m+1) \times (m+1) = \pi^D(t) \cap (m+1) \times (m+1).$$

In particular, if $[s]_{\equiv_{m+1}^D}$ and $[t]_{\equiv_{m+1}^D}$ have the same immediate predecessor (so $s \equiv_m^D t$), then $\pi([s]_{\equiv_{m+1}^D}) = \pi([t]_{\equiv_{m+1}^D})$. □

Now, let us define the cell decomposition of a predilator morphism.

Proposition D.1.13. *For a predilator morphism $\iota \colon D \to E$, we define* $\mathrm{Cell}(\iota)$ *by*

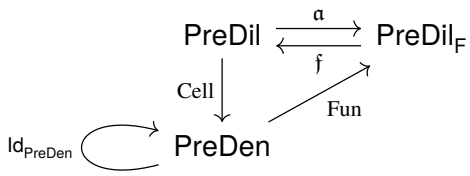
$$\mathrm{Cell}(\iota)([s]_{\equiv_m^D}) = [\iota(s)]_{\equiv_m^E}.$$

Then $\mathrm{Cell}(\iota)$ *is a well-defined predendrogram morphism.*

Proof. ι preserves the term distance $\mathbf{p}^D$, so it also preserves $\equiv_m^D$. It shows $\mathrm{Cell}(\iota)$ is well-defined. The remainder part follows from that ι preserves π^D and $\equiv_m^D$. □

The next goal is to show that Fun and Cell are category equivalences between the category of predilators and that of predendrograms. However, we should be careful that Fun and Cell cannot be inverse of each other since the codomain of Fun and the domain

of Cell differ, each is $\mathsf{PreDil}_{\mathsf{F}}$ and PreDil respectively. Despite that, we can still say they form an equivalence in the sense that every functor from one category to the other in the following diagram is naturally isomorphic:



In particular, we claim that $\mathfrak{a} \cong \mathrm{Fun} \circ \mathrm{Cell}$ and $\mathsf{Id}_{\mathsf{PreDen}} \cong \mathrm{Cell} \circ \mathfrak{f} \circ \mathrm{Fun}$.

Proposition D.1.14. *For a predilator D and a linear order X, let us define* $\mathsf{edil}_{D,X} \colon \mathfrak{a}(D)(X) \to \mathrm{Fun}(\mathrm{Cell}(D))(X)$ *by*

$$\mathsf{edil}_{D,X}(t(a)) = \langle [t]_{\equiv_0^D}, \xi_0, [t]_{\equiv_1^D}, \xi_1, \cdots, [t]_{\equiv^D_{\mathrm{arity}\, t}} \rangle,$$

where $\langle \xi_i \mid i < \mathrm{arity}\, t \rangle$ *is the enumeration of* a *that is* $\pi\!\!\pi^D(t)$-coherent *in the sense that for every* $i, j <$ arity t,

$$X \vDash \xi_i < \xi_j \iff i <_{\pi\!\!\pi^D(t)} j.$$

Then edil_D *is an F-predilator isomorphism natural in* D.

Proof. We first claim that $\mathsf{edil}_{D,X}$ is increasing. Suppose that $s, t \in D$, $a \in [X]^{\mathrm{arity}\, s}$, $b \in [X]^{\mathrm{arity}\, t}$, $\langle \xi_i \mid i < \mathrm{arity}\, s \rangle$ the $\pi\!\!\pi^D(s)$-coherent enumeration of a, and $\langle \eta_i \mid i < \mathrm{arity}\, t \rangle$ the $\pi\!\!\pi^D(t)$-coherent enumeration of b. If $D(X) \vDash s(a) < t(b)$, then Theorem C.3.5 implies we have one of the following:

1. $\xi_i = \eta_i$ for every $i < \mathbf{p}^D(s,t)$ and $s <^D t$, or
2. There is $j < \mathbf{p}^D(s,t)$ such that $\xi_j < \eta_j$ and $\xi_i = \eta_i$ for every $i < j$.

Note that $[s]_{\equiv_i^D} = [t]_{\equiv_i^D}$ for every $i < \mathbf{p}^D(s,t)$, so both cases imply

$$\langle [s]_{\equiv_0^D}, \xi_0, [s]_{\equiv_1^D}, \xi_1, \cdots, [s]_{\equiv^D_{\mathrm{arity}\, s}} \rangle <_{\mathsf{KB}} \langle [t]_{\equiv_0^D}, \eta_0, [t]_{\equiv_1^D}, \eta_1, \cdots, [t]_{\equiv^D_{\mathrm{arity}\, t}} \rangle.$$

We can see that $\mathsf{edil}_{D,X}$ is natural both in X and D by simple calculation. To see that edil_D is an F-predilator morphism, we need to check that it commutes with the support transformation, which is also direct to check. To finish the proof, it remains to show that $\mathsf{edil}_{D,X}$ is onto, which guarantees $\mathsf{edil}_{D,X}$ is a linear order isomorphism. However, the definition of Fun and Cell forces that every member of $\mathrm{Fun}(\mathrm{Cell}(D))(X)$ takes the form $\langle [t]_{\equiv_0^D}, \xi_0, [t]_{\equiv_1^D}, \xi_1, \cdots, [t]_{\equiv_{\mathrm{arity}\, t}^D} \rangle$ for some $t \in \mathrm{field}(D)$ and a $\pi^D(t)$-coherent sequence $\xi_0, \cdots, \xi_{\mathrm{arity}\, t-1} \in X$, which finishes the proof. □

Now, let us construct the inverse morphism. We first examine the behavior of the denotation system corresponding to $\mathrm{Fun}(C)$:

Lemma D.1.15. *Let C be a predendrogram. Then $\mathfrak{f} \circ \mathrm{Fun}(C)$ is a denotation system satisfying the following: First, for each $x \in \mathrm{term}(C)$, we define*

$$\mathbf{n}(x) = (\mathrm{lh}\, x, \langle x_0, e_0, \cdots, e_{\mathrm{lh}\, x-1}, x_{\mathrm{lh}\, x} \rangle) \in \mathrm{Tr}(\mathrm{Fun}(C)),$$

($\mathbf{n}$ comes from the normal form*) where $\langle x_i \mid i \leq \mathrm{lh}\, x \rangle$ is the sequence of predecessors of x, and $\langle e_i \mid i < \mathrm{lh}\, x \rangle$ is the enumeration of $\mathrm{lh}\, x$ that is $\pi^C(x)$-coherent (or equivalently, in $\pi^C(x)$-increasing way.) Then*

1. $\mathrm{field}\,(\mathfrak{f} \circ \mathrm{Fun}(C))$ *is the set of* $\mathbf{n}(x)$ *for* $x \in \mathrm{term}(C)$.
2. $\mathrm{arity}^{\mathfrak{f} \circ \mathrm{Fun}(C)}(\mathbf{n}(x)) = \mathrm{lh}\, x$.
3. $\pi^{\mathfrak{f} \circ \mathrm{Fun}(C)}(\mathbf{n}(x)) = \pi^C(x)$.
4. $\mathbf{p}^{\mathfrak{f} \circ \mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y))$ *is the least* $m \leq \min(\mathrm{lh}\, x, \mathrm{lh}\, y)$ *such that* $x \restriction m \neq y \restriction m$.
5. $\mathbf{n}(x) <^{\mathfrak{f} \circ \mathrm{Fun}(C)} \mathbf{n}(y)$ *iff for* $m = \mathbf{p}^{\mathfrak{f} \circ \mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y))$, $C \vDash x \restriction m \lessdot y \restriction m$.

Proof. The first two are obvious. For the remainder, let us observe the following: First, if $\langle \xi_i \mid i < \mathrm{lh}\, x \rangle$ is a $\pi^C(x)$-coherent sequence of elements of a linear order X, then

$e_i \mapsto x_i$ is an order isomorphism from $\mathrm{lh}\, x$ to $\{\xi_0, \cdots, \xi_{\mathrm{lh}\, x-1}\}$. Also, if $x, y \in \mathrm{term}(C)$, $\mathrm{pred}(x) = \langle x_0, \cdots, x_{\mathrm{lh}\, x}\rangle$, $\mathrm{pred}(y) = \langle y_0, \cdots, y_{\mathrm{lh}\, y}\rangle$, $\langle \xi_i \mid i < \mathrm{lh}\, x\rangle$ a $\pi^C(x)$-coherent sequence of members of X, and $\langle \eta_i \mid i < \mathrm{lh}\, y\rangle$ a $\pi^C(y)$-coherent sequence of members of X, then the following are equivalent:

1. $\mathfrak{f} \circ \mathrm{Fun}(C)(X) \vDash \mathbf{n}(x)(\{\xi_0, \cdots, \xi_{\mathrm{lh}\, x-1}\}) < \mathbf{n}(y)(\{\eta_0, \cdots, \eta_{\mathrm{lh}\, y-1}\})$.
2. $\mathrm{Fun}(C)(X) \vDash \langle x_0, \xi_0, \cdots, \xi_{\mathrm{lh}\, x-1}, x_{\mathrm{lh}\, x}\rangle <_{\mathsf{KB}} \langle y_0, \eta_0, \cdots, \eta_{\mathrm{lh}\, y-1}, y_{\mathrm{lh}\, y}\rangle$.

This follows from the definition of the comparison rules of $\mathfrak{f}$. In particular, when we compare $\mathbf{n}(x)(a)$ and $\mathbf{n}(x)(b)$ for $a, b \in [X]^{\mathrm{lh}\, x}$, we first compare the e_0th elements of a and b, then compare the e_1th elements of a and b, etc. This shows $\pi^{\mathfrak{f}\circ\mathrm{Fun}(C)}(x) = \pi^C(x)$. We also have that $\mathbf{p}^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y)) \leq m$ if m is the least number such that $x_m \neq y_m$, because if the first m parameters (i.e., ξ_i and η_i) in the comparison

$$\langle x_0, \xi_0, \cdots, \xi_{\mathrm{lh}\, x-1}, x_{\mathrm{lh}\, x}\rangle \quad (<_{\mathsf{KB}} \text{ or } >_{\mathsf{KB}}) \quad \langle y_0, \eta_0, \cdots, \eta_{\mathrm{lh}\, y-1}, y_{\mathrm{lh}\, y}\rangle$$

are the same, then the comparison only depends on that of x_m and y_m. To see the reverse inequality $\mathbf{p}^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y)) \geq m$, we claim that $k < m$ implies $k < \mathbf{p}^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y))$: By the amalgamation theorem, we can find a linear order A amalgamating $\pi^C(x)$ and $\pi^C(y)$ with the overlapped part $\pi^C(x) \cap (m \times m) = \pi^C(y)(m \times m)$. More precisely, we have a linear order A and embeddings $f_x \colon \pi^C(x) \to A$, $f_y \colon \pi^C(y) \to A$ such that for $i < m$, $f_x(i) = f_y(i)$. Then over $\mathrm{Fun}(C)(A \times 2)$,

1. $\langle x_0, \langle f_x(0), 0\rangle, \cdots, x_k, \langle f_x(k), 0\rangle, \cdots\rangle <_{\mathsf{KB}} \langle y_0, \langle f_y(0), 0\rangle, \cdots, y_k, \langle f_y(k), 1\rangle, \cdots\rangle$
2. $\langle x_0, \langle f_x(0), 1\rangle, \cdots, x_k, \langle f_x(k), 0\rangle, \cdots\rangle >_{\mathsf{KB}} \langle y_0, \langle f_y(0), 1\rangle, \cdots, y_k, \langle f_y(k), 1\rangle, \cdots\rangle$

In the first inequality, every parameter component takes the form $\langle f(i), 0\rangle$ except for $\langle f_y(k), 1\rangle$. In the second inequality, every parameter component takes the form $\langle f(i), 1\rangle$

except for $\langle f_x(k), 0\rangle$. Hence, the comparison depends on the kth parameter component, which means $k < \mathbf{p}^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x), \mathbf{n}(y))$.

The last equivalence is immediate since we used $\lessdot$ to compare x_i and y_i. □

Proposition D.1.16. *For a predendrogram C, let us define* $\mathsf{eden}_C \colon C \to (\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun})(C)$ *by* $\mathsf{eden}_C(x) = [\mathbf{n}(x')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x}}$, *where* $x' \in \mathrm{term}(C)$ *is any terminal node extending x. Then* eden_C *is a well-defined predendrogram isomorphism natural in C.*

Proof. $\mathsf{eden}_C(x)$ is well-defined when x is a terminal node of C. Otherwise, suppose that x', x'' are two terminal nodes extending x. Then $x = x' \restriction \mathrm{lh}\, x = x'' \restriction \mathrm{lh}\, x$, so $\mathbf{p}^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x'), \mathbf{n}(x'')) > \mathrm{lh}\, x$ by Lemma D.1.15. It means $[\mathbf{n}(x')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x}} = [\mathbf{n}(x'')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x}}$.

Then we claim that eden_C is a predendrogram isomorphism. Clearly, eden_C is onto. For the remaining preservation conditions,

1. Suppose that $C \vDash x \multimap y$. Fix a terminal node y' extending y. Then $y' \restriction (\mathrm{lh}\, x) = x$ and $y' \restriction (\mathrm{lh}\, x + 1) = y$, so $\mathsf{eden}_C(x) = [\mathbf{n}(y')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x}}$ and $\mathsf{eden}_C(y) = [\mathbf{n}(y')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x+1}}$. Hence $\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun}(C) \vDash \mathsf{eden}_C(x) \multimap \mathsf{eden}_C(y)$ by the definition of $\multimap$ over the cell decomposition.

2. For the terminal point preservation, suppose that $\mathsf{eden}_C(x)$ is not a terminal. Then we have $y \in C$ such that $\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun}(C) \vDash \mathsf{eden}_C(x) \multimap \mathsf{eden}_C(y)$. If $\mathsf{eden}_C(x) = [\mathbf{n}(x')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x}}$ and $\mathsf{eden}_C(y) = [\mathbf{n}(y')]_{\equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, y}}$, then $\mathrm{lh}\, y = \mathrm{lh}\, x + 1$ and $\mathbf{n}(x') \equiv^{\mathfrak{f}\circ\mathrm{Fun}(C)}_{\mathrm{lh}\, x} \mathbf{n}(y')$. By Lemma D.1.15, this implies $x = x' \restriction \mathrm{lh}\, x = y' \restriction \mathrm{lh}\, x$, so $x \multimap y$.

3. Now suppose that $C \vDash x \lessdot y$. Let x', y' be terminal extensions of x and y, respectively. Then m is the least natural number such that $C \vDash x' \restriction m \neq y' \restriction m$.

By Lemma D.1.15, $\mathbf{n}(x') <^{\mathfrak{f}\circ\mathrm{Fun}(C)} \mathbf{n}(y')$. We also have $\mathbf{n}(x') \equiv_{m-1}^{\mathfrak{f}\circ\mathrm{Fun}(C)} \mathbf{n}(y')$, so the definition of the cell decomposition implies the desired result.

4. By Lemma D.1.15, $\pi^{\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathbf{n}(x)) = \pi^C(x)$ for every $x \in \mathrm{term}(C)$. By the definition of the cell decomposition, we have $\pi^{\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathsf{eden}_C(x)) = \pi^C(x)$. For a non-terminal $x \in C$, let x' be a terminal extension of x. Then $\pi^C(x) = \pi^C(x') \cap (\mathrm{lh}\, x \times \mathrm{lh}\, x)$ and

$$\pi^{\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathsf{eden}_C(x)) = \pi^{\mathrm{Cell}\circ\mathfrak{f}\circ\mathrm{Fun}(C)}(\mathsf{eden}_C(x')) \cap (\mathrm{lh}\, x \times \mathrm{lh}\, x).$$

Hence, we get a desired result.

The naturality of eden_C follows from a direct calculation, so we leave it to the reader. □

Example D.1.17. In this example, we show that the predendrogram presented in Example D.1.2 represents the square dilator $D(X) = X^2$. First, we can see that the square dilator consists of the following three terms:

1. A term $p_0(x, y)$, representing $\langle x, y\rangle$ for $x > y$.
2. A term $d(x)$, representing $\langle x, x\rangle$.
3. A term $p_1(x, y)$, representing $\langle x, y\rangle$ for $x < y$.

We can also check the following with some laborious computation:

1. $\pi^D(p_0) = \{1 < 0\}$, $\pi^D(p_1) = \{0 < 1\}$.
2. The distance between two different terms is 1.
3. $p_0 <^D d <^D p_1$.

This information is enough to calculate the cell decomposition of D, which is isomorphic to Figure D.1.

D.2 Characterizing dilators with predendrograms

In this section, we characterize predendrograms coding dilators. Throughout this section, we use the axiom of countable choice. We can avoid the use of the axiom of countable choice when the target predendrogram C is countable since we have the canonical choice function (namely, the well-order over field(C)). The following condition gives a necessary condition for dilator-encoding predendrograms:

Definition D.2.1. Let C be a predendrogram. We say C is *locally well-founded* if, for every $x \in C$, the set of immediate successors of x is well-ordered under $\lessdot$.

Not every locally well-founded predendrogram gives a dilator:

Example D.2.2. Let $C \subseteq 2^{<\omega}$ be the set of non-empty binary tuples of the form either $\langle 0, 0, \cdots, 0, 0\rangle$ or $\langle 0, 0, 0, \cdots, 0, 1\rangle$. That is, C is the set of binary tuples whose all but the last component is 0. We equip the natural immediate successor relation $\multimap$, and the local comparison relation $\lessdot$ (namely, Kleene-Brouwer order). Then term(C) is the set of tuples whose last component is 1. For each $\sigma \in C$ of length m, we set $\overline{\pi}(\sigma)$ to be the order

$$0 <_{\overline{\pi}(\sigma)} 1 <_{\overline{\pi}(\sigma)} \cdots <_{\overline{\pi}(\sigma)} (m-1).$$

C is locally well-founded since it is a forest with two trees, and each node of C has at most two immediate successors. However, Fun$(C)(\omega)$ is ill-founded since if we take

$$\sigma_i = \langle\langle 1\rangle, 0, \langle 0, 1\rangle, 1, \langle 0, 0, 1\rangle, 2 \cdots, (i-1), \langle \underbrace{0, 0, \cdots, 0}_{i \text{ times}}, 1\rangle\rangle,$$

then $\sigma_0 >_{\mathsf{KB}} \sigma_1 >_{\mathsf{KB}} \sigma_2 >_{\mathsf{KB}} \cdots$.

We also need the following notion to isolate dilators:

Definition D.2.3. Let C be a predendrogram and $B = \langle x_i \mid i < \omega\rangle$ be an infinite branch of C, that is, $x_0 \multimap x_1 \multimap x_2 \multimap \cdots$. A *thread associated with* B is the linear order

$\bigcup_{i<\omega} \varpi^C(x_i)$ over $\mathbb{N}$. A *thread of* C is a thread associated with some infinite branch B of C.

It turns out that locally well-foundedness and threads are sufficient to isolate dilator-coding predendrograms:

Theorem D.2.4. *Let C be a predendrogram over X. Then* $\mathrm{Fun}(C)$ *is a dilator if and only if C is locally well-founded and every thread of C is ill-founded.*

We divide the proof of Theorem D.2.4 into a few parts. We start with a sufficient condition for local well-foundedness:

Lemma D.2.5. *Suppose that C is a predendrogram such that* $\mathrm{Fun}(C)(\alpha)$ *is well-founded for $\alpha < \omega^2$. Then C is locally well-founded.*

Proof. Suppose that C is not locally well-founded, so we have $x \in C$ such that the set $\{y \in C \mid x \multimap y\}$ is ill-founded. Pick an infinite $\lessdot$-descending sequence $\langle y_n \mid n < \omega\rangle$ over the set. For each $n < \omega$, we choose a terminal $x_n \in C$ such that $y_n \multimap^* x_n$.

Now let us find a well-order α of ordertype $< \omega^2$ making $\mathrm{Fun}(C)(\alpha)$ ill-founded. Let $m = \mathrm{lh}\, x + 1$. First, let $\langle \xi_0, \cdots, \xi_{m-1}\rangle$ be the $\varpi(y_i)$-coherent (equivalently, $\varpi(y_0)$-coherent) enumeration of $\{\omega, \omega \cdot 2, \cdots, \omega \cdot m\}$. Then we can expand $\langle \xi_0, \cdots, \xi_{m-1}\rangle$ into $\langle \xi_0, \cdots, \xi_{m-1}, \xi^n_m, \cdots, \xi^n_{\mathrm{lh}\, x_n - 1}\rangle \in (\omega^2)^{<\omega}$ in a way that the sequence is $\varpi(x_n)$-coherent.

Observe that for each n and $i < m$, we have $x_n \restriction i = x \restriction i$ and $x_n \restriction m = y_n \restriction m$. Now for each $i < \omega$, let

$$\sigma_n = \langle x_n \restriction 0, \xi_0, x_n \restriction 1, \xi_1, \cdots, \xi_{m-1}, x_n \restriction m, \xi^n_m, x_n \restriction (m+1), \cdots, \xi^n_{\mathrm{lh}\, x_n - 1}, x_n\rangle.$$

Then $\sigma_n \in \mathrm{Fun}(C)(\omega^2)$ and the first $2m$ components of σ_n is independent of the choice of n. Hence, $x_n \restriction m = y_n$ implies $\langle \sigma_n \mid n < \omega\rangle$ is $<_{\mathsf{KB}}$-decreasing. □

Note that Example D.2.2 gives a counterexample of the converse of Lemma D.2.5, so the converse fails. The next lemma is due to Freund [23, Theorem 3.6]:

Lemma D.2.6. *Suppose that C is a locally well-founded predendrogram. If* $\mathrm{Fun}(C)(\alpha)$ *is ill-founded for some well-order α, then C has a well-founded thread of ordertype $\leq \alpha$.*

Proof. Suppose that we have a $<_{\mathsf{KB}}$-infinite descending sequence $\langle \sigma_n \mid n \in \mathbb{N} \rangle$ over $\mathrm{Fun}(C)(\alpha)$ of the form

$$\sigma_n = \langle x_0^n, \xi_0^n, \cdots, \xi_{l(n)-1}^n, x_{l(n)}^n \rangle$$

for $x_i^n \in C$ and $\xi_i^n \in \alpha$. By local well-foundedness of C and the well-foundeness of α, the length of σ_n must be unbounded. Moreover, for each $i \in \mathbb{N}$ we have $N(i) \in \mathbb{N}$ and $x_i \in X$, $\xi_i \in \alpha$ such that $x_i^n = x_i$ and $\xi_i^n = \xi_i$ for $n > N(i)$. Hence $\mathrm{Fun}(C)(\alpha)$ as a tree has an infinite branch $\langle x_0, \xi_0, x_1, \xi_1, \cdots \rangle$. Clearly $x_i \in C$ and $x_0 \multimap x_1 \multimap x_2 \multimap \cdots$. Then $\pi^C(x_i)$ is a linear order over i given by

$$j <_{\pi^C(x_i)} k \iff \xi_j < \xi_k,$$

then $\pi^C(x_0) \subseteq \pi^C(x_1) \subseteq \pi^C(x_2) \subseteq \cdots$. The thread $\bigcup_i \pi^C(x_i)$ is well-ordered since the map $k \mapsto \xi_k$ is an order-preserving map into α. □

The next result states the converse of Lemma D.2.6 also due to Freund [23, Proposition 3.4, Theorem 3.6]:

Proposition D.2.7. *Suppose that C is a predendrogram. If C has a well-founded thread of ordertype α, then* $\mathrm{Fun}(C)((\omega \cdot (1 + (2 \cdot \alpha))))$ *is ill-founded.*

Proof. To illustrate the idea, let us first consider the case when C is Example D.2.2; For notational convenience, write $s_i = \langle \underbrace{0, 0, \cdots, 0}_{i \text{ times}} \rangle$ and $t_i = \langle \underbrace{0, 0, \cdots, 0}_{i-1 \text{ times}}, 1 \rangle$ In this case, the

unique thread has ordertype ω.

$$\mathrm{Fun}(C)(\omega) \vDash \langle t_0\rangle > \langle s_0, 1, t_1\rangle > \langle s_0, 0, s_1, 2, t_2\rangle > \langle s_0, 0, s_1, 2, s_2, 3, t_3\rangle > \cdots.$$

We mimic this type of descending sequence, more precisely, an infinite descending sequence of the form

$$(2^\omega, <_{\mathsf{KB}}) \vDash \langle 1\rangle > \langle 0, 1\rangle > \langle 0, 0, 1\rangle > \langle 0, 0, 0, 1\rangle > \cdots,$$

but we have to consider additional parameters.

Suppose that $\langle x_n \mid n \in \mathbb{N}\rangle$ is an infinite branch of C with the associated well-founded thread $(\mathbb{N}, \prec)$ of ordertype α. For each n, we choose a terminal extension y_n of x_n of length $l(n)$. For notational convenience, take $\langle y_n^i \mid i < l(n)\rangle$ to be the sequence of predecessors of y_n. Now let $\langle \xi_n \mid n \in \mathbb{N}\rangle$ be an enumeration of α such that $n \mapsto \xi_n$ is an isomorphism from $(\mathbb{N}, \prec)$ to α.

Now we will find $a_n \subseteq [\omega \cdot (1 + 2 \cdot \alpha)]^{l(n)}$ such that in, $\mathrm{Fun}(C)(\omega \cdot (1 + 2 \cdot \alpha))$, if we take

$$\sigma_n = \langle y_n^0, a_n(\Sigma_{y_n}^{\mathrm{f}\circ\mathrm{Fun}(C)}(0)), y_n^1, \cdots, a_n(\Sigma_{y_n}^{\mathrm{f}\circ\mathrm{Fun}(C)}(l(n)-1)), y_n^{l(n)}\rangle,$$

then $\sigma_0 >_{\mathsf{KB}} \sigma_1 >_{\mathsf{KB}} \cdots$. Note that we can informally understand σ_n as $y_n(a_n)$, if we view y_n as a predilator term.

For each n, let us set

$$b_n = \{\omega \cdot (1 + 2 \cdot \xi_i) \mid i < n - 1\} \cup \{\omega \cdot (1 + 2 \cdot \xi_{n-1} + 1)\}.$$

Then we can find $a_n = b_n \cup \{\eta_n^i \mid n \leq i < l(n)\}$ such that for each i, η_n^i is a successor ordinal, and moreover, the map

$$i \mapsto \begin{cases} \omega \cdot (1 + 2 \cdot \xi_i) & i < n - 1 \\ 1 + 2 \cdot \xi_{n-1} + 1 & i = n - 1 \\ \eta_n^i & n \leq i < l(n) \end{cases}$$

is an order-preserving map from $\pi^C(y_n)$ to $\omega \cdot (1 + 2 \cdot \alpha)$. Then we have $\sigma_n >_{\mathsf{KB}} \sigma_{n+1}$ since

$$\langle y_n^0, \omega \cdot (1 + 2 \cdot \xi_0), y_n^1, \cdots, \omega \cdot (1 + 2 \cdot \xi_{n-1} + 1), y_n^n, \cdots \rangle$$
$$> \langle y_n^0, \omega \cdot (1 + 2 \cdot \xi_0), y_n^1, \cdots, \omega \cdot (1 + 2 \cdot \xi_{n-1}), y_{n+1}^n, \cdots \rangle,$$

as desired. □

Now, let us state a proof of the main theorem:

Proof of Theorem D.2.4. For one direction, suppose that $\mathrm{Fun}(C)$ is not a dilator. If C is locally well-founded, then Lemma D.2.6 implies C has a well-founded thread. Hence, C is either not locally well-founded or has a well-founded thread. Conversely, suppose that $\mathrm{Fun}(C)$ is a dilator. Then Lemma D.2.5 implies C is locally well-founded, and Proposition D.2.7 implies every thread of C is ill-founded. □

It should be correct to define dendrograms based on Theorem D.2.4 as follows:

Definition D.2.8. A predendrogram C is a *dendrogram* if C is locally well-founded and every thread of C is ill-founded.

Note that predendrogram morphism preserves local well-foundedness and the absence of a well-founded thread; More precisely, if C is locally well-founded or has no well-founded thread, then so is its image under a predendrogram morphism. Also, we have the following:

Corollary D.2.9. *Let D be a predilator. Then D is a dilator iff $\mathrm{Cell}(D)$ is a dendrogram.*

□

Note that the previous corollary is not provable in RCA_0. More precisely, the proof of Proposition D.2.7 uses the fact that a tree T is well-founded iff $(T, <_{\mathsf{KB}})$ is a well-order. Its proof relies on Σ^0_1-Comprehension and Weak Kőnig's lemma (See [89, Lemma V.1.3.].) Indeed, Aguilera-Freund-Pakhomov [1, Lemma 3.2] proved that if RCA_0 proves a recursive predilator D is a dilator, then there is a natural number n such that the arity of every D-term is bounded by n. In particular, the exponential dilator $X \mapsto 2^X$ is not a dilator in a model of RCA_0. Similarly, [1, Theorem 4.1] shows that if a recursive predilator D is a dilator in every ω-model of RCA_0, then D embeds into $(\omega + \mathsf{Id})^\alpha$ for some recursive well-order α. This shows there is an ω-model of RCA_0 in which the exponential dilator $X \mapsto 2^X$ is not a dilator. The failure of the exponential being a dilator is witnessed by an ill-founded linear order with no infinite descending sequence inside the (ω-)model.

D.3 Developing the theory of predendrograms

In this section, we describe the properties of predilators in terms of predendrograms. We also develop the theory of predilators with the help of predendrograms, some of which will have a role in defining 2-ptykes.

Proposition D.3.1. *Let C be a predendrogram over X. Then* $\mathrm{Fun}(C)$ *is a preflower iff either every $x \in C$ has length 0 or there is $x_{\mathsf{M}} \in C$ of length 0 such that*

1. *If $x \in C$ has length 0, then $x \leq x_{\mathsf{M}}$.*
2. *If $x \in C$ has length at least 1, then $x_{\mathsf{M}} \multimap^* x$.*
3. *For every x of length at least 1, $\pi^C(x)$ evaluates 0 as the largest member.*

Proof. Let us prove the converse first, which is easier to prove. If every $x \in C$ has length 0, then $\mathrm{Fun}(C)$ is constant. Otherwise, suppose that A is a linear order and B is an end extension of A. Suppose that

$$\sigma = \langle x_{\mathsf{M}}, \xi_0, x_1, \cdots, \xi_{m-1}, x_m \rangle \in \mathrm{Fun}(C)(B).$$

Since $\xi_0, \cdots, \xi_{m-1}$ is $\pi^C(x_m)$-cohenrent and $\pi^C(x_m)$ evaluates 0 as the largest member, $\xi_0 > \xi_1, \cdots, \xi_{m-1}$. If $\tau = \langle x_{\mathsf{M}}, \eta_0, y_1, \cdots, \eta_{n-1}, y_n \rangle \in \mathrm{Fun}(C)(A)$ and $\sigma <_{\mathsf{KB}} \tau$, then $\eta_1, \cdots, \eta_{n-1} < \eta_0 < \xi_0$, so every parameter components of σ is in A. In sum, every element of $\mathrm{Fun}(C)(B)$ less than an element of $\mathrm{Fun}(C)(A)$ is also an element of $\mathrm{Fun}(C)(A)$, that is, $\mathrm{Fun}(C)(A)$ is an initial segment of $\mathrm{Fun}(C)(B)$. Hence $\mathrm{Fun}(C)$ is a preflower.

To show the converse, let us calculate its cell decomposition for a preflower F. Proposition C.4.11 implies the following:

1. For a nullary term s and a non-nullary term t of F, $s <^F t$.
2. If s and t are non-nullary terms of F, $\mathbf{p}^F(s,t) \geq 1$ and $\pi^F(s)$ evaluates 0 as the largest element.

Hence, we have the desired result if we take $x_{\mathsf{M}} = [s]_{\equiv_0^F}$ for any non-nullary term s. □

The next proposition, whose detail is left to exercise, describes familiar predilators in terms of predendrogram:

Proposition D.3.2.

1. *Let I be a linear order and $\{D_i \mid i \in I\}$ be a family of predilators. Then the ordered sum $\sum_{i \in I} D_i$ is isomorphic to a predendrogram C whose field is*

$$\{\langle i, x \rangle \mid i \in I \wedge x \in \mathrm{Cell}(D_i)\}$$

with the following relations:

- *For* $x, y \in \mathrm{Cell}(D_i)$*, we set* $C \vDash \langle i, x\rangle \multimap \langle i, y\rangle$ *if* $\mathrm{Cell}(D_i) \vDash x \multimap y$*.*
- $C \vDash \langle i, x\rangle \multimap \langle j, y\rangle$ *if either* $I \vDash i < j$*, or* ($i = j$ *and* $D_i \vDash x \lessdot y$)*.*
- $\pi^C(\langle i, x\rangle) = \pi^{\mathrm{Cell}(D_i)}(x)$*.*

2. *Let* D *be a predilator. Then* $\int D$ *is isomorphic to a predendrogram* $C = \mathrm{Cell}(D) \cup \{\star\}$ *for a fresh node* $\star$*. Every relation of* C *is the same as that of* $\mathrm{Cell}(D)$ *and* $C \vDash \star \multimap x$ *for every* $x \in \mathrm{Cell}(D)$ *(so the length of a node from* $\mathrm{Cell}(D)$ *is increased by one in* C*.) We also define* $\pi^C(x)$ *by*

$$i \leq_{\pi^C(x)} j \iff [i, j \neq 0 \to (i-1) <_{\pi^{\mathrm{Cell}(D)}(x)} (j-1)].$$

3. *Let* D *be a predilator. Then* D *is perfect if and only if there is a unique node of length 0.* □

We can visualize various types of predilators as a predendrogram as follows. The detail is left to the reader.

Proposition D.3.3. *Let* C *be a predendrogram. Then*

1. $\mathrm{Fun}(C)$ *is of type 0 iff* C *is a singleton.*
2. $\mathrm{Fun}(C)$ *is of type 1 iff the set of elements of* C *of length 0 has a maximum* x_{M}*, and* x_{M} *is a terminal node.*
3. $\mathrm{Fun}(C)$ *is of type* ω *iff the set of elements of* C *of length 0 has no maximum.*
4. $\mathrm{Fun}(C)$ *is of type* Ω *iff the set of elements of* C *of length 0 has a maximum* x_{M}*, and* x_{M} *is not a terminal node.* □

Pairwise comparison between two increasing maps has a role in defining predilators. That is, a semidilator F is a predilator if $f \leq g$ for $f, g \colon X \to Y$, we have $F(f) \leq F(g)$. We have an analogous notion of pairwise comparison also for predilator embeddings:

Definition D.3.4. Let $f, g \colon D \to E$ be predilator embeddings. We say $f \leq g$ if $f(t) \leq^E g(t)$ for every $t \in \mathrm{field}(D)$.

Pointwise dominance of predilator morphism has the following predendrogram characterization:

Proposition D.3.5. *Let C and D be predendrograms and $h, k \colon C \to D$ be predendrogram morphisms. Then $\mathfrak{f} \circ \mathrm{Fun}(h) \leq \mathfrak{f} \circ \mathrm{Fun}(k)$ if and only if for every $x \in C$, $h(x) \trianglelefteq k(x)$.*

Equivalently, for two predilators F, G and predilator morphisms $f, g \colon F \to G$, $f \leq g$ iff for every $x \in \mathrm{Cell}(F)$, $\mathrm{Cell}(f)(x) \trianglelefteq \mathrm{Cell}(g)(x)$.

Proof. We have the following by Lemma D.1.15: For every $u, v \in \mathrm{term}(D)$, $\mathbf{n}(u) \leq^{\mathfrak{f} \circ \mathrm{Fun}(D)} \mathbf{n}(v)$ if and only if for the least m such that $u \restriction m \neq v \restriction m$, $D \vDash u \restriction m \trianglelefteq v \restriction m$. Also, let us note that $\mathfrak{f} \circ \mathrm{Fun}(h)(\mathbf{n}(x)) = \mathbf{n}(h(x))$.

For one direction, suppose that $h(x) \trianglelefteq k(x)$ for every $x \in C$. Then the previous claim implies $\mathfrak{f} \circ \mathrm{Fun}(h)(\sigma) \leq^{\mathfrak{f} \circ \mathrm{Fun}(D)} \mathfrak{f} \circ \mathrm{Fun}(k)(\sigma)$ holds for every σ in the field of $\mathfrak{f} \circ \mathrm{Fun}(C)$. Conversely, suppose that $h(x) \trianglerighteq k(x)$ for some $x \in C$. Take a terminal extension y of x, and consider $\sigma = \mathbf{n}(y)$. Then we have $\mathfrak{f} \circ \mathrm{Fun}(h)(\sigma) >^{\mathfrak{f} \circ \mathrm{Fun}(D)} \mathfrak{f} \circ \mathrm{Fun}(k)(\sigma)$. □

Natural numbers represent finite linear orders in the sense that the category of natural numbers is a skeleton of the category of finite linear orders. Let us define the dilator version of natural number objects.

Definition D.3.6. A predendrogram C is *natural* if the field of C is a subset of $\mathbb{N}^{<\omega}\setminus\{\langle\rangle\}$, and

1. C is closed under non-empty initial segments.
2. For $x, y \in C$ $x \multimap y$ iff there is $i \in \mathbb{N}$ such that $y = x^\frown\langle i\rangle$.
3. For $x, y \in C$, $x \lessdot y$ if x and y have the same length, they agree except for the last component, and the last component of x is less than that of y.
4. For each $x \in C \cup \{\langle\rangle\}$, the set $\{i \in \mathbb{N} \mid x^\frown\langle i\rangle \in C\}$ is an initial segment of $\mathbb{N}$.

A *natural predilator* is a predilator of the form $\mathfrak{f}\circ\mathrm{Fun}(C)$ for some natural predendrogram C.

Note that $\mathrm{Fun}(C)$ is a finite predilator when C is natural, so $\mathrm{Fun}(C)$ is a dilator. Hence, we can use the word *natural dilator*. Of course, we should distinguish natural predilator and natural dilator when we work on a weak fragment of second-order arithmetic like RCA_0, on which we do not know if every finite predilator is a dilator. The same caution also applies to natural predendrograms and natural dendrograms. We claim that the category of natural predilators is a skeleton of the category of finite predilators:

Proposition D.3.7. *Let C and D be isomorphic natural predendrograms. Then $C = D$.*

Proof. Let $h\colon C \to D$ be an isomorphism. We claim by induction on the length of $x \in C$ that $h(x) = x$. For nodes of length 0, let $N_C = \{k \in \mathbb{N} \mid \langle k\rangle \in C\}$. Then we have a bijective increasing map $f_C\colon N_C \to N_D$ such that $h(\langle k\rangle) = \langle f_C(k)\rangle$. N_C and N_D are initial segments of $\mathbb{N}$, so f_C is the identity and $N_C = N_D$.

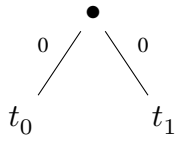
Now suppose that our claim holds for C-nodes of length n. For $x \in C$ of length n, consider the set $N_{C,x} = \{k \in \mathbb{N} \mid x^\frown\langle k\rangle \in C\}$. Then we have a bijective increasing

map $f_{C,x} \colon N_C \to N_D$ given satisfying $h(x^\frown\langle k\rangle) = x^\frown\langle f_{C,x}(k)\rangle$. $f_{C,x}$ is the identity map, and $N_{C,x} = N_{D,x}$. □

Definition D.3.8. By Proposition D.3.7, every finite predendrogram is isomorphic to a unique natural predendrogram. For a finite predendrogram C, we let $\|C\|$ be the unique natural predendrogram isomorphic to C. $\mathrm{en}_C \colon \|C\| \to C$ is the unique isomorphism between $\|C\|$ and C.

Then let us discuss the amalgamation theorem of predendrograms. An amalgamation theorem of linear orders roughly states that if we are given two linear orders with a designated intersection, then we can find a linear order merging the two while preserving the given intersection. In the case of linear orders, the 'intersection' is a suborder of the two linear orders. We can expect similar to predendrograms, but we should be aware that the meaning of the 'intersection' of two predendrograms is not necessarily a sub-predendrogram.

Example D.3.9. Consider the dendrogram D given by



(Note that its functorization is $X \mapsto 2 \cdot X$, but this fact is irrelevant to the following explanation.) Then consider a subdendrogram D_i ($i = 0, 1$) given by the branch ending with t_i. Then the overlapping part of D_0 and D_1 is the bullet node. However, the overlapping part has no terminal node in either D, D_0, or D_1. That is, the overlapping part does not occur as a subdendrogram.

To express the 'overlapping part' in the previous example, we introduce the following weaker notion of morphism:

Definition D.3.10. A function $h\colon D \to E$ is a *predendrogram quasimorphism* if h preserves $\lessdot$, $\multimap$, and the priority orders. (h may not preserve terminal points.) If h is the inclusion map, we say D *is a quasi-subpredendrogram of* E.

We can state the amalgamation theorem of two predendrograms as follows:

Theorem D.3.11. *Let D, E_0, E_1 be predendrograms and $f_0\colon D \to E_0$, $f_1\colon D \to E_1$ be predendrogram quasimorphisms. Then we can find a predendrogram F and predendrogram quasimorphisms $g_0\colon E_0 \to F$, $g_1\colon E_1 \to F$ such that the following diagram commutes*

$$\begin{array}{ccc} D & \xrightarrow{f_1} & E_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle g_1} \\ E_0 & \xrightarrow[g_0]{} & F \end{array}$$

and moreover, $\operatorname{ran} g_0 \cap \operatorname{ran} g_1 = \operatorname{ran}(g_0 \circ f_0) = \operatorname{ran}(g_1 \circ f_1)$. *Furthermore, if each $x \in D$, $f_0(x)$ is a terminal node in E_1 iff $f_1(x)$ is a terminal node of E_1, then we can make g_0 and g_1 proper morphisms.*

Proof. By replacing elements of E_i if necessary, we may assume that both f_0 and f_1 are inclusion maps, so D is a quasi-subpredendrogram of E_0 and E_1 respectively, and $D = E_0 \cap E_1$. We want to turn $E_0 \cup E_1$ into a predendrogram. The immediate successor relation and the priority orders are already determined, so the only thing we need to consider is the local comparison order. The relation $\lessdot^{E_0} \cup \lessdot^{E_1}$ is a partial order, which is undefined between $x_0 \in E_0 \setminus D$ and $x_1 \in E_1 \setminus D$ such that either both have length 0, or have the common immediate predecessor in D. We divide $\lessdot$ by parts, extend each part, then join them to get the desired local comparison relation.

1. First, let us consider $\lessdot^{E_0} \cup \lessdot^{E_1}$ over the set of roots of $E_0 \cup E_1$, which is the union of the set of roots of E_0 and that of E_1. By Theorem A.1.8, we can amalgamate

$\lessdot^{E_0}$ over the set of E_0-roots and $\lessdot^{E_1}$ over the set of E_1-roots, thus forming a linear order over the set of roots of $E_0 \cup E_1$

2. Then let us consider $\lessdot^{E_0} \cup \lessdot^{E_1}$ over the set of all immediate successors of some $x \in D$. We can apply the same argument we previously did to get a linear order over the set.

Then take $\lessdot^F$ to be the union of $\lessdot^{E_0} \cup \lessdot^{E_1}$ and the union of all the previously constructed orders. Note that we can canonically choose each extension since the proof of the amalgamation theorem of linear orders constructs an amalgamation in a definable manner. Let us call the resulting predendrogram F.

It remains to show that both E_0 and E_1 are subpredendrograms of F when E_0 and E_1 agree with points of D being terminal. We have assumed that $D \cap \mathrm{term}(E_0) = D \cap \mathrm{term}(E_1)$, and $D = E_0 \cap E_1$. Hence, if $x \in E_0 \cap E_1$ is a terminal point in E_0, then so is $x \in E_1$, and so also $x \in F$. This shows E_0 is a subpredendrogram of F. □

We know that the category of natural numbers forms a skeleton of the category of finite linear orders. Likewise, we have the following, which immediately follows from the uniqueness of $\|C\|$ and the isomorphism $\mathrm{en}_C : \|C\| \to C$:

Proposition D.3.12. *Let* qFinPreDen *be the category of predendrograms with predendrogram quasimorphisms. Then its subcategory* qNat^1 *of natural predenrograms with predendrogram quasimorphisms forms a skeleton.*

In previous sections, we have extended linear orders to build the desired commutative diagrams. In particular, for a given finite linear order X, we considered a suborder of ω^2 isomorphic to an extension of X. We frequently used this extension argument to prove the abstract construction of predilators. We will need a similar extending-dilator

argument to prove the abstract construction of 2-preptykes, but there is no clear ω^2-like object for dilators. Also, building such dilators by hand is often cumbersome: The main reason is that when we expand predilators (rather, a predendrogram), we consider not only the nodes of the same length, but also those of different lengths. The following $\mathbb{Q}$-like predendrogram often allows us to avoid headache-causing arguments.

Proposition D.3.13. *There is a primitive recursive predendrogram $\mathbb{Q}_1$ such that the following holds:*

1. *(ω-homogeneity) Suppose that we are given*

   - *Finite predendrograms $D \subseteq E$. (D and E may disagree with terminal points.)*
   - *A predendrogram quasimorphism $f \colon D \to \mathbb{Q}_1$.*
   - *A finite subpredendrogram $F \Subset \mathbb{Q}_1$ containing* ran f.

   Then we have a predendrogram quasimorphism $f' \colon E \to \mathbb{Q}_1$ extending f and $F \cap \operatorname{ran} f' = \operatorname{ran} f$.

   Furthermore, if we assume x is a terminal in E iff $f(x)$ is a terminal in $\mathbb{Q}_1$ for every $x \in D$, we can further assume $f' \colon C \to \mathbb{Q}_1$ respects terminal points, so it is a predendrogram morphism.

2. *(Universality) Every countable predendrogram embeds into $\mathbb{Q}_1$.*

Proof. We construct $\mathbb{Q}_1$ as follows: Let $\mathtt{P}_0$ be the set of linear orders whose field is some natural number. For $a, b \in \mathtt{P}_0$, we say $a \multimap b$ if $\operatorname{field}(b) - \operatorname{field}(a) = 1$ and b restricted to $\operatorname{field}(a)$ is a. We recursively define $\langle A_{a,i} \mid a \in \mathtt{P}_0, i \in 2 \rangle$ as follows:

1. $\mathbb{Q} = A_{\emptyset,0} \cup A_{\emptyset,1}$ is a disjoint union and each $A_{\emptyset,i}$ is dense.

2. $A_{a,0} = \bigcup_{a \multimap b} (A_{b,0} \cup A_{b,1})$ is a disjoint union and each $A_{b,i}$ is dense.

We can make the whole $\langle A_{a,i} \mid a \in \mathtt{P}_0, i \in 2 \rangle$ primitive recursive, so we assume the enumeration is primitive recursive. $A_{a,0}$ will be associated with the set of non-terminal nodes of priority order a, and $A_{a,1}$ will be associated with the set of terminal nodes of priority order a.

Then we define the predendrogram $\mathbb{Q}_1$ defined as follows:

1. The field of $\mathbb{Q}_1$ is the set of $\langle q_0, \cdots, q_m \rangle$ such that there are $a_0, \cdots, a_m \in \mathtt{P}_0$ with $a_0 \multimap a_1 \multimap \cdots \multimap a_m$ such that $q_i \in A_{a_i,0}$ for each $i < m$, and $q_m \in A_{a_m,0} \cup A_{a_m,1}$.
2. For $x, y \in \mathbb{Q}_1$, $x \multimap y$ if y end-extends x by length 1.
3. For $x, y \in \mathbb{Q}_1$, $x \lessdot y$ if x and y have the same length, same components except for the last one, and the last component of x is less than that of y.
4. For $x = \langle q_0, q_1, \cdots, q_m \rangle \in \mathbb{Q}_1$, we define $\pi^{\mathbb{Q}_1}(x) = a$ if $a \in \mathtt{P}_0$ is the unique linear order such that $q_m \in A_{a,0} \cup A_{a,1}$.

Then we can see that $\mathbb{Q}_1$ satisfies ω-homogeneity, essentially by the 'forth argument' of the usual back-and-forth argument. Let us sketch how the argument works: Let us enumerate $E \setminus D$ $\{u_i \mid i < N\}$ in a way that $u_i \multimap u_j$ implies $i < j$. We extend a quasimorphism $f \colon D \to \mathbb{Q}_1$ into another quasimorphism $f_i \colon D \cup \{u_j \mid j < i\} \to \mathbb{Q}_1$ recursively as follows: We take $f_0 = f$. Now suppose that f_i is defined.

1. If u_i is a root in E, then let us choose $v := f_{i+1}(u_i) \in \mathbb{Q}_1 \setminus (F \setminus \operatorname{ran} f)$ in a way that for every root $y \in D \cup \{u_j \mid j < i\}$, $E \vDash y \lessdot u_i$ iff $\mathbb{Q}_1 \vDash f_i(y) \lessdot v$.

2. If $E \vDash x \multimap u_i$ for some $x \in D$, then $f(x)$ is also not a terminal in $\mathbb{Q}_1$ by the assumption on f and g. Hence, we can appropriately choose $v := f_{i+1}(u_i) \in \mathbb{Q}_1 \setminus (F \setminus \operatorname{ran} f)$ in a way that

   - $\mathbb{Q}_1 \vDash f_i(x) \multimap v$.
   - $\pi^E(u_i) = \pi^{\mathbb{Q}_1}(v)$.
   - If $E \vDash u_i \multimap y$ and $\mathbb{Q}_1 \vDash v \multimap y'$ for some y and y', then $\pi^E(y) = \pi^{\mathbb{Q}_1}(y')$.
   - For every $y \in D \cup \{u_j \mid j < i\}$ such that $E \vDash x \multimap y$, $E \vDash y \lessdot u_i$ iff $\mathbb{Q}_1 \vDash f_i(y) \lessdot v$.
   - v is a terminal in $\mathbb{Q}_1$ iff u_i is a terminal in E. (If we have assumed "x is a terminal in E iff $f(x)$ is a terminal in $\mathbb{Q}_1$ for every $x \in D$.")

The universality of $\mathbb{Q}_1$ follows from the ω-homogeneity. □

We can easily see that a countable ω-homogeneous predilator is unique up to bi-embeddability. It is not unique up to isomorphism since, for example, we can do the same construction with $\mathbb{Q} \cap [0, 1]$ instead of $\mathbb{Q}$, which gives a non-isomorphic ω-homogeneous predendrogram $\mathbb{Q}_1'$. (For example, $\mathbb{Q}_1'$ has a $\preccurlyeq^{\mathbb{Q}_1'}$-maximal and minimal terms, while $\mathbb{Q}_1$ is not. See Theorem C.1.2 for the definition of $\preccurlyeq$.) Also, $\mathbb{Q}_1$ is not a dendrogram since it also embeds predendrograms that are not dendrograms. We may ask if the existence of a dilator embeds every countable dilator, and the answer depends on set-theoretic assumptions. More precisely, we can find such a dilator if we have either the axiom of choice or every real has a sharp. See Section 4.3 for more details.

We often need to explicate the 'hidden terms' of a predendrogram, which the following completion operation can do:

Definition D.3.14. Let C be a predendrogram. We define its *completion* C^* as follows: For each non-terminal $x \in C$, we add a new element $x^* \in C^*$. x and x^* have the same

immediate predecessor and priority orders, and for $C \vDash x \lessdot y$, we let $C^* \vDash x \lessdot x^* \lessdot y \lessdot y^*$. Informally, we attach x^* right next to x.

The completion operator extends to a predendrogram quasimorphisms as follows:

Proposition D.3.15. *Let $f\colon C \to D$ be a predendrogram quasimorphism. Let us define $f^*\colon C^* \to D^*$ as follows: For $x \in C$,*

$$f^*(x) = \begin{cases} f(x), & \text{if } x \in \text{term}(C) \leftrightarrow f(x) \in \text{term}(D), \\ f(x)^*, & \text{if } x \in \text{term}(C) \wedge f(x) \notin \text{term}(D). \end{cases}$$

and $f^*(x^*) = f(x)^*$ when $x \notin \text{term}(C)$. (Note that x^* exists in C^* only when x is not terminal in C.) Then f^* is a predendrogram morphism.

Proof. Note that if $C \vDash x \lessdot y$, then $D^* \vDash f(x) \lessdot f(x)^* \lessdot f(y) \lessdot f(y)^*$. This implies f^* always preserves the local comparison relation. The definition of completion also implies that f^* preserves all other relations, so it suffices to see that f^* preserves terminal points, which is easy to check. □

Proposition D.3.16. *Let $f\colon C \to D$ and $g\colon D \to E$ be predendrogram quasimorphisms. Then $(g \circ f)^* = g^* \circ f^*$.*

Proof. Let $x \in C$. If $x \in \text{term}\, C$ and $(g \circ f)(x) \in \text{term}\, E$, then $f(x) \in \text{term}\, D$ too, otherwise $(g \circ f)(x)$ cannot be terminal in E. Hence in this case, $(g \circ f)^*(x) = (g \circ f)(x) = g(f(x))$. Similarly, we can see that if $x \notin \text{term}\, C$, then $f(x) \notin \text{term}\, D$ and $g(f(x)) \notin \text{term}\, E$, so $(g \circ f)(x) = g(f(x))$.

Now assume that $x \in C$, $x \in \text{term}\, C$, but $(g \circ f)(x) \notin \text{term}\, E$. If $f(x) \in \text{term}\, D$, then $g^*(f(x)) = g(f(x))^*$, and $(g \circ f)^*(x) = (g \circ f)(x)^*$, which are equal. If $f(x) \notin \text{term}\, D$, then $f^*(x) = f(x)^*$, so $g^*(f^*(x)) = g^*(f(x)^*) = (g \circ f(x))^*$ and $(g \circ f)^*(x) = (g \circ f)(x)^*$, which are equal. The remaining case is immediate by definition. □

Every completion comes with a canonical inclusion embedding $\mathsf{cem}_C \colon C \to C^*$ acting identically (so $\mathsf{cem}_C(x) = x$.) cem_C is natural in the following sense:

Proposition D.3.17. *Let $f \colon C \to D$ be a predendrogram quasimorphism. Then $\mathsf{cem}_D \circ f(x) = f^* \circ \mathsf{cem}_C(x)$ holds when $x \in \operatorname{term} C \leftrightarrow f(x) \in \operatorname{term} D$. In particular, if f is a predendrogram morphism, then the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \mathsf{cem}_C}\downarrow & & \downarrow{\scriptstyle \mathsf{cem}_D} \\ C^* & \xrightarrow[f^*]{} & D^* \end{array}$$

Proof. This follows directly from a calculation and the definition of f^*. f respecting the terminality of x plays a crucial role in the calculation. □

C^* is a completion of C in the sense that it unravels every intermediate node of C. The choice of local comparison order is natural in the following sense:

Proposition D.3.18. *For a predendrogram C and a linear order X, let $\operatorname{Fun}^*(C)(X)$ is defined as like in Definition D.1.3 $\operatorname{Fun}(C)(X)$ but we do not demand the last component x_m of*

$$\langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m \rangle$$

to be a terminal node of C. We also equip $\operatorname{Fun}^*(C)(X)$ the Kleene-Brouwer order. Then the map $\operatorname{fun}^*_{C,X} \colon \operatorname{Fun}^*(C)(X) \to \operatorname{Fun}(C^*)(X)$ given by

$$\langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m \rangle \mapsto \begin{cases} \langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m^* \rangle & x_m \notin \operatorname{term}(C) \\ \langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x_m \rangle & x_m \in \operatorname{term}(C) \end{cases}$$

is an order isomorphism natural in C and X. (In particular, $\operatorname{Fun}^*(C) \cong \operatorname{Fun}(C^*)$ as predilators.)

Proof. It is easy to see that $\mathrm{fun}^*_{C,X}$ is order-preserving, so it is injective. To see $\mathrm{fun}^*_{C,X}$ is surjective, observe that if $x \in C^*$ is a terminal, then either $x \in C$ and is also terminal in C, or $x = y^*$ for some $y \in C$ that is not terminal in C. Hence every chain $y_0 \multimap \cdots \multimap y_m$ in C^* takes the form either $x_0 \multimap \cdots \multimap x_m$ for a terminal x_m in C, or $x_0 \multimap \cdots \multimap x_{m-1} \multimap x^*_m$ for a non-terminal x_m in C. The naturality follows from direct calculations. □

D.4 Semidendrograms

The abstract construction of predilators yields predendroids and predendrograms. Similarly, the abstract construction of semidilators yields the corresponding notions we will call *semidendroids and semidendrograms.* We only introduce semidendrograms in this chapter, but the reader may formulate semidendroids, relevant notions, and relevant theorems on their own.

Definition D.4.1. A *semidendrogram* is a structure $(C, \leqslant, \multimap, \pi, \wp)$ satisfying the following:

1. $(C, \leqslant, \multimap, \pi)$ satisfies the defintion of a predendrogram.
2. For each $x \in \mathrm{field}(C)$, we have the *set of positively compared indices* $\wp_x \subseteq \mathrm{lh}\, x$ such that $x \multimap y$ implies $\wp_x \subseteq \wp_y$.

Like in semidilators, $\wp_x$ indicates how to compare parameters we substituted; $i \notin \wp_x$ means we compare the argument with the ith priority in the reversed order. When $\wp_x = \mathrm{lh}\, x$, we identify it with the predendrogram obtained by dropping $\wp$.

Let us define the functorialization of a semidendrogram. We need a 'reverse comparison' corresponding to negatively compared indices, so it is convenient to consider

the following semidilator for parameters:

Definition D.4.2. Let us define $\mathrm{sgn} = \mathsf{opp} + \mathsf{Id}$, where opp is the opposition semidilator defined in Example B.6.8. We denote two unary terms of sgn by $-$ and $+$ respectively.

Definition D.4.3. Let C be a semidendrogram over X. Let us define $\mathrm{Fun}(C)(X)$ to be the set of

$$\langle x_0, \pm_0 \xi_0, \cdots, \pm_{m-1} \xi_{m-1}, x_m \rangle \in (C \cup \mathrm{sgn}(X))^{<\omega}$$

such that $x_m \in \mathrm{term}(C)$, $\langle x_0, x_1, \cdots, x_m \rangle$ is the sequence of predecessors of x_m, $\xi_0, \cdots, \xi_{m-1} \in X$ is $\pi^C(x_m)$-coherent, and $\pm_i = +$ iff $i \in \wp^C_{x_m}$. We equip $\mathrm{Fun}(C)(X)$ the Kleene-Brouwer order. For an increasing $f \colon \alpha \to \beta$, we define

$$\mathrm{Fun}(f)(\langle x_0, \pm_0 \xi_0, \cdots, \pm_{m-1} \xi_{m-1}, x_m \rangle) = \langle x_0, \pm_0 f(\xi_0), \cdots, \pm_{m-1} f(\xi_{m-1}), x_m \rangle$$

and $\mathrm{supp}^{\mathrm{Fun}(C)}_\alpha(\langle x_0, \pm_0 \xi_0, \cdots, \pm_{m-1} \xi_{m-1}, x_m \rangle) = \{\xi_0, \cdots, \xi_{m-1}\}$.

It is easy to check that $\mathrm{Fun}(C)$ is an F-semidilator. We can also define a semidendrogram morphism as a function preserving every structure of a semidilator and terminal nodes. It is also easy to see that Proposition D.1.8 holds for semidendrogram morphisms. The proof of the next proposition is similar to that of Proposition D.1.9, so we omit it:

Proposition D.4.4. *Let $h \colon C \to C'$ be a semidendrogram morphism. Then we can define* $\mathrm{Fun}(h)$ *in an obvious manner, and it is a support-preserving natural transformation.*

For a given semidilator D, we can calculate its cell decomposition. We can define $\equiv^D_n$ also for a semidilator D by following Definition D.1.10, and prove the following:

Proposition D.4.5. *For a semidilator D, let us define* $\mathrm{Cell}(D)$ *by following Proposition D.1.12 with the additional clause*

5. $\wp([s]_{\equiv^D_m}) = \wp^D_s \cap m$.

Then $\mathrm{Cell}(D)$ *is a semidendrogram.* □

We can also show that the category of semidengrograms and the category of semidilators are category equivalent. The main argument is quite similar to the predendrogram cases, so we only state focal definitions and propositions to prove the category equivalence.

Proposition D.4.6. *For a semidilator* D *and a linear order* X*, let us define* $\mathsf{edil}_{D,X} \colon \mathfrak{a}(D)(X) \to \mathrm{Fun}(\mathrm{Cell}(D))(X)$ *by*

$$\mathsf{edil}_{D,X}(t(a)) = \langle [t]_{\equiv_0^D}, \pm_0 \xi_0, [t]_{\equiv_1^D}, \pm_1 \xi_1, \cdots, [t]_{\equiv_{\mathrm{arity}\, t}^D} \rangle,$$

where $\langle \xi_i \mid i < \mathrm{arity}\, t \rangle$ *is the enumeration of* a *that is* $\pi^D(t)$*-coherent, and* $\pm_i = +$ *iff* $i \in \wp_t^D$*. Then* edil_D *is an F-semidilator isomorphism natural in* D*.* □

Lemma D.4.7. *Let* C *be a semidendrogram. Then* $\mathfrak{f} \circ \mathrm{Fun}(C)$ *is a denotation system satisfying the conditions in Lemma D.1.15 plus the following:*

6. $\wp^{\mathfrak{f} \circ \mathrm{Fun}(C)}(\mathbf{n}(x)) = \wp_x^C$. □

The definition of eden_C for a semidendrogram C is identical to that for predendrograms. Like predendrograms, we can consider the natural semidendrograms by mimicking the definition of a natural predendrogram. Like in the predendrogram case, every two isomorphic natural semidendrograms are the same, and every finite semidilator is isomorphic to a unique natural semidilator. Its proof is identical to the predendrogram case. We can also define semidendrogram quasimorphisms and prove the amalgamation theorem for semidendrograms. We can also show that the category qSDen of semidendrograms with quasimorphisms has its subcategory qSNat^1 of natural semidendrograms with quasimorphisms as its skeleton.

D.5 Predendroids

In this section, we introduce Girard's dendroid. We do not need dendroids in my thesis, and the only purpose of this section is to sketch what dendroids are, which shows the connection between dendrograms and quasidendroids. Many details in this section will thus be skipped. The reader may skip this section at their preference. Girard only defined dendroids and did not define what we should call predendroids, which should correspond to predilators. In this section, we define predendroids first and define dendroids from them.

Definition D.5.1. Let α, X be linear orders. An *α-predendroid over X* is a set $D \subseteq [\alpha \cup X]^{<\omega}$ such that

1. Every element of D has odd length.
2. D does not contain an initial segment of its elements; That is, if $\sigma \in D$ and $\tau \subsetneq \sigma$ then $\tau \notin D$.
3. Every element of D is of the form $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle$ for $x_0, \cdots, x_m \in X$ and $\xi, \cdots, \xi_{m-1} \in \alpha$.

Now let D^* be the closure of D under the initial segment. For an element $\sigma \in D^*$, we call the odd-th component a *term part of σ*, and the even-th component a *parameter part of σ*. Then

4. If $\sigma = \langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle \in D^*$, $\sigma^\frown\langle\xi\rangle, \sigma^\frown\langle\eta\rangle \in D^*$ and $\xi \leq \eta$, then for every $i < m$, either $\xi_i < \xi \leq \eta$ or $\xi \leq \eta < \xi_i$ holds.

The last condition roughly states that the relative order of the parameter part of $\sigma \in D^*$ only depends on the term part of σ.

We want to understand an α-predendroid as an object of the form $\mathrm{Fun}(C)(\alpha)$ for a predendrogram C. Although not every α-predendroid takes the form $\mathrm{Fun}(C)(\alpha)$, every α-predendroid appearing in literature takes this form. Also, the reader should be cautious that the enumeration $\xi_0, \cdots, \xi_{m-1}$ of parameter parts is *not necessarily an increasing enumeration.*

Definition D.5.2. An α-predendroid D over X is *locally well-founded* if for every $\sigma = \langle x_0, \xi_0, \cdots, \xi_{m-1} \rangle \in D^*$, the set $\{x \in X \mid \sigma^\frown\langle x\rangle \in D\}$ is a well-order. An α-predendroid D over X is an *α-dendroid over X* if D is locally well-founded and D^* has no infinite branch.

Like predilators, α-predendroids also admit morphisms:

Definition D.5.3. Let D be an α-predendroid over X, and D' be an α'-predendroid over X'. We say $(f, g) \colon D \to D'$ is an (α, α')-predendroid morphism if

1. $f \colon \alpha \to \alpha'$, $g \colon D^* \to (D')^*$, and $g[D] \subseteq D'$.
2. $g(\langle\rangle) = \langle\rangle$ and $g(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m, \xi\rangle) = g(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle)^\frown\langle f(\xi)\rangle$.
3. If σ has even length, $x, y \in X$, $g(\sigma^\frown\langle x\rangle) = g(\sigma)^\frown\langle x'\rangle$, $g(\sigma^\frown\langle y\rangle) = g(\sigma)^\frown\langle y'\rangle$, then $x \leq_X y$ implies $x' \leq_{X'} y'$.

Hence, an (α, α')-predendroid morphism comprises a map between parameter parts and a map $g \colon D^* \to (D')^*$ such that g follows f for the parameter parts and preserves the order of a term part. α-predendroids only encodes the 'local' part $D(\alpha)$ of a predilator X. To turn α-predendroids to global information for a predendroid, we need a way to relate α-predendroids and β-predendroids for $\alpha \leq \beta$. We introduce two ways of connection:

Definition D.5.4. Let $f \colon \alpha \to \beta$ be an increasing map, and D be an β-predendroid over X. Then we define the *multilation* ${}^fD$ as follows:

$$ {}^fD = \{\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \mid \langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m\rangle \in D\}. $$

We also define the *mutilation function* $\mathsf{m}_f^D \colon ({}^f D)^* \to D^*$ by

1. $\mathsf{m}_f^D(\langle\rangle) = \langle\rangle$.
2. $\mathsf{m}_f^D(\sigma^\frown\langle\xi\rangle) = \mathsf{m}_f^D(\sigma)^\frown\langle f(\xi)\rangle$ when σ has odd length.
3. $\mathsf{m}_f^D(\sigma^\frown\langle x\rangle) = \mathsf{m}_f^D(\sigma)^\frown\langle x\rangle$ when σ has even length.

It is not hard to prove that ${}^f D$ is an α-predendroid over X, and $(f, \mathsf{m}_f^D) \colon {}^f D \to D$. We can describe the construction of ${}^f D$ as follows:

1. From D delete all elements having a component $\xi \notin \operatorname{ran} f$.
2. From the new D, replace every components of the form $f(\xi)$ by ξ.

But mutilation only gives a way to produce an α-predendroid from a β-predendroid for a larger β. Can we produce an α-predendroid from any β-predendroid?

Definition D.5.5. Let D be an α-predendroid over X. For a linear order β, let us define the *expansion* D_β by

$$D_\beta = \big\{\langle x_0, \eta_0, \cdots, \eta_{m-1}, x_m\rangle \in [X \cup \beta]^{<\omega} \;\big|\; \exists \xi_0, \cdots \xi_{m-1} \in \alpha \\ \big[\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \in D \wedge \xi_i \mapsto \eta_i \text{ is an order-} \\ \text{preserving map } \{\xi_0, \cdots \xi_{m-1}\} \to \{\eta_0, \cdots \eta_{m-1}\}\big]\big\}.$$

Then we can prove the following:

Lemma D.5.6. *If α is an α-predendroid over X and if $f \colon \alpha \to \beta$ is an embedding, then $D \subseteq {}^f(D_\beta)$.*

However, we cannot expect equality in general. To ensure this, let us introduce a new condition for predendroids:

Definition D.5.7. An α-predendroid D is *homogeneous* if for every increasing $f, g \colon \gamma \to \alpha$, ${}^{f}D = {}^{g}D$.

The following lemma is just a restatement of the definition, but provides a useful insight into the homogeneity of predendroids:

Lemma D.5.8. *For an α-predendroid D over X, the following are equivalent:*

1. *D is homogeneous.*
2. *For two increasing $f, g \colon \gamma \to \alpha$ and $\xi_0, \cdots, \xi_{m-1} \in \gamma$, we have*

$$\langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle \in D \iff \langle x_0, g(\xi_0), \cdots, g(\xi_{m-1}), x_m \rangle \in D.$$

Then we have

Proposition D.5.9. *Let D be a homogeneous α-predendroid over X, and $f \colon \alpha \to \beta$. Then $D = {}^{f}(D_\beta)$.*

Proof. We claim ${}^{f}(D_\beta) \subseteq D$. Let $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle \in {}^{f}(D_\beta)$, so

$$\langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle \in D_\beta.$$

By definition of D_β, we have $\langle x_0, \eta_0, \cdots, \eta_{m-1}, x_m \rangle \in D$ such that $h \colon \eta_i \mapsto f(\xi_i)$ is an order-preserving map from $\{\eta_0, \cdots, \eta_{m-1}\}$ to $\{f(\xi_0), \cdots, f(\xi_{m-1})\}$. Now fix an increasing map $g \colon m \to \{\eta_0, \cdots, \eta_{m-1}\}$, then Lemma D.5.8 applied to g, $h \circ g$ implies

$$\langle x_0, \eta_0, \cdots, \eta_{m-1}, x_m \rangle \in D \iff \langle x_0, h(\eta_0), \cdots, h(\eta_{m-1}), x_m \rangle \in D$$

Hence $\langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle \in D$. Lemma D.5.8 applied to $f \restriction \{\xi_0, \cdots, \xi_{m-1}\}$ and the identity map over $\{\xi_0, \cdots, \xi_{m-1}\}$ gives

$$\langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle \in D \iff \langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle \in D. \qquad \square$$

Then can we recover D by expanding a mutilation? It is not always possible since if we mutilate D by a function $f\colon n \to \alpha$ for $n \in \mathbb{N}$ then every node of D of length longer than $(2n+1)$ would be lost in ${}^{f}D$, and we cannot recover them by expansion. However, we can recover the whole D from ${}^{f}D$ if the domain of f is $\mathbb{N}$:

Proposition D.5.10. *Let $f\colon \mathbb{N} \to \alpha$ be an increasing map. If D is a homogeneous α-predendroid over X, then $({}^{f}D)_{\alpha} = D$.*

Proof. For $D \subseteq ({}^{f}D)_{\alpha}$, suppose that $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \in D$. Pick an increasing function $g\colon n \to \alpha$ of range $\{\xi_i \mid i < n\}$. Then we have

$$\langle x_0, g^{-1}(\xi_0), \cdots g^{-1}(\xi_{m-1}), x_m\rangle \in {}^{g}D = {}^{f\restriction n}D \subseteq {}^{f}D.$$

Thus we have

$$\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle = \langle x_0, g(g^{-1}(\xi_0)), \cdots g(g^{-1}(\xi_{m-1})), x_m\rangle \in ({}^{f}D)_{\alpha}.$$

For $({}^{f}D)_{\alpha} \subseteq D$, suppose that $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \in ({}^{f}D)_{\alpha}$. This implies we have a finite subset $\{k_0, \cdots, k_{m-1}\} \subseteq \mathbb{N}$ and an increasing map $g\colon k_i \mapsto \xi_i$ such that $\langle x_0, k_0, \cdots, k_{m-1}, x_m\rangle \in {}^{f}D$, so

$$\langle x_0, f(k_0), \cdots, f(k_{m-1}), x_m\rangle \in D.$$

Let $h\colon m \to \{k_0, \cdots, k_{m-1}\}$ be an increasing enumeration. Then Lemma D.5.8 applied to $f \circ h$ and $g \circ h$ gives

$$\begin{aligned} &\langle x_0, (f\circ h)(h^{-1}(k_0)), \cdots, (f\circ h)(h^{-1}(k_{m-1})), x_m\rangle \in D \\ \iff\ &\langle x_0, (g\circ h)(h^{-1}(k_0)), \cdots, (g\circ h)(h^{-1}(k_{m-1})), x_m\rangle \in D, \end{aligned}$$

so $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \in D$. □

Hence, we have the following, stating that for a homogeneous $\mathbb{N}$-predendroid D, its expansion D_{α} is the only way to 'expand' D to an α-predendroid:

Corollary D.5.11. *Let D be a homogeneous $\mathbb{N}$-predendroid over X. If $f\colon \mathbb{N} \to \alpha$ and D' is an α-predendroid such that ${}^{f}(D') = D$, then $D' = D_{\alpha}$.* □

We will see later that homogeneous $\mathbb{N}$-predendroids correspond to predilators. Thus, let us call $\mathbb{N}$-predendroids just *predendroids*.

Definition D.5.12. A *predendroid* is a homogeneous $\mathbb{N}$-predendroid.

The following notion will be a counterpart of dilators:

Definition D.5.13. Let D be a locally well-founded predendroid over X. We say D is a *dendroid* if for every infinite well-order α, there is an α-dendroid D' such that ${}^{f}(D') = D$ (or equivalently, the expansion D_{α} is an α-dendroid.)

Note that Girard [26] called our dendroid a *strongly homogeneous α-dendroid*. The current definition of an (α, α')-predendroid morphism $(f, g)\colon D \to D'$ is a mixture of a map f between linear orders and the g part. We will separate the g part, which will correspond to an embedding between two predilators:

Definition D.5.14. Let D, D^* be predendroids. We say $g\colon D \to D'$ is a predendroid morphism if $(\mathsf{Id}_{\mathbb{N}}, g)\colon D \to D'$ is a $(\mathbb{N}, \mathbb{N})$-predendroid morphism respecting the mutilation function; That is,

1. $g\colon D^* \to (D')^*$ and $g[D] \subseteq D'$.
2. $g(\langle\rangle) = \langle\rangle$ and $g(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m, \xi\rangle) = g(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle)^\frown\langle\xi\rangle$.
3. If σ has even length, $x, y \in X$, $g(\sigma^\frown\langle x\rangle) = g(\sigma)^\frown\langle x'\rangle$, $g(\sigma^\frown\langle y\rangle) = g(\sigma)^\frown\langle y'\rangle$, then $x \leq_X y$ implies $x' \leq_{X'} y'$.
4. $\mathsf{m}^{D'}_h \circ g = g \circ \mathsf{m}^{D}_h$ for every increasing $h\colon \mathbb{N} \to \mathbb{N}$.

D.6 Linearizing and Branching of predendroids

Definition D.6.1. PreDen is the category of predendroids, whose objects are predendroids and morphisms are predendroid morphisms.

We will see that PreDen is category equivalent to the category of predilators. Let us start with the following observations on the category of ω-predendroids:

Lemma D.6.2. *Let D, D' be predendroids over X and $g\colon D \to D'$ be a predendroid morphism. If $\{a_i \mid i < m\}, \{b_i \mid i < m\} \subseteq \mathbb{N}$ be two subsets such that $a_i \mapsto b_i$ is order-preserving, then for $x_0, \cdots, x_m, y_0, \cdots, y_m \in X$,*

$$g(\langle x_0, a_0, \cdots, a_{m-1}, x_m\rangle) = \langle y_0, a_0, \cdots, a_{m-1}, y_m\rangle$$
$$\implies g(\langle x_0, b_0, \cdots, b_{m-1}, x_m\rangle) = \langle y_0, b_0, \cdots, b_{m-1}, y_m\rangle.$$

Proof. Fix an increasing function $h\colon m \to \{a_0 \cdots a_{m-1}\}$ and extend h to an increasing map $h\colon \omega \to \omega$. Then we have

$$\begin{aligned} g(\langle x_0, a_0, \cdots, a_{m-1}, x_m\rangle) &= (g \circ \mathsf{m}_h^D)(\langle x_0, h^{-1}(a_0), \cdots, h^{-1}(a_{m-1}), x_m\rangle) \\ &= \mathsf{m}_h^{D'}(g(\langle x_0, h^{-1}(a_0), \cdots, h^{-1}(a_{m-1}), x_m\rangle)). \end{aligned}$$

Similarly, if $k\colon \omega \to \omega$ is an increasing map such that $k \restriction m$ enumerates $\{b_0, \cdots, b_{m-1}\}$, then

$$g(\langle x_0, b_0, \cdots, b_{m-1}, x_m\rangle) = \mathsf{m}_k^{D'}(g(\langle x_0, k^{-1}(b_0), \cdots, k^{-1}(b_{m-1}), x_m\rangle))$$

and observe that

$$g(\langle x_0, h^{-1}(a_0), \cdots, h^{-1}(a_{m-1}), x_m\rangle) = g(\langle x_0, k^{-1}(b_0), \cdots, k^{-1}(b_{m-1}), x_m\rangle).$$

Hence if $g(\langle x_0, h^{-1}(a_0), \cdots, h^{-1}(a_{m-1}), x_m\rangle) = \langle y_0, h^{-1}(a_0), \cdots, h^{-1}(a_{m-1}), y_m\rangle$, then we have

- $g(\langle x_0, a_0, \cdots, a_{m-1}, x_m\rangle) = \langle y_0, a_0, \cdots, a_{m-1}, y_m\rangle$, and
- $g(\langle x_0, b_0, \cdots, b_{m-1}, x_m\rangle) = \langle y_0, b_0, \cdots, b_{m-1}, y_m\rangle$. □

The previous lemma allows us to extend a predendroid morphism $g\colon D \to D'$ to an (α, α)-predendroid morphism:

Definition D.6.3. Let D, D' be predendroids and $g\colon D \to D'$ a predendroid morphism. Let us define $g_\alpha\colon D_\alpha \to D'_\alpha$ by

$$g_\alpha(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle) = \langle y_0, \xi_0, \cdots, \xi_{m-1}, y_m\rangle,$$

when for the increasing enumeration $h\colon m \to \{\xi_0, \cdots, \xi_{m-1}\}$, we have

$$g(\langle x_0, h^{-1}(\xi_0), \cdots, h^{-1}(\xi_{m-1}), x_m\rangle) = \langle y_0, h^{-1}(\xi_0), \cdots, h^{-1}(\xi_{m-1}), y_m\rangle.$$

Then we can see the previous definition is correct in the following sense:

Lemma D.6.4. *If $g\colon D \to D'$ is a predendroid morphism, then $(\mathsf{Id}_\alpha, g_\alpha)\colon D_\alpha \to D'_\alpha$ is an (α, α)-predendroid morphism.*

We want to turn $\alpha \mapsto D_\alpha$ as a functor from $\mathsf{LO} \to \mathsf{LO}$. The obvious way to turn D_α a linear order is taking a Kleene-Brouwer order $\mathrm{Lin}(D)(\alpha) = (D_\alpha, <_{\mathsf{KB}})$. To make $\mathrm{Lin}(D)$ a functor, we need to define how $\mathrm{Lin}(D)$ maps an increasing map between linear orders:

Definition D.6.5. For a predendroid D and an increasing map $f\colon \alpha \to \beta$ between two linear orders α, β, let us define $\mathrm{Lin}(D)(f) = D_f\colon D_\alpha \to D_\beta$ by

$$D_f(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle) = \langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m\rangle.$$

It is easy to see that $D_f\colon (D_\alpha, <_{\mathsf{KB}}) \to (D_\beta, <_{\mathsf{KB}})$ is increasing. Hence $\mathrm{Lin}(D)$ is a functor from LO to LO. Now let us turn $\mathrm{Lin}(D)$ into a predilator by imposing a support transformation:

Definition D.6.6. Let us define $\mathrm{supp}_\alpha^{\mathrm{Lin}(D)} \colon \mathrm{Lin}(D)(\alpha) \to [\alpha]^{<\omega}$ by

$$\mathrm{supp}_\alpha^{\mathrm{Lin}(D)}(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle) = \{\xi_0, \cdots, \xi_{m-1}\}.$$

Then it is easy to check that $\mathrm{supp}^{\mathrm{Lin}(D)}$ satisfies the support condition. Moreover, we can turn predendroid morphism $g \colon D \to D'$ to a natural transformation from $\mathrm{Lin}(D) \to \mathrm{Lin}(D')$:

Proposition D.6.7. *If $g \colon D \to D'$ is a predendroid morphism, then $D'_f \circ g_\alpha = g_\beta \circ D_f$, so $\langle g_\alpha \mid \alpha \in \mathsf{LO}\rangle$ is a natural transformation from $\mathrm{Lin}(D)$ to $\mathrm{Lin}(D')$.*

Proof. Let $\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle \in D_\alpha$. Fix the increasing $h \colon n \to \{\xi_0, \cdots, \xi_{m-1}\}$ and $y_0, \cdots, y_{m-1}$ such that

$$g(\langle x_0, h^{-1}(\xi_0), \cdots, h^{-1}(\xi_{m-1}), x_m\rangle) = \langle y_0, h^{-1}(\xi_0), \cdots, h^{-1}(\xi_{m-1}), y_m\rangle.$$

Then we have

$$(D'_f \circ g_\alpha)(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle) = \langle y_0, f(\xi_0), \cdots, f(\xi_{m-1}), y_m\rangle$$

and

$$\begin{aligned}(g_\beta \circ D_f)(\langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m\rangle) &= g_\beta(\langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m\rangle)\\ &= \langle y_0, f(\xi_0), \cdots, f(\xi_{m-1}), y_m\rangle\end{aligned}$$

since f is order-preserving. □

Finally, let us check that $\mathrm{Lin}(D)$ is not only a semidilator but a predilator:

Proposition D.6.8. *Let $f, g \colon \alpha \to \beta$ be increasing functions. If $f(\xi) \leq g(\xi)$ for every $\xi \in \alpha$, then $\mathrm{Lin}(D)(f)(\sigma) \leq \mathrm{Lin}(D)(g)(\sigma)$ for every $\sigma \in \mathrm{Lin}(D)(\alpha)$.*

Proof. If $\sigma = \langle x_0, \xi_0, \cdots, \xi_{m-1}, x_m \rangle$, then $\mathrm{Lin}(D)(f)(\sigma) = \langle x_0, f(\xi_0), \cdots, f(\xi_{m-1}), x_m \rangle$. Hence if $f \leq g$ holds pointwise, then $\mathrm{Lin}(D)(f)(\sigma) \leq \mathrm{Lin}(D)(g)(\sigma)$. □

Thus Lin a functor $\mathsf{PreDen} \to \mathsf{PreDil}$. Furthermore, Lin maps dendroids to dilators:

Lemma D.6.9. *Let D be a locally well-founded predendroid. Then D_α is also locally well-founded.*

Proof. If D_α is not locally well-founded, then there is $\langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1} \rangle \in D^*_\alpha$ such that the set

$$\{x \in X \mid \langle x_0, \xi_0, \cdots, x_{m-1}, \xi_{m-1}, x \rangle \in D^*_\alpha\}$$

is ill-founded. Hence for the increasing $h \colon m \to \{\xi_0, \cdots, \xi_{m-1}\}$ then set

$$\{x \in X \mid \langle x_0, h^{-1}(\xi_0), \cdots, x_{m-1}, h^{-1}(\xi_{m-1}), x \rangle \in D^*\}$$

is also ill-founded. □

Proposition D.6.10. *If D is a dendroid, then* $\mathrm{Lin}(D)$ *is a dilator.*

Proof. We prove that $(D_\alpha, <_{\mathsf{KB}})$ is a well-order for a well-order α. D is locally well-founded by definition of being a dilator. By Lemma D.6.9, D_α is locally well-founded. Moreover, D_α has no infinite branch for each well-order α. Hence, a typical argument for the well-orderedness of a Kleene-Brouwer order shows $(D_\alpha, <_{\mathsf{KB}})$ is a well-order. □

Conversely, we can construct a predendroid from a predilator. Let us recall that by the abstract construction of a predilator, every predilator F is characterized by $<^F, \Sigma^F, \mathbf{p}^F$.

Definition D.6.11. For a predilator F and $n \in \mathbb{N}$, define $\equiv^F_n$ over $\mathrm{field}(F)$ by

$$s \equiv^F_n \iff \mathbf{p}^F(s,t) > n \vee s = t.$$

Then $\equiv_n^F$ respects $<^F$ in the following sense:

Lemma D.6.12. *$\equiv_n^F$-equivalence classes form an interval on $(F, <^F)$.*

Proof. For $s, t, u \in \text{field}(F)$, suppose that $s <^F t <^F u$ and $\mathbf{p}^F(s, u) > n$. Then $\min(\mathbf{p}^F(s, t), \mathbf{p}^(t, u)) = \mathbf{p}^F(s, u) > n$, so s, t, u are all $\equiv_n^F$-equivalent. □

Hence $(F, <^F)/\equiv_n^F$ also forms a linear order. Let us define the 'underlying order' $\mathsf{U}(F)$, which will be an underlying order of a predendorid derived from F:

Definition D.6.13.

$$\mathsf{U}(F) = \sum_{n<\omega} (F, <^F)/\equiv_n^F .$$

We are ready to define an ω-predendroid $\text{Bch}(F)$ from a predilator F:

Definition D.6.14. For a predilator F, let us define $\text{Bch}(F)$ over by the set of all

$$\langle [t]_{\equiv_0^F}, a_{\Sigma_t^F(0)}, \cdots, a_{\Sigma_t^F(m-1)}, [t]_{\equiv_m^F} \rangle$$

for $t \in \text{field}(F)$, $\text{arity}(t) = m$, $a_0 < \cdots < a_{m-1} < \omega$.

Then we can see that $\text{Bch}(F)$ is a predendroid over $\mathsf{U}(F)$. To turn Bch to a functor, let us define

Definition D.6.15. Let $\iota \colon F \to G$ be a predilator embedding. We define $\text{Bch}(\iota) \colon \text{Bch}(F) \to \text{Bch}(G)$ by

$$\text{Bch}(\iota)(\langle [t]_{\equiv_0^F}, a_{\Sigma_t^F(0)}, \cdots, a_{\Sigma_t^F(m-1)}, [t]_{\equiv_m^F} \rangle) = \langle [\iota(t)]_{\equiv_0^G}, a_{\Sigma_{\iota(t)}^G(0)}, \cdots, a_{\Sigma_{\iota(t)}^G(m-1)}, [\iota(t)]_{\equiv_m^G} \rangle.$$

Note in the previous definition that $\Sigma_t^F = \Sigma_{\iota(t)}^G$, so $\text{Bch}(\iota)$ does not change the parameter part. Thus we can see that $(\mathsf{Id}_\omega, \text{Bch}(\iota)) \colon \text{Bch}(F) \to \text{Bch}(G)$ is an ω-dendroid morphism. Moreover,

Proposition D.6.16. $\mathrm{Bch}(\iota)$ *is a predendroid morphism.*

Proof. It suffices to see that for every increasing $h\colon \omega \to \omega$, $\mathsf{m}_h^{\mathrm{Bch}(G)} \circ \mathrm{Bch}(\iota) = \mathrm{Bch}(\iota) \circ \mathsf{m}_h^{\mathrm{Bch}(F)}$, which follows from a tedious computation. □

Thus $\mathrm{Bch}\colon \mathsf{PreDil} \to \mathsf{PreDen}$ is also a functor. The next goal is to show that Bch maps a dilator to a strongly homogeneous dendroid, but let us postpone its proof by proving the category equivalence between PreDen and PreDil.

Theorem D.6.17. *There are natural isomorphisms* $\mathsf{edil}\colon \mathsf{Id}_{\mathsf{PreDen}} \to \mathrm{Lin} \circ \mathrm{Bch}$ *and* $\mathsf{eden}\colon \mathsf{Id}_{\mathsf{PreDil}} \to \mathrm{Bch} \circ \mathrm{Lin}$.

Its proof is decomposed into parts. First, let us define edil as follows:

Definition D.6.18. Let F be a predilator. For a given linear order α, let us define

$$\mathsf{edil}_{F,\alpha}(t(a)) = \langle [t]_{\equiv_0^F}, a_{\Sigma_t^F(0)}, \cdots, a_{\Sigma_t^F(m-1)}, [t]_{\equiv_m^F} \rangle$$

for $t \in F$, $a = \{a_0, \cdots, a_{m-1}\} \in [\alpha]^{<\omega}$.

Then $\mathsf{edil}_{F,\alpha}\colon F(\alpha) \to (\mathrm{Bch} \circ \mathrm{Lin}(F))_\alpha$ is onto. To see $\mathsf{edil}_{F,\alpha}$ is an order-preserving map, observe that for $s, t \in F$ and $a, b = \{b_0, \cdots, b_{n-1}\}$, $F(\alpha) \vDash s(a) < t(b)$ iff either

1. $s <^F t$ and for every $i < \mathbf{p}^F(s,t)$, $a_{\Sigma_s^F(i)} = b_{\Sigma_t^F(i)}$, or
2. there is $q < \mathbf{p}^F(s,t)$ such that $a_{\Sigma_s^F(q)} < b_{\Sigma_t^F(q)}$ and for every $i < q$, $a_{\Sigma_s^F(i)} = b_{\Sigma_t^F(i)}$.

We can see in either cases, we have $\mathsf{edil}_{F,\alpha}(s(a)) < \mathsf{edil}_{F,\alpha}(t(b))$:

1. In the first case, let $p = \mathbf{p}^F(s,t)$. Then

$$\mathsf{U}(F) \vDash [s]_{\equiv_p^F} < [t]_{\equiv_p^F} \tag{D.2}$$

since $s <^F t$ and $s \not\equiv_p^F t$. Since $s \equiv_i^F t$ and $a_{\Sigma_s^F(i)} = b_{\Sigma_t^F(i)}$ for every $i < p$, $\mathsf{edil}_{F,\alpha}(s(a))$ and $\mathsf{edil}_{F,\alpha}(t(b))$ differs first at $(2p+1)$-th position, which is (D.2).

2. A similar argument works by replacing $p = q$. In this case, $\mathsf{edil}_{F,\alpha}(s(a))$ and $\mathsf{edil}_{F,\alpha}(t(b))$ differs first at $(2q)$-th position.

Thus $\mathsf{edil}_{F,\alpha} \colon F(\alpha) \cong (\mathrm{Lin} \circ \mathrm{Bch}(F))(\alpha)$. Moreover, a simple computation gives

Lemma D.6.19.

1. $\mathsf{edil}_F \colon F \to \mathrm{Lin} \circ \mathrm{Bch}(F)$ *is a predilator isomorphism.*
2. edil_F *is natural in* F.

Hence $\mathsf{edil} \colon \mathsf{Id}_{\mathsf{PreDil}} \to \mathrm{Lin} \circ \mathrm{Bch}$ is a natural isomorphism. Before defining eden, let us recall the following:

Remark D.6.20. For a predendroid D over X, elements of $\mathrm{Tr}(\mathrm{Lin}(D))$ are precisely of the form $(m, \langle x_0, a_0, \cdots, a_{m-1}, x_m \rangle)$ such that $\{a_0, \cdots, a_{m-1}\} = m$ and $x_0, \cdots, x_m \in X$.

We have defined $\mathrm{Lin}(D)$ as a predilator à la Freund, and we can turn it into a denotation system of field $\mathrm{Tr}(F)$. (See [39, §3.4] for the details.) Thus for each $\langle x_0, a_0, \cdots, a_{m-1}, x_n \rangle \in D$ if we define

$$\mathrm{term}(\langle x_0, a_0, \cdots, a_{m-1}, x_n \rangle) = \langle x_0, \mathrm{en}_a^{-1}(a_0), \cdots, \mathrm{en}_a^{-1}(a_{m-1}), x_n \rangle$$

for an increasing map $\mathrm{en}_a \colon m \to a = \{a_0, \cdots, a_{m-1}\}$, we may view term as a function taking an element of D that can be identified with $\mathrm{Lin}(D)(\omega)$, and separating the term part of the element.

Definition D.6.21. For $\sigma = \langle x_0, a_0, \cdots, a_{m-1}, x_m \rangle \in D$, let us define $\mathsf{eden}_D \colon D \to (\mathrm{Bch} \circ \mathrm{Lin})(D)$ by

$$\mathsf{eden}_D(\sigma) = \langle [\mathrm{term}(\sigma)]_{\equiv_0^{\mathrm{Lin}(D)}}, a_0, \cdots, a_{m-1}, [\mathrm{term}(\sigma)]_{\equiv_m^{\mathrm{Lin}(D)}} \rangle.$$

Again, by definition, eden_D is onto. To see eden_D is a predendroid morphism, let us check the following:

Lemma D.6.22. *Suppose that* $\sigma = \langle x_0, a_0, \cdots, a_{m-1}, x_m \rangle$ *and* $\tau = \langle y_0, b_0, \cdots, b_{n-1}, y_n \rangle$ *are elements of* $\mathrm{Lin}(D)$ *such that* $(m, \sigma), (n, \tau) \in \mathrm{Tr}(\mathrm{Lin}(D))$. *Then*

1. $\Sigma_\sigma^{\mathrm{Lin}(D)}(i) = a_i$, $\Sigma_\tau^{\mathrm{Lin}(D)}(j) = b_j$ *for every* $i < m$, $j < n$.
2. *If there is* $k \leq \min(m, n)$ *such that* $\sigma \restriction (2k) = \tau \restriction (2k)$ *and* $x_k < y_k$. *Then* $\mathbf{p}^{\mathrm{Lin}(D)}(\sigma, \tau) = k$ *and* $\sigma <^{\mathrm{Lin}(D)} \tau$.

Proof. For the first claim, observe that $m = \{a_0, \cdots, a_{m-1}\}$. If we view σ as a $\mathrm{Lin}(D)$-term and $f \colon m \to \mathbb{N}$, then $\sigma(\mathrm{ran}\, f)$ is $\mathrm{Lin}(D)(f)(\sigma) = \langle x_0, f(a_0), \cdots, f(a_{m-1}), x_m \rangle$. We can see that when we compare $\sigma(\mathrm{ran}\, f)$ and $\sigma(g)$, then we compare the first component of the tuple, which is the a_0th element of the range of $\mathrm{ran}\, f$ and $\mathrm{ran}\, g$. This shows $\Sigma_\sigma^{\mathrm{Lin}(D)}(0) = a_0$. By a similar reason, we also have $\Sigma_\sigma^{\mathrm{Lin}(D)}(i) = a_i$.

For the second claim, observe that for $f \colon m \to \mathbb{N}$, $g \colon n \to \mathbb{N}$, if $f(a_i) = g(b_i)$ for $i < k$, then $\mathrm{Lin}(D)(\mathbb{N}) \vDash \sigma(\mathrm{ran}\, f) < \tau(\mathrm{ran}\, g)$ no matter what the values of other parts of f, g are. It shows $\mathbf{p}^{\mathrm{Lin}(D)}(\sigma, \tau) \geq k$. However, if $f(a_i) > g(b_i)$ for some $i < k$, then $\mathrm{Lin}(D)(\mathbb{N}) \vDash \sigma(\mathrm{ran}\, f) > \tau(\mathrm{ran}\, g)$. Hence k is the least secure number relative to σ and τ, so $\mathbf{p}^{\mathrm{Lin}(D)}(\sigma, \tau) = k$. $\sigma <^{\mathrm{Lin}(D)} \tau$ follows automatically. □

Lemma D.6.23. *For an increasing* $h \colon \omega \to \omega$, *we have* $\mathsf{m}_h^{\mathrm{Bch}(\mathrm{Lin}(D))} \circ \mathsf{eden}_D = \mathsf{eden}_D \circ \mathsf{m}_h^D$. *Furthermore,* eden_D *is natural in* D.

Proof. For the equality, suppose that $\sigma = \langle x_0, a_0, \cdots, a_{m-1}, x_m \rangle \in D$. Then

$$\mathsf{m}_h^D(\sigma) = \langle x_0, h(a_0), \cdots, h(a_{m-1}), x_m \rangle.$$

Let us observe that $\mathrm{term}(\sigma) = \mathrm{term}(\mathsf{m}_h^D(\sigma))$ since $\mathrm{en}_a^{-1}(a_i) = \mathrm{en}_{h[a]}^{-1}(h[a]_i)$ by monotonicity of h. Hence

$$\begin{aligned}\mathsf{eden}_D(\mathsf{m}_h^D(\sigma)) &= \langle [\mathrm{term}(\sigma)]_{\equiv_0^{\mathrm{Lin}(D)}}, h(a_0), \cdots, h(a_{m-1}), [\mathrm{term}(\sigma)]_{\equiv_m^{\mathrm{Lin}(D)}} \rangle \\ &= (\mathsf{m}_h^{\mathrm{Bch}(\mathrm{Lin}(D))} \circ \mathsf{eden}_D)(\sigma).\end{aligned}$$

For the naturality, suppose that $g \colon D \to D'$ is a predendroid morphism. For $\sigma = \langle x_0, a_0, \cdots, a_{m-1}, x_m \rangle \in D$, we have

$$\mathsf{eden}_{D'}(g(\sigma)) = \langle [\mathrm{term}(g(\sigma))]_{\equiv_0^{\mathrm{Lin}(D)}}, a_0, \cdots, a_{m-1}, [\mathrm{term}(g(\sigma))]_{\equiv_m^{\mathrm{Lin}(D)}} \rangle.$$

But

$$\begin{aligned}&\mathrm{Bch}(\mathrm{Lin}(g))(\mathsf{eden}_D(\sigma)) \\ &\qquad = \langle [\mathrm{Lin}(g)(\mathrm{term}(\sigma))]_{\equiv_0^{\mathrm{Lin}(D)}}, a_0, \cdots, a_{m-1}, [\mathrm{Lin}(g)(\mathrm{term}(\sigma))]_{\equiv_m^{\mathrm{Lin}(D)}} \rangle,\end{aligned}$$

and $\mathrm{Lin}(g)(\mathrm{term}(\sigma)) = \mathrm{term}(g(\sigma))$, proving $\mathsf{eden}_{D'} \circ g = \mathrm{Bch}(\mathrm{Lin}(g)) \circ \mathsf{eden}_D$. □

Hence $\mathsf{eden} \colon \mathsf{Id}_{\mathsf{PreDen}} \to \mathrm{Bch} \circ \mathrm{Lin}$ is a natural isomorphism, which proves Theorem D.6.17. We finish this subsection with the following promised theorem:

Theorem D.6.24. *If F is a dilator, then $\mathrm{Bch}(F)$ is strongly homogeneous.*

First, let us prove that $\mathrm{Bch}(F)$ is locally well-founded:

Lemma D.6.25. *If F is a dilator, then $(F, <^F)/\equiv_n^F$ is well-ordered. Hence $\mathsf{U}(F)$ is also well-ordered and so $\mathrm{Bch}(F)$ is locally well-founded.*

Proof. If $(F, <^F)/\equiv^F_n$ is ill-founded, then we have a sequence of F-terms $t_0 >^F t_1 >^F t_2 >^F \cdots$ such that for each i, $\mathbf{p}^F(t_i, t_{i+1}) \leq n$. Let us fix $m \leq n$ such that infinitely many i satisfies $\mathbf{p}^F(t_i, t_{i+1}) = m$, and we may assume that the equality holds for every i.

To get a contradiction, we claim that $F(\omega \cdot (m+1))$ is ill-founded: From $\mathbf{p}^F(t_i, t_{i+1}) = m$, we have that $\mathrm{arity}(t_i) \geq m$. For each i, take $a_i = \{a_i(k) \mid k < \mathrm{arity}\, t_i\} \subseteq \omega \cdot (m+1)$ such that

$$\{a_i(\Sigma_{t_i}(k)) \mid k < m\} = \{\omega \cdot (1+k) \mid k < m\}.$$

The idea for the choice of a_i is that a_i and a_j agree with the first m elements of the highest comparison priority, so the comparison between $t_i(a_i)$ and $t_{i+1}(a_{i+1})$ only follows $<^F$. That is, we have $F(\omega \cdot (m+1)) \vDash t_i(a_i) > t_{i+1}(a_{i+1})$ for every i, a contradiction. □

Proof. Proof of Theorem D.6.24 We claim that the expansion $\mathrm{Bch}(F)_\alpha$ has no infinite branch. Since $\mathrm{Bch}(F)_\alpha$ is also locally well-founded by Lemma D.6.9, the assertion '$\mathrm{Bch}(F)_\alpha$ has no infinite branch' is equivalent to the well-foundedness of $(\mathrm{Bch}(F)_\alpha, <_{\mathsf{KB}}) = \mathrm{Lin}(\mathrm{Bch}(F))(\alpha)$. However, $\mathrm{Lin}(\mathrm{Bch}(F))(\alpha) \cong F(\alpha)$, so it is well-founded. □

Note that the well-foundedness of $\mathsf{U}(F)$ is equivalent to the local well-foundedness:

Proposition D.6.26. *For a predilator F, if $\mathsf{U}(F)$ is well-founded, then $\mathrm{Bch}(F)$ is locally well-founded. Conversely, if D is a locally well-founded predendroid, then $\mathsf{U}(\mathrm{Lin}(D))$ is well-founded.*

Proof. The first assertion is clear. For the second assertion, if $\mathsf{U}(\mathrm{Lin}(D))$ is ill-founded, then one of $(\mathrm{Lin}(D), <^{\mathrm{Lin}(D)})/\equiv^{\mathrm{Lin}(D)}_n$ must be ill-founded for some $n \in \mathbb{N}$. Observe that we can view $\mathrm{Lin}(D)$ as a predilator with the set of terms $\mathrm{Tr}(\mathrm{Lin}(D))$, and

for $(m,\sigma),(m',\sigma') \in \mathrm{Tr}(\mathrm{Lin}(D))$,

$$(m,\sigma) <^{\mathrm{Lin}(D)} (m',\sigma') \iff \exists k \leq \min(m,m')[\sigma \restriction 2k = \sigma' \restriction 2k \wedge \sigma(2k) < \sigma'(2k)].$$

Let $\langle (m_i,\sigma_i) \mid i \in \mathbb{N}\rangle$ be an $<^{\mathrm{Lin}(D)}$-decreasing sequence such that $(m_i,\sigma_i) \not\equiv_n^{\mathrm{Lin}(D)} (m_j,\sigma_j)$ for all $i<j \in \mathbb{N}$, which is equivalent to $\sigma_i \restriction (2n+2) \neq \sigma_j \restriction (2n+2)$ by Lemma D.6.22.

For each $i<j \in \mathbb{N}$, let $k=k(i,j)$ be the unique natural number satisfying $\sigma_i \restriction 2k \neq \sigma_j \restriction 2k$ and $\sigma_i(2k) \neq \sigma_j(2k)$. Since $(m_i,\sigma_i) >^{\mathrm{Lin}(D)} (m_j,\sigma_j)$, we have$\sigma_i(2k) > \sigma_j(2k)$. Then $k(i,j) \leq n$ for every $i<j$, so we may assume that $k(i,j)$ is constant by Ramsey's theorem for pairs. Hence $\{x \mid \sigma_0 \restriction (2k)^\frown\langle x\rangle \in D^*\}$ has an infinite descending sequence $\sigma_0(2k) > \sigma_1(2k) > \cdots$, so D is not locally well-founded. □

D.7 Quasidendroids

Defining β-logic requires a 'functorial' proof tree for well-orders. Girard [26] formulated functorial trees under the name *quasidendroid*. However, quasidendroids correspond to dilators and not predilators, and we also need a pre- notion for quasidendroids. Hence, we define the prequasidendroid first, and this makes our definition slightly different from that of Girard since Girard's definition does not fit in the pre-notion of quasidendroids.

Definition D.7.1. Let X and α be linear orders. An *α-prequasidendroid over X* is a subset D of a tree over $X \sqcup \alpha$, the disjoint union of X and α, satisfying the following:

1. D does *not* contain an initial segment of an element of D.

The next condition for the definition requires new terminology. We call elements of X the *term part of D*, and elements of α the *parameter part of D*. We use underlined

elements $\underline{\xi}, \underline{\eta}$ to distinguish parameter parts from term parts.

Now let D^* be the closure of D under an initial segment. The next condition says the 'successor' of a node in D^* must be either all underlined or all non-underlined:

2. For every $\sigma \in D^*$, it cannot be the case that there are $x \in X$ and $\xi \in \alpha$ such that $\sigma^\frown\langle x\rangle \in D^*$ and $\sigma^\frown\langle \underline{\xi}\rangle \in D^*$.

We say D is an α-*quasidendroid* if D additionally satisfies the following:

3. (Local well-foundedness) For each $\sigma \in D^*$, the set

$$\{x \in X \mid \exists \xi \in \alpha[\sigma^\frown\langle (x, \xi)\rangle \in D^*]\}$$

   is well-ordered.

4. There is no infinite branch over D^*.

Girard had not defined the 'pre-' notion for the quasidendroid. We define it since an 'ill-founded prequasidendroid' has a more significant rule in this paper. Local well-foundedness in our definition of quasidendroid is important; Otherwise, the 'linearized' quasidendroid can have an infinite decreasing sequence even when there is no infinite branch. Girard only considered the case when the term part X is a set of ordinals. But we let X be a general linear order, which should allow every predilator to be expressed by a prequasidendroid.

We can get a new α-prequasidendroid from a β-prequasidendroid for $\beta > \alpha$ by *mutilation*:

Definition D.7.2. Let D be a β-prequasidendroid over X and $f : \alpha \to \beta$ be an increasing map. Let us construct the *mutilation* ${}^f D$ as follows: Starting from D,

1. Remove $\sigma \in D$ if σ has a component of the form $\underline{\xi}$ for some $\xi \notin \operatorname{ran} f$.
2. Replace every $\underline{f(\eta)}$ with $\underline{\eta}$.

If we extend f to a function from $X \sqcup \alpha$ to $X \sqcup \beta$ by letting $f(x) = x$ for $x \in X$, then we can rephrase the previous construction of ${}^fD$ into the following equality:

$$ {}^fD = \{\langle \sigma_i \mid i < n\rangle \mid \langle f(\sigma_i) \mid i < n\rangle \in D\}, $$

We also define the *mutilation function* $\mathsf{m}_f^D \colon ({}^fD)^* \to D^*$ by

1. $\mathsf{m}_f^D(\langle\rangle) = \langle\rangle$.
2. $\mathsf{m}_f^D(\sigma^\frown\langle x\rangle) = \mathsf{m}_f^D(\sigma)^\frown\langle x\rangle$ when x is a term part.
3. $\mathsf{m}_f^D(\sigma^\frown\langle\underline{\xi}\rangle) = \mathsf{m}_f^D(\sigma)^\frown\langle\underline{f(\xi)}\rangle$ when $\underline{\xi}$ is a parameter part.

It is easy to check that ${}^fD$ is an α-prequasidendroid over X. Moreover, ${}^fD$ is an α-quasidendroid if D is β-quasidendroid, and $\sigma \in {}^fD$ iff $\mathsf{m}_f^D(\sigma) \in D$.

In the definition of the mutilation function, we see that m_f^D does not change the term part but changes the parameter part. We will often see modifying parameter parts of $\sigma \in D$ by an increasing function for a parameter, so let us employ the following notation:

Definition D.7.3. Let $f \colon \alpha \to \beta$ be an increasing function between two linear orders α and β, and suppose that every element of α and β is underlined. For a linear order X whose elements are *not* underlined, we define the function $\mathsf{m}_f \colon [X \sqcup \alpha]^{<\omega} \to [X \sqcup \beta]^{<\omega}$ by

1. $\mathsf{m}_f(\langle\rangle) = \langle\rangle$.
2. $\mathsf{m}_f(\sigma^\frown\langle x\rangle) = \mathsf{m}_f(\sigma)^\frown\langle x\rangle$ when x is not underlined.
3. $\mathsf{m}_f(\sigma^\frown\langle\underline{\xi}\rangle) = \mathsf{m}_f(\sigma)^\frown\langle\underline{f(\xi)}\rangle$ when $\underline{\xi}$ is underlined.

From an α-prequasidendroid D and $\beta > \alpha$, we can get a new β-prequasidendroid by *instantiation*:

Definition D.7.4. For an α-prequasidendroid D over X and $\beta \geq \alpha$, define its β-*instantiation* D_β as follows: Starting from D,

1. For each $\sigma \in D$, fix the set $\mathrm{Occ}(\sigma)$ of underlied elements occurring in σ.
2. For every increasing $f\colon \mathrm{Occ}(\sigma) \to \beta$, consider a copy of σ in which every underlined element is replaced by its value by f.
3. Collect every modified σ for each $f\colon \mathrm{Occ}(\sigma) \to \beta$, and form the set D_β.

Again, if we expand f into a function $X \sqcup \alpha \to X \sqcup \beta$ by letting $f(x) = x$, then we can rephrase the definition of D_β as follows:

$$D_\beta = \{\langle f(\sigma_i) \mid i < n\rangle \mid \langle \sigma_i \mid i < n\rangle \in D \wedge f\colon \mathrm{Occ}(\langle \sigma_i \mid i < n\rangle) \to \beta \text{ increasing}\}.$$

It is also easy to see that D_β is a β-prequasidendroid. However, we do not know if D_β is a β-quasidendroid when D is an α-quasidendroid, since we do not know if D_β has an infinite branch even when D has no infinite branch. Note that if D is locally well-founded, then so is D_β, so the presence of an infinite branch is the only reason why D_β can fail to be a β-quasidendroid.

The instantiation should be the inverse operator of mutilation, and the following lemma gives a partial positive answer for the expectation:

Lemma D.7.5. *Let D be an α-prequasidendroid over X and $\beta \geq \alpha$. For an increasing $f\colon \alpha \to \beta$, we have $D \subseteq {}^f(D_\beta)$.*

Proof. For $D \subseteq {}^{f}(D_\beta)$, let $\langle \sigma_i \mid i < n \rangle \in D$. Then $\langle f(\sigma_i) \mid i < n \rangle \in D_\beta$ by definition of D_β, so $\langle \sigma_i \mid i < n \rangle \in {}^{f}(D_\beta)$. □

However, there is no reason in general that the reversed inclusion also holds. Even worse, there is no guarantee that the instantiation of a mutilation 'restores' an α-prequasidendroid. It requires an additional property called *homogeneity*:

Definition D.7.6. An α-prequasidendroid over X is *homogeneous* if for every increasing map $f, g \colon \gamma \to \alpha$, ${}^{f}D = {}^{g}D$.

Note that the homogeneity of D is equivalent to the following condition: For two increasing $f, g \colon \gamma \to \alpha$ and any finite sequence σ over $X \sqcup \gamma$, we have $\mathsf{m}_f^D(\sigma) \in D$ iff $\mathsf{m}_g^D(\sigma) \in D$. Then we can say that the mutilation of an instantiation restores a homogeneous prequasidendroid:

Lemma D.7.7. *Let D be a homogeneous α-prequasidendroid over X and $f \colon \alpha \to \beta$. Then $D = {}^{f}(D_\beta)$.*

Proof. We claim that ${}^{f}(D_\beta) \subseteq D$. Suppose that $\sigma \in {}^{f}(D_\beta)$, so $\mathsf{m}_f(\sigma) \in D_\beta$. By definition of D_β, there is $\tau \in D$ and an order isomorphism $h \colon \operatorname{Occ}(\tau) \to \operatorname{Occ}(\mathsf{m}_f(\sigma))$ such that $\mathsf{m}_h(\tau) = \mathsf{m}_f(\sigma)$. Observe that $|\operatorname{Occ}(\tau)| = |\operatorname{Occ}(\mathsf{m}_f(\sigma))| = |\operatorname{Occ}(\sigma)|$, and say it m. Then fix an increasing enumeration $g \colon m \to \operatorname{Occ}(\tau)$ and $k \colon m \to \operatorname{Occ}(\sigma)$, and let $\sigma' = \mathsf{m}_k^{-1}(\sigma)$, $\tau' = \mathsf{m}_g^{-1}(\tau)$. Since $h \circ g$ and $f \circ k$ have the same domain and range that are finite, we have $h \circ g = f \circ k$. From $\mathsf{m}_{h \circ g}(\tau') = \mathsf{m}_{f \circ k}(\sigma')$, we get $\sigma' = \tau' \in {}^{g}D$. Homogeneity implies ${}^{g}D = {}^{k}D$, so $\sigma' \in {}^{k}D$, implying $\sigma = \mathsf{m}_k(\sigma) \in D$. □

The next lemma says instantiating a mutilation of a homogeneous α-prequasidendroid results in the previous α-prequasidendroid when $\alpha = \omega$:

Lemma D.7.8. *If $f\colon \omega \to \beta$ is an increasing map and D is a homogeneous β-predendroid, then $({}^{f}D)_{\beta} = D$.*

Proof. Let us prove $D \subseteq ({}^{f}D)_{\beta}$: Suppose that $\sigma \in D$. Take $m = |\mathrm{Occ}(\sigma)|$, and fix an increasing enumeration $h\colon m \to \mathrm{Occ}(\sigma)$. Then we have

$$\mathsf{m}_{h}^{-1}(\sigma) \in {}^{h}D = {}^{f\restriction n}D \subseteq {}^{f}D.$$

For $({}^{f}D)_{\beta} \subseteq D$, suppose that $\sigma \in ({}^{f}D)_{\beta}$. Thus we have $\tau \in {}^{f}D$ and an increasing map $g\colon \mathrm{Occ}(\tau) \to \beta$ satisfying $\mathsf{m}_{g}(\tau) = \sigma$. If $\imath\colon \mathrm{Occ}(\tau) \to \alpha$ is the inclusion map, then ${}^{f\circ\imath}D = {}^{g}D \ni \tau$. Hence $\sigma = \mathsf{m}_{f\circ\imath}(\tau) \in D$. □

Hence, if we are given a homogeneous ω-prequasidendroid, we can produce an α-prequasidendroid for every linear order α. It is similar to that for a predilator F, we can compute $F(\alpha)$ for every α from $F(\omega)$. Thus let us impose a homogeneous ω-prequasidendroid a special name:

Definition D.7.9. D is a *prequasidendroid* if D is a homogeneous ω-prequasidendroid. A prequasidendroid D is a *quasidendroid* if D_{α} is an α-quasidendroid for every well-order α.

Note that Girard [26] called our quasidendroid a *strongly homogeneous quasidendroid*.

Remark D.7.10. It is convenient to view a prequasidendroid as a *functorial* sequence of α-prequasidendroid $\langle D_{\alpha} \mid \alpha \in \mathsf{WO}\rangle$, where the functoriality means that ${}^{f}(D_{\beta}) = D_{\alpha}$ holds for every α, β and an increasing $f\colon \alpha \to \beta$. In one direction, the ω-prequasidendroid D_{ω} in the functorial sequence is homogeneous, so it is a prequasidendroid. In the other direction, Lemma D.7.8 implies we can recover D_{α} from D_{ω} for $\alpha \geq \omega$ by expansion, and the combination of Lemmas D.7.5 and D.7.7 implies we can

obtain D_n from D_ω for $n < \omega$. Understanding prequasidendroids as functorial sequences of α-prequasidendroids is what prior materials for dilators and β-logics take.

Now let D be a prequasidendroid over X. If X is a subset of $\mathbb{N}$, then we can code D as a subset of $\mathbb{N}$. This allows us to talk about D being recursive:

Definition D.7.11. Let D be a prequasidendroid over $\mathbb{N}$. We say D is *(primitive) recursive* if D as a subset of $\mathbb{N}$ is (primitive) recursive.

We can turn a prequasidendroid into a predilator. Girard used the *linearization* $\mathrm{Lin}(D)$ given by $\mathrm{Lin}(D)(\alpha) := (D_\alpha, <_{\mathsf{KB}})$. We will use a slightly different linearization construction, which is more useful in our context.

Definition D.7.12. Let D be a prequasidendroid. Define the *hull-linearization* $\mathrm{Lin}^*(D)$ *of* D by $\mathrm{Lin}^*(D)(\alpha) = (D^*_\alpha, \leq_{\mathsf{KB}})$. For $f\colon \alpha \to \beta$, we define $\mathrm{Lin}^*(D)(f)(\sigma) = \mathsf{m}^D_f(\sigma)$.

Then we have the following:

Proposition D.7.13. *Let D be a prequasidendroid. Then $\mathrm{Lin}^*(D)$ is a predilator. Moreover, for a locally well-founded D and a well-order α, D^*_α has an infinite branch iff $\mathrm{Lin}^*(D)(\alpha)$ is ill-founded.*

Proof. We will define an appropriate support transformation to turn $\mathrm{Lin}^*(D)$ into a semidilator. For $\sigma \in D^*_\alpha$, define $\mathrm{supp}_\alpha(\sigma) = \mathrm{Occ}(\sigma)$. Then the direction computation gives $[f]^{<\omega} \circ \mathrm{supp}_\alpha = \mathrm{supp}_\beta \circ \mathrm{Lin}^*(D)(f)$. To check the support condition, fix $f\colon \alpha \to \beta$ and $\tau \in \mathrm{Lin}^*(D)(\beta)$ such that $\mathrm{supp}_\beta(\tau) \subseteq \mathrm{ran}\, f$. Then every underlined component of τ is a member of f, so $\mathsf{m}^{-1}_f(\tau)$ is well-defined. Clearly $\tau = \mathsf{m}_f(\mathsf{m}^{-1}_f(\tau)) \in \mathrm{ran}(\mathrm{Lin}^*(D)(f))$.

To see $\mathrm{Lin}^*(D)$ is a predilator, let us check the monotonicity condition: That is, for two increasing $f, g\colon \alpha \to \beta$ such that $f \leq g$ pointwise, we have $\mathrm{Lin}^*(D)(f) \leq$

$\mathrm{Lin}^*(D)(g)$ pointwise. However, one can see that for $\sigma \in \mathrm{Lin}^*(D)(\alpha)$ and $i < |\sigma|$, the ith component of $\mathsf{m}_f(\sigma)$ is less than or equal to $\mathsf{m}_g(\sigma)$, so $\mathsf{m}_f(\sigma) \leq_{\mathsf{KB}} \mathsf{m}_g(\sigma)$.

The last condition follows from the fact that if T is a tree over a well-order, then T has an infinite branch if and only if $(T, \leq_{\mathsf{KB}})$ is ill-founded. □

As a corollary, we have

Corollary D.7.14. *Let D be a prequasidendroid. Then D is a quasidendroid iff* $\mathrm{Lin}^*(D)$ *is a dilator. Furthermore, if D is locally well-founded and* $\mathrm{Lin}^*(D)$ *is not a dilator, then* $\mathsf{Clim}(\mathrm{Lin}^*(D))$ *is the least ordinal such that D^*_α has an infinite branch.* □

APPENDIX E

PTYKES

Sur les crédences, au salon vide: nul ptyx,
Aboli bibelot d'inanité sonore,
Car le Maître est allé puiser des pleurs au Styx
Avec ce seul objet dont le Néant s'honore.

Stéphane Mallarmé, "Sonnet en X"

Ptykes are algebraic objects defined by Girard to represent Π^1_{n+1}-formulas. Its name originated from the puzzling word 'ptyx' that appeared in Stéphane Mallarmé's poem *Sonnet en X*. Girard defined n-ptykes as a functor from the category of $(n-1)$-ptykes to the category of well-orders preserving direct limits and pullbacks. 0-ptykes are well-orders, and 1-ptykes are dilators.[1] Girard and Ressayre analyzed basic properties of n-ptykes in [31], which is written in French, which is unfortunate for non-French speakers.

In this chapter, we develop the theory of 2-ptykes. Most literature calls 2-ptykes just ptykes, but we reserve the word ptyx for general objects. The reader could find that most arguments in Appendix B and Appendix C carry over in a more complex form. One reason for the complexity is that we simultaneously use semidilators and semidendrograms, so we need to convert one into the other. Another reason is that semidilators (or better to say, semidendrograms) have a tree structure, and we often need to consider 'hidden' terms to convert proofs for dilators to proofs for ptykes. I could not complete every detail on general ptykes in this dissertation, but at least for n-ptykes, most basic facts about 2-ptykes also hold for n-ptykes *mutatis mutandis*. Many ideas of the first three sections

[1] I was asked why n-ptykes are not named as n-dilators. I would agree with this suggestion, but it is too late to change the terminology. For future terminologies, we should avoid making new words for the ptyx-version of dilator notions. For example, a 2-ptyx version of flower should be 2-flower, and should not use a new word.

are from [31], and Section E.4 is influenced by Freud [22].

E.1 Denotation system of 2-semiptykes

Definition E.1.1. A *1-arity diagram* Д is a commutative diagram over the category of natural semidendrogramoids with *semidendrogram quasimorphisms* of the form

$$\text{Д} = \begin{array}{ccc} a_\cap & \overset{f_{\cap,1}}{\dashrightarrow} & a_1 \\ {\scriptstyle f_{\cap,0}}\,\vdots & & \downarrow{\scriptstyle f_{1,\cup}} \\ a_0 & \xrightarrow[f_{0,\cup}]{} & a_\cup \end{array} \tag{E.1}$$

such that both $f_{0,\cup}$ and $f_{1,\cup}$ are morphisms (so they preserve terminal points), the above diagram is a pullback, and $\operatorname{ran} f \cup \operatorname{ran} g = a_\cup$. We say a 1-arity diagram is *trivial* if $a_\cap = a_0 = a_1 = a_\cup$. For a 1-arity diagram of the form (E.1), the diagram $-$Д is

$$-\text{Д} = \begin{array}{ccc} a_\cap & \overset{f_{\cap,0}}{\dashrightarrow} & a_0 \\ {\scriptstyle f_{\cap,1}}\,\vdots & & \downarrow{\scriptstyle f_{0,\cup}} \\ a_1 & \xrightarrow[f_{1,\cup}]{} & a_\cup \end{array}$$

As we did in Appendix B, we understand a 1-arity diagram as a functor from the category $\mathtt{L}_2$ to the category of natural semidendrogramoids with semidengrogram quasimorphisms.

Remark E.1.2. The condition '$f_{\cap,0}$ and $f_{\cap,1}$ are semidendrogram quasimorphisms' in Definition E.1.1 cannot be fortified by '$f_{\cap,0}$ and $f_{\cap,1}$ are semidendrogram morphisms,' otherwise we cannot consider the term configurations of two finite semidilators with their overlapping parts hidden, like illustrated in Example D.3.9.

We also use the following notation as we did in Appendix B:

$$\text{Д} = \begin{array}{ccc} \text{Д}(\bot) & \dashrightarrow & \text{Д}(1) \\ \vdots & & \downarrow{\scriptstyle \text{Д}(1,\top)} \\ \text{Д}(0) & \xrightarrow[\text{Д}(0,\top)]{} & \text{Д}(\top) \end{array}$$

Definition E.1.3. Let $a_0, \cdots, a_{m-1}$ be natural semidendrograms and let $\mathtt{L}_m$ be the free distributive lattice generated by $\{0, 1, \cdots, m-1\}$. A *1-IU diagram* Д *for* $a_0, \cdots, a_{m-1}$ is a collection of objects $\{a_i \mid i \in \mathtt{L}_m\}$ and semidendrogramoid quasimorphisms $f_{ij} \colon a_i \to a_j$ for $i \leq j$ such that the following holds:

1. f_{ii} is the identity map.
2. For $i \leq j \leq k$, $f_{ik} = f_{jk} \circ f_{ij}$.
3. For each $i, j \in \mathtt{L}_m$, the following diagram is a pullback and ran $f_{i,i\vee j} \cup$ ran $f_{j,i\vee j} = a_{i\vee j}$:

$$\text{Д}[i,j] = \begin{array}{ccc} a_{i\wedge j} & \overset{f_{i\wedge j,j}}{\dashrightarrow} & a_j \\ {\scriptstyle f_{i\wedge j,i}}\vdots & & \downarrow{\scriptstyle f_{j,i\vee j}} \\ a_i & \xrightarrow[f_{i,i\vee j}]{} & a_{i\vee j} \end{array}$$

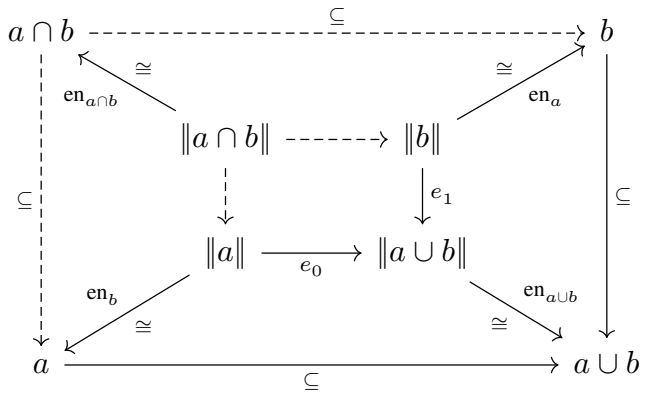
   Moreover, $f_{i,i\vee j}$ and $f_{j,i\vee j}$ are semidendrogram morphisms.

Alternatively, we can think of an IU diagram as a functor from $\mathtt{L}_n$ as a category to the category of natural semidendrogramoids with quasimorphisms.

Definition E.1.4. Let $a, b \subseteq C$ be two finite sub-semidendrograms of C. *The 1-diagram* $\text{Diag}_C(a,b)$ *of* a, b *over* C is the unique 1-arity diagram isomorphic to the inclusion diagram

$$\begin{array}{ccc} a \cap b & \overset{\subseteq}{\dashrightarrow} & b \\ {\scriptstyle \subseteq}\vdots & & \downarrow{\scriptstyle \subseteq} \\ a & \xrightarrow[\subseteq]{} & a \cup b \end{array}$$

Here note that $a \cap b$ may *not* be a sub-semidendrogram of C since a terminal point of $a \cap b$ may not be a terminal of C. More precisely, $\mathrm{Diag}_C(a, b)$ is the innermost diagram in the below commutative diagram:



In general, for finite semidendrograms $a_0, \cdots, a_{n-1} \subseteq C$ of a semidendrogram C, $\mathrm{Diag}_C(a_0, \cdots, a_{n-1})$ is the unique IU diagram isomorphic to the inclusion diagram $\mathcal{J} : \mathtt{L}_n \to \mathcal{P}(a_0^* \cup \cdots \cup a_{n-1}^*)$, where $\mathcal{J}$ can be thought of a lattice homomorphism from $(\mathtt{L}_n, \wedge, \vee)$ to $(\mathcal{P}(a_0^* \cup \cdots \cup a_{n-1}^*), \cap, \cup)$ satisfying $\mathcal{J}(i) = a_i$ for every $i < n$.

Definition E.1.5. A *2-semiptyx* is a set P of *P-terms* t with a natural semidendrogram $\mathrm{arity}(t)$. For two P-terms t_0, t_1 and a 1-arity diagram

$$\text{Д} = \begin{array}{ccc} a_\cap & \dashrightarrow & a_1 \\ \vdots & & \downarrow \\ a_0 & \longrightarrow & a_\cup \end{array}$$

such that $a_i = \mathrm{arity}\, t_i$, we have a relation $t_0 <_{\text{Д}} t_1$. We call Д a *1-arity diagram for* t_0 *and* t_1. P also satisfies the following:

1. (Irreflexivity) If $t_0 = t_1$ and Д is trivial, then $t_0 <_{\text{Д}} t_0$ does not hold.
2. (Linearity) If $t_0 \neq t_1$ or Д is not trivial, then one of $t_0 <_{\text{Д}} t_1$ or $t_1 <_{-\text{Д}} t_0$ must hold.
3. (Transitivity) For three P-terms t_0, t_1, t_2 such that $\mathrm{arity}(t_i) = a_i$, and an IU diagram $\mathbf{Д}$ for t_0, t_1, t_2, if $t_0 <_{\mathbf{Д}[0,1]} t_1$ and $t_1 <_{\mathbf{Д}[1,2]} t_2$, then $t_0 <_{\mathbf{Д}[0,2]} t_2$.

We say P is *positive* if $\mathrm{arity}(t)$ is a natural predendrogram for every P-term t.

Like semidilators, we can first-order axiomatize the definition of 2-semiptykes over the multi-sorted language. Also, we have the following:

Proposition E.1.6. *Let P be a 2-semiptyx, $t_0, t_1 \in \mathrm{field}(P)$, and Д a 1-arity diagram for t_0, t_1. Then neither $t_0 <_{Д} t_1$ nor $t_1 <_{-Д} t_0$ both hold.*

Proof. Suppose not, assume that $t_0 <_{Д} t_1 <_{-Д} t_0$. Assume that Д has of the form (E.1), and consider the 1-IU diagram $\mathbf{Д} = \mathrm{Diag}_{a_\cup}(\mathrm{ran}\, f, \mathrm{ran}\, g, \mathrm{ran}\, f)$. Then we have $\mathbf{Д}[0,1] = Д$, $\mathbf{Д}[1,2] = -Д$, and $\mathbf{Д}[0,2]$ is the trivial diagram for a_0. By transitivity of a 2-semiptyx, we have $t_0 <_{\mathbf{Д}[0,2]} t_0$, so $t_0 <_{Ж} t_0$ for a trivial diagram Ж, contradicting with the irreflexivity. □

We can also talk about the subobject of a 2-semiptyx:

Definition E.1.7. Let P be a 2-semiptyx and $X \subseteq \mathrm{field}(P)$. A *sub-2-semiptyx $P \restriction X$ induced by X* is a 2-semiptyx whose field is X with the same arity function and comparison rules for P.

E.2 Category-theoretic aspects of 2-semiptykes

Semidilators can be thought of as functors from LO to LO. Likewise, 2-semiptykes should be functors from the category SDil of semidilators to LO. However, unlike semidilators in which arities and arity diagrams are defined in terms of finite linear orders, we defined 2-semiptykes with finite semidendrograms and not finite semidilators. It makes the technical definition of 2-semiptykes slightly different from our expectation:

Definition E.2.1. From a given 2-semiptyx P and a semidendrogram C, let us define $P(C)$, the *application of P to C* by

$$P(C) = \{(t, a) \mid t \text{ is a } P\text{-term, } a \Subset C, \text{ and } \|a\| = \text{arity}\, t\}. \tag{E.2}$$

We write $t(a)$ instead of (t, a). The order of $P(C)$ is given by

$$P(C) \vDash s(a) < t(b) \iff P \vDash s <_{\mathrm{Diag}_C(a,b)} t$$

For a semidendrogram morphism $f \colon C \to C'$, consider the map $P(f) \colon P(C) \to P(C')$ given by $P(f)(t, a) = (t, f[a])$.

Proposition E.2.2. *Let P be a 2-semiptyx and C be a semidendrogram. Then $P(C)$ is a linear order. Also, P induces a functor* $\mathsf{SDen} \to \mathsf{LO}$.

Proof. Let us prove that $P(C)$ is a linear order. The irreflexivity and the linearity of a semidilator imply that the comparison relation over $P(C)$ satisfies the irreflexivity and the linearity, respectively. To see the transitivity of $<_{P(C)}$, suppose that we have $P(C) \vDash s_0(a_0) < s_1(a_1) < s_2(a_2)$. Now let Д be the 1-IU diagram isomorphic to a diagram induced by intersections and unions of a_0, a_1, a_2, and the insertion maps. In particular, $\text{Д}(i, j)$ is the induced diagram $\mathrm{Diag}_C(a_i, a_j)$ from a_i and a_j for $i \neq j < 2$. Then we have

$$s_0 <_{\text{Д}(0,1)} s_1 \wedge s_1 <_{\text{Д}[0,2]} s_2.$$

Thus we have $s_0 <_{\text{Д}[0,2]} s_2$ by the transitivity of a semidilator, which implies $P(C) \vDash s_0(a_0) < s_2(a_2)$. To see that P induces a functor, it suffices to show that the following holds:

1. $P(\mathsf{Id}_C) = \mathsf{Id}_{P(C)}$,
2. $P(f \circ g) = P(f) \circ P(g)$.

Both of them are easy to verify, so we leave their proof to the reader. □

Definition E.2.3. Let P and Q be two 2-semiptykes. A map $\iota\colon P \to Q$ is an *embedding* or a *2-semiptyx morphism* if it satisfies the following conditions:

1. ι is a function field P to field Q.
2. ι preserves the arity: i.e., $\mathrm{arity}(\iota(t)) = \mathrm{arity}(t)$ for every $t \in \mathrm{field}\, P$,
3. For each two terms $t_0, t_1 \in \mathrm{field}\, P$ and a 1-arity diagram Д between them, we have $t_0 <_{\text{Д}} t_1$ iff $\iota(t_0) <_{\text{Д}} \iota(t_1)$.

An embedding ι is an *isomorphism* if whenever $\iota\colon \mathrm{field}\, P \to \mathrm{field}\, Q$ is a bijection and ι^{-1} is also an embedding. We denote $P \cong Q$ if there is an isomorphism between P and Q. We also write $P \leq Q$ if there is an embedding from P to Q.

Like the semidilator case, a 2-semiptyx morphism is also one-to-one, and a bijective embedding is an isomorphism:

Lemma E.2.4. *If $\iota\colon P \to Q$ is an embedding, then $\iota\colon \mathrm{field}\, P \to \mathrm{field}\, Q$ is injective.*

Proof. Suppose the contrary that we have two different P-terms t_0 and t_1 such that $\iota(t_0) = \iota(t_1)$. Then $\mathrm{arity}\, t_0 = \mathrm{arity}\, \iota(t_0) = \mathrm{arity}\, t_1$, and let Д be the trivial diagram for t_0. Then either $t_0 <_{\text{Д}} t_1$ or $t_1 <_{\text{Д}} t_0$ holds, so we get $\iota(t_0) <_{\text{Д}} \iota(t_1)$ or $\iota(t_1) <_{\text{Д}} \iota(t_0)$. It contradicts the irreflexivity of Q. □

Lemma E.2.5. *If $\iota\colon P \to Q$ is an embedding and a bijection, then $\iota^{-1}\colon Q \to P$ is an embedding.*

Proof. Suppose that $Q \vDash t_0 <_{\text{Д}} t_1$. Then we can find $s_0, s_1 \in \mathrm{field}(P)$ such that $\iota(s_0) = t_0$ and $\iota(s_1) = t_1$. If $P \vDash s_1 <_{-\text{Д}} s_0$, then $E \vDash t_1 <_{-\text{Д}} t_0$, so $Q \vDash t_1 <_{-\text{Д}} t_0 <_{\text{Д}} t_1$, which is impossible. Hence $P \vDash s_0 <_{\text{Д}} s_1$. □

We can view an embedding between two semidilators as a natural transformation in the following manner:

Definition E.2.6. For a 2-semiptyx morphism $\iota\colon P \to Q$ and a semidendrogram C, define $\iota_D\colon P(C) \to Q(C)$ by

$$\iota_D(t, a) = (\iota(t), a).$$

Proposition E.2.7. *Let $\iota\colon P \to Q$ be 2-semiptyx morphism between two 2-semiptykes P and Q. Then $\iota_C\colon P(C) \to Q(C)$ is a strictly increasing map. Furthermore, if $f\colon C \to C'$ is a semidendrogram morphism from C to C', then the following diagram commutes:*

$$\begin{array}{ccc} P(C) & \xrightarrow{P(f)} & P(C') \\ {\scriptstyle \iota_C}\downarrow & & \downarrow{\scriptstyle \iota_{C'}} \\ Q(C) & \xrightarrow[Q(f)]{} & Q(C') \end{array}$$

Proof. For ι_C being strictly increasing, suppose that we have $P(C) \vDash s(a) < t(b)$, and let Д be the induced diagram from a and b. Then we have $s <_{\text{Д}} t$, so $\iota(s) <_{\text{Д}} \iota(t)$. Since $\operatorname{arity} s = \operatorname{arity} \iota(s)$ and $\operatorname{arity} t = \operatorname{arity} \iota(t)$, $(\iota(s), a), (\iota(t), b)$ are in $Q(C)$ and moreover we have $Q(C) \vDash \iota(s)(a) < \iota(t)(b)$, which is $Q(C) \vDash \iota_C(s(a)) < \iota_C(t(b))$. The commutativity is easy to check. □

E.3 2-Preptykes and 2-ptykes

In this section, we define 2-preptykes and 2-ptykes. Girard's original definition of 2-ptykes is a functor from the category Dil of dilators to the category of well-orders preserving direct limits and pullbacks, and 2-preptykes are a direct limit of 2-ptykes. Every Girard-styled 2-ptyx extends to a functor from the category PreDil of predilators to LO, and we can further extend it by 'ignoring' non-predilator parts. Although we will not

follow Girard's original definition, we follow some spirit of his definition; i.e., we will always require 2-preptykes and 2-ptykes to be *positive*, so they ignore non-predilator parts.

Definition E.3.1. A 2-semiptyx P is *positive* if for every $t \in \mathrm{field}(P)$, $\mathrm{arity}(t)$ is a predendrogram.

Definition E.3.2. Let us say a 1-arity diagram

$$\text{Д} = \begin{array}{ccc} a_\cap & \dashrightarrow & a \\ \vdots & & \downarrow{\scriptstyle e_1} \\ a & \xrightarrow[e_0]{} & a_\cup \end{array} \tag{E.3}$$

is *monotone* if a is a natural predendrogram and $e_0(x) \preceq_{\mathsf{KB}} e_1(x)$ for every $x \in a$, where $x \preceq_{\mathsf{KB}} y$ if and only

1. Either $x = y$, or
2. If m is the least number such that $x \restriction m \neq y \restriction m$, we have $a \vDash x \restriction m \lessdot y \restriction m$.

A 2-semiptyx P is *monotone* if it satisfies the *monotonicity condition*: For every $t \in \mathrm{field}(P)$ and a non-trivial monotone 1-arity diagram Д of the form (E.3), $t <_{\text{Д}} t$ holds. A positive monotone 2-semiptyx P is called a *2-preptyx*.

Like predilators, we can formulate the definition of 2-preptyx in a multi-sorted first-order language of 2-semiptykes. Also, we can formulate a 2-preptyx in a structural manner: In the following proposition, $f \leq g$ for two 2-preptyx morphisms $f, g \colon C \to C'$ means for every $x \in C$, $f(x) \preceq_{\mathsf{KB}} g(x)$.

Proposition E.3.3. *A positive 2-semiptyx P is a 2-preptyx iff for every predendrogram morphisms $f, g \colon C \to C'$, if $f \leq g$ then $P(f) \leq P(g)$.*

Proof. For the left-to-right direction, suppose that $f, g \colon C \to C'$ are two predendrogram morphisms such that $f \leq g$. We claim that for each $x \in P(C)$, $P(f)(x) \leq P(g)(x)$.

Every element of $P(C)$ has the form $t(a)$ for some $t \in \mathrm{field}(P)$ and $a \Subset C$ such that $\|a\| = \mathrm{arity}\, t$. Hence

$$P(C') \vDash P(f)(t(a)) \leq P(g)(t(a)) \iff P(C') \vDash t(f[a]) \leq t(g[a]).$$

If $f[a] = g[a]$, then $P(C') \vDash t(f[a]) \leq t(g[a])$ clearly holds. Now consider the case $f[a] \neq g[a]$: Let $c = \mathrm{arity}\, t$ and $Д = \mathrm{Diag}_C(f[a], g[a])$. Then $Д$ is not trivial. Moreover, if $Д$ has the form (E.3), then for $x \in a$,

$$e_0(x) \leq_{\mathsf{KB}} e_1(x) \iff \mathrm{en}_{f[a]}(x) \leq_{\mathsf{KB}} \mathrm{en}_{g[a]}(x).$$

The latter inequality holds by the assumption $f \leq g$ and the fact that $\mathrm{en}_{f[a]} = f \circ \mathrm{en}_a$. This shows $e_0 \leq e_1$, so $Д$ is monotone. Since P is monotone, $P \vDash t <_Д t$. Therefore $P(C) \vDash t(f[a]) < t(g[a])$.

For the right-to-left direction, let $Д$ be a non-trivial monotone 1-arity diagram of the form (E.3), and t be a P-term of arity a. Then $e_0 \leq e_1$ implies $P(e_0) \leq P(e_1)$. In particular,

$$P(a_\cup) \vDash P\big(\mathrm{Dec}(e_0)\big)(t(a)) \leq P\big(\mathrm{Dec}(e_1)\big)(t(a)).$$

Observe that $P(e_i)(t(a)) = t(\mathrm{ran}\, e_i)$, so we get

$$P(a_\cup) \vDash t(\mathrm{ran}\, e_0) \leq t(\mathrm{ran}\, e_1).$$

By the non-triviality of $Д$, we have $\mathrm{ran}\, e_0 \neq \mathrm{ran}\, e_1$, so the inequality is strict. Since the induced diagram from $\mathrm{ran}\, e_0$ and $\mathrm{ran}\, e_1$ is again $Д$, we get $P \vDash t <_Д t$. □

Definition E.3.4. A positive 2-semiptyx P is a *2-ptyx* if for every dendrogram C, $P(C)$ is a well-order.

The following proposition follows the proof of Proposition B.3.5. However, we work with predendrograms instead of linear orders, so the proof is more convoluted:

Proposition E.3.5. *If P is a 2-ptyx, then P is a 2-preptyx.*

More precisely, a positive 2-semiptyx P is a 2-preptyx if $P(C)$ is well-founded for every locally well-founded predendrogram C of finite height such that for every $u \in C$, the ordertype of the set of immediate successors of u is less than $\omega \cdot 2$.

Proof. Suppose that P is a positive 2-semiptyx that is not a 2-preptyx. Then we have a non-trivial monotone 1-arity diagram Д of the form (E.3) and a P-term t such that $\operatorname{arity} t = a$. Similar to the proof of Proposition B.3.5, we decompose Д to simpler 1-arity diagrams.

Let $\{u_i \mid i < N\} \subseteq a$ be the set of $u \in a$ such that $e_0(u) \lessdot e_1(u)$. We enumerate $\{u_i \mid i < N\}$ in a way that if $i < j < N$, then

1. $\operatorname{lh} u_i \leq \operatorname{lh} u_j$.
2. If u_i and u_j have the same immediate predecessor, then $u_i \lessdot u_j$.

If $e_0(x) \lessdot_{\mathsf{KB}} e_1(x)$, then x is an extension of some u_i. In fact, if m is the least number satisfying $e_0(x) \restriction m \lessdot e_1(x) \restriction m$, then we can take $u_i = x \restriction m$. (Note that $e_0(x) \restriction m = e_0(x \restriction m)$ and the same for e_1.) For each $i \leq N$, define $f_i \colon a \to a_\cup$ by

$$f_i(x) = \begin{cases} e_1(x) & x \text{ extends } u_j \text{ for some } j \geq N-1-i, \\ e_0(x) & \text{Otherwise.} \end{cases}$$

For a technical convenience, we assume u_{-1} is extended to every node. Then $f_0 = e_0$ and $f_N = e_1$.

Claim E.3.6. Each f_i is a predendrogram morphism.

Proof. It suffices to show that if y is extended to u_j for some $j \geq N-1-i$, then $e_0(y) =$

$e_1(y)$. In fact, the definition of $\{u_j \mid j < N\}$ implies that $e_0(u_j)$ and $e_1(u_j)$ have the same immediate predecessor, so $e_0(u_j \restriction m) = e_1(u_j \restriction m)$ for every $m < \mathrm{lh}\, u_j$. □

Moreover, $f_i(x) \neq f_{i+1}(x)$ iff x extends u_{N-1-i}, and in this case, we have $f_i(x) \preccurlyeq_{\mathsf{KB}} f_{i+1}(x)$.

Now consider the IU diagram $\text{Д} = \mathrm{Diag}_{a_\cup}(\mathrm{ran}\, f_0, \mathrm{ran}\, f_1, \cdots, \mathrm{ran}\, f_N)$. Then $\text{Д}[0, N] = \text{Д}$, so the transitivity of a 2-preptyx implies there is $i < N$ such that $P \vDash t \not\preccurlyeq_{\text{Д}[i,i+1]} t$. Now let us write $\text{Ж} = \text{Д}[i, i+1]$. If we assume Ж takes the form

$$\text{Ж} = \begin{array}{ccc} a'_\cap & \dashrightarrow & a \\ \vdots & & \downarrow{\scriptstyle g_1} \\ a & \xrightarrow[g_0]{} & a'_\cup \end{array}$$

then $g_0(x) \leq_{\mathsf{KB}} g_1(x)$ for every $x \in a$, and they differ iff x extends $u := u_{N-1-i}$.

Now let us build a locally well-founded predendrogram C of finite height with a sequence of subpredendrograms $\{a_i \mid i \in \mathbb{N}\}$ such that $\mathrm{Diag}_C(a_i, a_j) = \text{Ж}$ for $i < j$. If we have such C and a_i, then we have

$$P(C) \vDash t(a_0) > t(a_1) > t(a_2) > \cdots,$$

so $P(C)$ is ill-founded. The basic idea of the construction of C is 'copying' parts below u from a ω many times; See Figure E.1 for the illustration of the construction.

Let us start from a natural predendrogram a. We first replace every component of a with a multiple of ω, that is, we consider a new predendrogram a' such that the map

$$\langle e_0, e_1, \cdots, e_{l-1} \rangle \mapsto \langle \omega \cdot e_0, \omega \cdot e_1, \cdots, \omega \cdot e_{l-1} \rangle$$

is an isomorphism $a \to a'$. We denote this map by $x \mapsto \omega \cdot x$. For $x \in a$ extending u, we define a new tuple

$$\omega \cdot x +_u i = \langle \omega \cdot x_0, \cdots, \omega \cdot x_{l-2}, (\omega \cdot x_{l-1} + i), \omega \cdot x_l, \cdots, \omega \cdot x_{m-1} \rangle, \tag{E.4}$$

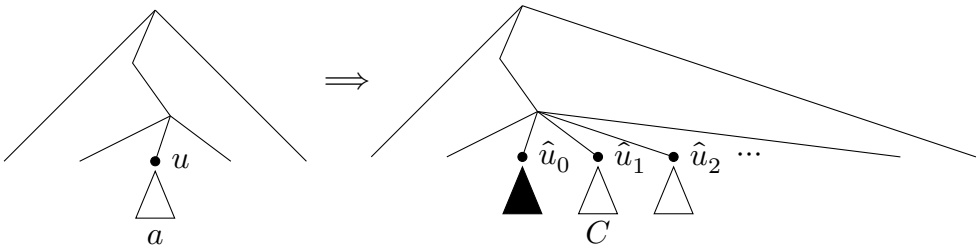


Figure E.1: The construction of C from a. The filled triangle, $\widehat{u}_0$, and the other uncopied part constitute a_0. Here, $\widehat{u}_i = \omega \cdot u +_u i$.

where $x = \langle x_0, \cdots, x_{m-1} \rangle$ and $u = \langle x_0, \cdots, x_{l-1} \rangle$ for $l \leq m$. Then let us consider the new predendrogram $a'' = a' \cup \{\omega \cdot x +_u i \mid i < \omega \wedge u \multimap^* x\}$ with the following additional information:

1. If $a \vDash v \multimap u$, then $a'' \vDash \omega \cdot v \multimap \omega \cdot u +_u i$. For $x, y \in a$ extending u such that $a \vDash x \multimap y$, $a'' \vDash \omega \cdot x +_u i \multimap \omega \cdot y +_u j$.
2. $a'' \vDash \omega \cdot u +_u i \lessdot \omega \cdot u +_u j$ iff $i < j$.
3. $\pi^{a''}(\omega \cdot x +_u i) = \pi^a(x)$.

Then consider the subpredendrogram $a_i \Subset a''$ given by

$$a_i = \{\omega \cdot x \mid u \not\multimap^* x\} \cup \{\omega \cdot x + i \mid u \multimap^* x\}. \qquad \square$$

Conversely, we can show that every *finite* 2-preptyx is a 2-ptyx. Let us start with counting the number of 1-arity diagrams for two copies of a predendrogram:

Lemma E.3.7. *For a natural predendrogram a, there are finitely many 1-arity diagrams between a and a.*

Proof. We may calculate the precise number of the 1-arity diagrams between a and a, but we only provide a dull upper bound. We will first find a finite family of natural

predendrograms embedding two copies of a, then count the number of such families. A natural dendrogram C is *h-short N-branching* if for $u \in C$

1. $\mathrm{lh}\, u \leq h$, and
2. Either u is a terminal point or $u^\frown\langle i\rangle \in C$ for every $i < N$.

We first set the upper bound of the number of h-short N-branching natural predendrograms. Let us imagine a forest of height h, with each node branching N times until it reaches level h. We color the edges of each level: In Level 0, we color 0 or 'terminates,' which means we will ignore successors following the edge. In Level 1, we color the edge in either 01, 10, or 'terminates,' Similarly, in Level k, we color the edge in an appropriate permutation of $(k+1)$ or 'terminates.' Some of the colorings give an h-short N-branching natural predendrograms, so we have at most

$$(1!+1)^N + (2!+1)^{2N} + \cdots + (h!+1)^{hN}$$

many h-short N-branching natural predendrograms. We can also see that for a 1-arity diagram Д of the form (E.3), if we take

- N to be the largest number of immediate successors of a node of C, and
- h is the maximum length of a node of C,

then $a_\cup$ embeds to a h-short N-branching natural dendrogram. Hence, the number of 1-arity diagrams of the form (E.3) is less than or equal to the number of pairs of subsets of an h-short N-branching natural dendrogram, which is at most

$$(2^{(1!+1)^N+(2!+1)^{2N}+\cdots+(h!+1)^{hN}})^2. \qquad \square$$

The proof of the next proposition uses Ramsey's theorem for pairs, so it is not provable over RCA_0 or ECA_0:

Proposition E.3.8. *Every finite 2-preptyx is a 2-ptyx. In fact, if P is a finite 2-preptyx, then $P(C)$ is a well-order for every locally well-founded predendrogram C.*

Proof. Suppose that P is a finite positive 2-semiptyx that is not a 2-ptyx. We claim that P is not a 2-preptyx. Fix a locally well-founded predendrogram C such that $P(C)$ is ill-founded, so we have a sequence $\langle t_i \mid i \in \mathbb{N} \rangle$ of P-terms and finite subpredendrograms a_i of C such that

$$P(C) \vDash t_0(a_0) > t_1(a_1) > t_2(a_2) > \cdots.$$

By the pigeonhole principle, we may assume every t_i is equal to some t. Then consider the coloring F of domain $[\mathbb{N}]^2$ given by $F(i,j) = \mathrm{Diag}_C(a_i, a_j)$. The range of F is finite by Lemma E.3.7, so by Ramsey's theorem for pairs, we have an infinite homogeneous set $H \subseteq \mathbb{N}$ with value Д.

For $i, j \in H$, $P(C) \vDash t(a_i) > t(a_j)$ implies $P \vDash t \not\leq_{\text{Д}} t$. Д is not trivial, otherwise $a_0 = a_1 = a_2 = \cdots$, contradicting with $t(a_0) > t(a_1) > t(a_2) > \cdots$. Now we claim that Д witnesses P is not a 2-preptyx: Suppose that Д has the form (E.3). We claim that $e_0 \leq e_1$ holds: Otherwise, we have $x \in a$ such that $e_0(x) \succcurlyeq_{\mathsf{KB}} e_1(x)$. Since $\mathrm{Diag}_C(a_i, a_j) = \text{Д}$, we also have $\mathrm{en}_{a_i}(x) \succcurlyeq_{\mathsf{KB}} \mathrm{en}_{a_j}(x)$ for every $i < j$. In particular, we have $\preccurlyeq_{\mathsf{KB}}$-descending sequence $\langle \mathrm{en}_{a_i}(x) \mid i \in \mathbb{N} \rangle$ over C of fixed length, i.e., the length of $\mathrm{en}_{a_i}(x)$ is constant. This is impossible since C is locally well-founded. □

Remark E.3.9. There is a non-trivial non-positive finite 2-semiptyx P such that $P(C) = 0$ for every predendrogram C. An easy example of P is when P has a single term of arity opp. Hence the positive condition of a 2-preptyx is essential to make the theory of 2-preptykes more similar to that of predilators.

E.4 2-Semiptykes as functors with support transformation

In this section, we provide an alternative formulation of 2-semiptykes close to the Freund-style definition of semidilators.

Definition E.4.1. An *F-2-semiptyx* is a pair (P, supp), where $P \colon \mathsf{SDen} \to \mathsf{LO}$ is a functor and $\mathrm{supp} \colon P \to [\cdot]^{<\omega}$ is a natural transformation satisfying the *support condition*

$$\{\sigma \in P(C) \mid \mathrm{supp}_C(\sigma) \Subset \mathrm{ran}(f)\} \subseteq \mathrm{ran}(F(f)) \tag{E.5}$$

for every semidendrogram morphism $f \colon C \to C'$. We often omit supp and simply write P to mean (P, supp).

To formulate the 2-preptyx à la Freund, we need a way to describe 'ignoring' the non-predendrogram part of a semidendrogram:

Definition E.4.2. Let C be a semidendrogram and $x \in C$. We say x is *positive* if $\wp_t = \mathrm{lh}(t)$. We define $\mathrm{pos}(C)$ to be the set of $x \in C$ such that x extends to a positive terminal node of C.

We can easily see that $\mathrm{pos}(C)$ forms a sub-predendrogram of C. Now, let us state the definition of a 2-preptyx with support transformation:

Definition E.4.3. A F-2-semiptyx P is a *F-2-preptyx* if it satisfies

1. P is *positive*, i,e, For every semidendrogram C, $P(C) = P(\mathrm{pos}(C))$.
2. P is monotone, i.e., $f \leq g$ implies $P(f) \leq P(g)$ for every predendrogram morphism $f, g \colon C \to C'$.

Definition E.4.4. The *category of F-2-semiptykes* $\mathsf{SPtyx}^2_{\mathsf{F}}$ is the category whose objects are F-2-semiptykes, and whose morphisms are natural transformations.

Like the semidilator case, every natural transformation between F-2-semiptykes preserves support transformation:

Proposition E.4.5. *Suppose that P, Q are F-2-semiptykes and $\epsilon\colon P \to Q$ is a natural transformation. Then* $\mathrm{supp}^Q = \iota \circ \mathrm{supp}^P$.

Proof. Let C be a semidendrogram and $\sigma \in P(C)$. We also let $\epsilon_\sigma\colon \mathrm{supp}^P_C(\sigma) \to C$ be the inclusion map and $\mathrm{en}_\sigma = \mathrm{en}_{\mathrm{supp}^P_C(\sigma)}$. For $a = \|\mathrm{supp}^P_C(\sigma)\|$, the support condition implies the existence of $\sigma_0 \in P(a)$ satisfying $\sigma = P(\epsilon_\sigma \circ \mathrm{en}_\sigma)(\sigma_0)$. By the naturality of ι and supp^Q, we have

$$\begin{aligned}\mathrm{supp}^Q_C(\iota_C(\sigma)) &= \mathrm{supp}^Q_C\Big(\iota_C \circ P(\epsilon_\sigma \circ \mathrm{en}_\sigma)(\sigma_0)\Big) = \mathrm{supp}^Q_D\Big(Q(\epsilon_\sigma \circ \mathrm{en}_\sigma) \circ \iota_a(\sigma_0)\Big)\\ &= [\epsilon_\sigma \circ \mathrm{en}_\sigma]^{<\omega}\Big(\mathrm{supp}^Q_a(\iota_a(\sigma_0))\Big) \subseteq \mathrm{ran}\,\epsilon_\sigma = \mathrm{supp}^P_D(\sigma).\end{aligned}$$

For the remaining inclusion, suppose that the inclusion fails and there is $x \in \mathrm{supp}^P_C(\sigma) \setminus \mathrm{supp}^Q_C(\iota_C(\sigma))$. Fix $u \in a$ such that $x = \mathrm{en}_\sigma(u)$, then we have $u \in \mathrm{supp}^P_a(\sigma_0) \setminus \mathrm{supp}^Q_a(\iota_a(\sigma_0))$ since

$$\begin{aligned}\mathrm{en}_\sigma(u) = (\epsilon_\sigma \circ \mathrm{en}_\sigma)(u) \in \mathrm{supp}^Q_C(\iota_C(\sigma))[\epsilon_\sigma \circ \mathrm{en}_\sigma]^{<\omega}[\mathrm{supp}^G_a(\iota_a(\sigma))]\\ \iff u \in \mathrm{supp}^G_a(\iota_a(\sigma)).\end{aligned}$$

Now let us consider a new semidendrogram b obtained from a by 'copying' u and its successors; More precisely, consider a semidendrogram $b \ni a$ with additional nodes v^+ for $v \in a$ extending u, and let us define

1. If w is an immediate predecessor of u, we take $b \vDash w \multimap u^+$. For two nodes $x, y \in a$ extending u such that $u \vDash x \multimap y$, we define $b \vDash x^+ \multimap y^+$.
2. If $x, y \in a$ satisfies $a \vDash x \lessdot u \lessdot y$, we define $b \vDash x \lessdot u \lessdot u^+ \lessdot y$.
3. $\pi^b(v^+) = \pi^a(v)$ and $\wp^b_{v^+} = \wp^a_v$ for every $v \in a$ extending u.

Now let $f_0 \colon a \to b$ be the inclusion map, and

$$f_1(x) = \begin{cases} x^+, & \text{if } x \text{ extends } u, \\ x, & \text{otherwise.} \end{cases}$$

Then, on the one hand, we have

- $u = f_0(u) \in [f_0]^{<\omega}[\mathrm{supp}_a^P(\sigma_0)] = \mathrm{supp}_b^P(P(f_0)(\sigma_0))$, and
- $u \notin \mathrm{ran}\, f_1 \supseteq [f_1]^{<\omega}[\mathrm{supp}_a^P(\sigma_0)] = \mathrm{supp}_b^P(P(f_1)(\sigma_0))$.

Hence $P(f_0)(\sigma_0) \neq P(f_1)(\sigma_0)$. By the injectivity of ι_b, we have

$$Q(f_0)(\iota_a(\sigma_0)) = \iota_b \circ P(f_0)(\sigma_0) \neq \iota_b \circ P(f_1)(\sigma_0) = Q(f_1)(\iota_a(\sigma_0)). \tag{E.6}$$

On the other hand, the support condition also implies the existence of $\tau_0 \in Q\Big(\|\mathrm{supp}_a^Q(\iota_a(\sigma_0))\|\Big)$ satisfying

$$\iota_a(\sigma_0) = Q(\epsilon_{\iota_a(\sigma_0)} \circ \mathrm{en}_{\iota_a(\sigma_0)})(\tau_0)$$

for the inclusion map $\epsilon_{\iota_a(\sigma_0)} \colon \mathrm{supp}_a^Q(\iota_n(\sigma_a)) \to a$ and $\mathrm{en}_{\iota_a(\sigma_0)} = \mathrm{en}_{\mathrm{supp}_a^Q(\iota_a(\sigma_0))}$. Since $u \notin \mathrm{ran}\, \epsilon_{\iota_a(\sigma_0)} = \mathrm{supp}_a^Q(\iota_a(\sigma_0))$, we have

$$f_0 \circ \epsilon_{\iota_a(\sigma)} \circ \mathrm{en}_{\iota_a(\sigma)} = f_1 \circ \epsilon_{\iota_a(\sigma)} \circ \mathrm{en}_{\iota_a(\sigma)}\,.$$

It implies

$$\begin{aligned} Q(f_0)(\iota_a(\sigma_0)) &= Q\Big(f_0 \circ \epsilon_{\iota_a(\sigma_0)} \circ \mathrm{en}_{\iota_a(\sigma_0)}\Big)(\tau_0) \\ &= Q\Big(f_1 \circ \epsilon_{\iota_a(\sigma_0)} \circ \mathrm{en}_{\iota_a(\sigma_0)}\Big)(\tau_0) = Q(f_1)(\iota_a(\sigma_0)), \end{aligned}$$

contradicting (E.6). □

Like the semidilator case, we want to show that the category of 2-semiptykes and that of F-2-semiptykes are equivalent. The following set will have a role of the set of terms in the denotation system we will construct from an F-2-semiptyx:

Definition E.4.6. The *trace* of an F-2-semiptyx P is given by

$$\mathrm{Tr}(P) = \{(a, \sigma) \mid a \text{ is a natural semidendrogram } \wedge\, \sigma \in P(a) \wedge \mathrm{supp}_a(\sigma) = a\}.$$

Proposition E.4.7. *For an F-2-semiptyx P, if we define $\mathfrak{f}(P)$ by* $\mathrm{field}(\mathfrak{f}(P)) = \mathrm{Tr}(P)$, $\mathrm{arity}^{\mathfrak{f}(P)}(a, t) = a$, *and for a 1-arity diagram*

$$\text{Д} = \begin{array}{ccc} a_\cap & \dashrightarrow & a_1 \\ \vdots & & \downarrow{\scriptstyle e_1} \\ a_0 & \xrightarrow[e_0]{} & a_\cup \end{array} \tag{E.7}$$

and $(a_0, t_0), (a_1, t_1) \in \mathrm{Tr}(P)$, let us define

$$(a_0, t_0) <_{\text{Д}} (a_1, t_1) \iff P(a_\cup) \vDash P(e_0)(t_0) < P(e_1)(t_1). \tag{E.8}$$

Then $\mathfrak{f}(P)$ is a 2-semiptyx.

Proof. It is easy to check that the comparison rule of $\mathfrak{f}(F)$ satisfies asymmetry and linearity. For transitivity, suppose that we have a 1-IU diagram **Д** for three objects and $t_0, t_1, t_2 \in \mathrm{field}(\mathfrak{f}(F))$ such that $(a_i, t_i) \in \mathrm{Tr}(F)$ for $i = 0, 1, 2$ and

$$\mathfrak{f}(F) \vDash t_0 <_{\text{Д}[0,1]} t_1 <_{\text{Д}[1,2]} t_2.$$

Now let $\text{Д}(i) = a_i$ and $\text{Д}(i, j) = e_{ij}$ for $i \leq j \in \mathtt{L}_3$ be morphisms of **Д**. Then we have

$$P(a_{0\vee1}) \vDash P(e_{0,0\vee1})(t_0) < P(e_{1,0\vee1})(t_1) \text{ and } P(a_{1\vee2}) \vDash P(e_{1,1\vee2})(t_1) < P(e_{2,1\vee2})(t_2).$$

By applying $P(e_{0\vee1,0\vee1\vee2})$ to the left side, we have

$$P(a_{0\vee1\vee2}) \vDash P(e_{0,0\vee1\vee2})(t_0) < P(e_{1,0\vee1\vee2})(t_1).$$

Similarly, we can see

$$P(a_{0\vee1\vee2}) \vDash P(e_{1,0\vee1\vee2})(t_1) < P(e_{2,0\vee1\vee2})(t_2).$$

Thus we have

$$P(a_{0\vee 1\vee 2}) \vDash P(e_{0,0\vee 1\vee 2})(t_0) < P(e_{2,0\vee 1\vee 2})(t_2).$$

Then by subtracting $P(e_{0\vee 2,0\vee 1\vee 2})$, we get

$$P(a_{0\vee 2}) \vDash P(e_{0,0\vee 2})(t_0) < P(e_{2,0\vee 2})(t_2),$$

which is equivalent to $P \vDash t_0 <_{\text{Д}[0,2]} t_2$. □

We can turn $\mathfrak{f}$ into a functor $\mathsf{SPtyx}^2_{\mathsf{F}} \to \mathsf{SPtyx}^2$ by defining $\mathfrak{f}(\iota)$ for a natural transformation $\iota \colon P \to Q$ by

$$\mathfrak{f}(\iota)(a, t) = (a, \iota_a(t)).$$

Lemma E.4.8. *$\mathfrak{f}(\iota)$ is a 2-semiptyx embedding from $\mathfrak{f}(P)$ to $\mathfrak{f}(Q)$.*

Proof. $\mathfrak{f}(\iota)$ preserves an arity of a term. To see the comparison rule preservation, suppose we are given a 1-arity diagram Д of the form (E.7), and assume that we are given $\mathfrak{f}(P) \vDash (a_0, t_0) <_{\text{Д}} (a_1, t_1)$, or equivalently,

$$P(a_\cup) \vDash P(e_0)(t_0) < P(e_1)(t_1).$$

Then let us apply $\iota_{a_\cup} \colon P(a_\cup) \to Q(a_\cup)$. Since ι is a natural transformation, we have $\iota_{a_\cup} \circ P(e_0) = Q(e_0) \circ \iota_{a_0}$ and $\iota_{a_\cup} \circ P(e_1) = Q(e_1) \circ \iota_{a_1}$. Hence

$$Q(a_\cup) \vDash Q(e_0)(\iota_{a_0}(t_0)) < Q(e_1)(\iota_{a_1}(t_1)),$$

which is the definition of $\mathfrak{f}(Q) \vDash \mathfrak{f}(\iota)(a_0, t_0) <_{\text{Д}} \mathfrak{f}(\iota)(a_1, t_1)$. □

Conversely, we can construct a functor $\mathfrak{a}(P)$ from an F-2-semiptyx in a natural way:

Definition E.4.9. Let P be a 2-semiptyx. Let us define $\mathfrak{a}(P)(C) = P(C)$ for a semidendrogram C. For a semidendrogram morphism $f \colon C \to C'$, let us define $\mathfrak{a}(P)(f) \colon \mathfrak{a}(P)(C) \to \mathfrak{a}(P)(C')$ by

$$\mathfrak{a}(P)(f)(t(a)) = t(f[a]).$$

for $t \in P$ and finite $a \Subset C$ such that $\operatorname{arity} t = \|a\|$.

Then $\mathfrak{a}(P) \colon \mathsf{SDil} \to \mathsf{LO}$ is a functor. Also, for a 2-semiptyx embedding $\iota \colon P \to Q$, let us define the natural transformation $\mathfrak{a}(\iota) \colon \mathfrak{a}(P) \to \mathfrak{a}(Q)$ by

$$\mathfrak{a}(\iota)_C(t(a)) = \iota(t)(a).$$

for $t \in \operatorname{field}(P)$ and $a \in [D]^{\operatorname{arity} t}$. To turn $\mathfrak{a}(P)$ an F-2-semiptyx, we need a support transformation: Let us define $\operatorname{supp}^{\mathfrak{a}(P)} \colon \mathfrak{a}(P) \to [\cdot]^{<\omega}$ by

$$\operatorname{supp}_D^{\mathfrak{a}(P)}(t(a)) = a.$$

Lemma E.4.10. $\operatorname{supp}^{\mathfrak{a}(P)}$ *satisfies the support condition, so* $\mathfrak{a}(P)$ *is an F-2-semiptyx.*

Proof. Let C, C' be semidendrograms, and $f \colon C \to C'$ be a semidendrogram embedding. Suppose that $t \in P$ and $a \Subset C$, $\|a\| = \operatorname{arity} t$, and

$$\operatorname{supp}_C^{\mathfrak{a}(E)}(t(a)) = a \Subset \operatorname{ran} f.$$

Since f is one-to-one, there is $b \in [D]^{<\omega}$ such that $a = f^{"}[b]$, so $t(a) = \mathfrak{a}(P)(f)(t(b))$ by definition. □

As we did in Section B.4, we constructed functors $\mathfrak{f} \colon \mathsf{SPtyx}_{\mathsf{F}}^2 \to \mathsf{SPtyx}^2$ and $\mathfrak{a} \colon \mathsf{SPtyx}^2 \to \mathsf{Ptyx}_{\mathsf{F}}^2$. We prove they are category equivalences:

Theorem E.4.11. *There is a natural isomorphism* $\mathsf{eaf} \colon \mathfrak{a} \circ \mathfrak{f} \to \mathsf{Id}_{\mathsf{SPtyx}_{\mathsf{F}}^2}$.

Proof. Let us describe $\mathfrak{a}(\mathfrak{f}(P))$ for an F-2-semiptyx P: $\mathfrak{a}(\mathfrak{f}(P))$ is a functor from SDil to LO, which maps a semidendrogram C to a linear order of field

$$\mathfrak{a}(\mathfrak{f}(P))(C) = \{(a,t)(c) \mid (a,t) \in \operatorname{Tr}(F) \wedge a \cong c \Subset C\}.$$

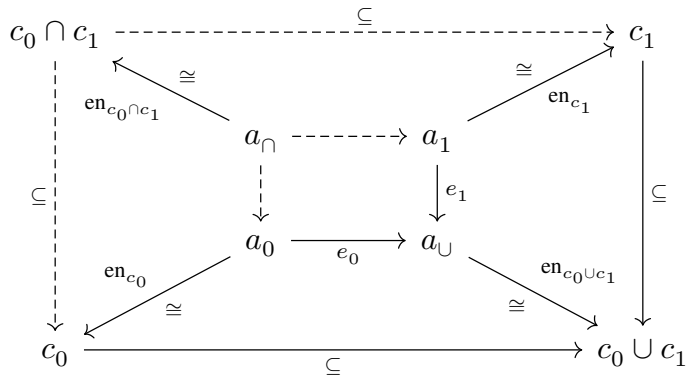
The comparison rule over $\mathfrak{a}(\mathfrak{f}(F))(X)$ is as follows: For $(a_0, t_0), (a_1, t_1) \in \mathrm{Tr}(F)$ and $c_0, c_1 \Subset C$,

$$\mathfrak{a}(\mathfrak{f}(P))(C) \vDash (a_0, t_0)(c_0) < (a_1, t_1)(c_1)$$
$$\iff \text{For } Д = \mathrm{Diag}_C(c_0, c_1),\ \mathfrak{f}(P) \vDash (a_0, t_0) <_Д (a_1, t_1).$$

and if $Д$ takes the form (E.7),

$$\mathfrak{f}(P) \vDash (a_0, t_0) <_Д (a_1, t_1) \iff P(a_\cup) \vDash P(e_0)(t_0) < P(e_1)(t_1).$$

Note that $Д = \mathrm{Diag}_C(c_0, c_1)$ implies $a_\cap = \|a_0 \cap a_1\|$, $a_\cup = \|a_0 \cup a_1\|$ and the following diagram commutes:



Hence applying $P(\mathrm{en}_{c_0 \cup c_1})$ to $P(n_\cup) \vDash P(e_0)(t_0) < P(e_1)(t_1)$ gives

$$\mathfrak{f}(P) \vDash (a_0, t_0) <_Д (a_1, t_1) \iff P(c_0 \cup c_1) \vDash P(\mathrm{en}_{c_0})(t_0) < P(\mathrm{en}_{c_1})(t_1).$$

Hence

$$\mathfrak{a}(\mathfrak{f}(P))(C) \vDash (a_0, t_0)(c_0) < (a_1, t_1)(c_1) \iff P(c_0 \cup c_1) \vDash P(\mathrm{en}_{c_0})(t_0) < P(\mathrm{en}_{c_1})(t_1).$$

Now let us define a natural transformation $\mathsf{eaf}_P \colon (\mathfrak{a} \circ \mathfrak{f})(P) \to P$ as follows: For each semidendrogram C, $\mathsf{eaf}_{P,C} \colon (\mathfrak{a} \circ \mathfrak{f})(P)(C) \to P(C)$ is the map

$$\mathsf{eaf}_{P,C}((a, t)(c)) = P(\mathrm{en}_c)(t).$$

Then $\mathsf{eaf}_{P,C}$ is an isomorphism: It is order-preserving since

$$\begin{aligned}
&(\mathfrak{a} \circ \mathfrak{f})(P)(C) \vDash (a_0, t_0)(c_0) < (a_1, t_1)(c_1) \\
\iff &P(c_0 \cup c_1) \vDash P(\mathrm{en}_{c_0})(t_0) < P(\mathrm{en}_{c_1})(t_1) \\
\iff &P(C) \vDash P(\mathrm{en}_{c_0})(t_0) < P(\mathrm{en}_{c_1})(t_1).
\end{aligned}$$

$\mathsf{eaf}_{P,C}$ is onto since for every $u \in P(C)$, if we let $c = \mathrm{supp}^P_C(u)$ and $t = P(\mathrm{en}_d^{-1})(u)$, then $(\|c\|, t) \in \mathrm{Tr}(P)$ and $\mathsf{eaf}_{P,C}((\|c\|, t)(c)) = u$.

To see eaf_P is a natural transformation from $\mathfrak{a}(\mathfrak{f}(P))$ to P, let us observe that what $\mathfrak{a}(\mathfrak{f}(P))(f)$ should be for a semidendrogram morphism $f \colon C \to C'$: By definition of $\mathfrak{a}$, we have

$$\mathfrak{a}(\mathfrak{f}(P))(f)((a, t)(c)) = (a, t)(f[c]).$$

Hence for $(a, t)(c) \in \mathfrak{a}(\mathfrak{f}(P))(C)$,

$$\begin{aligned}
P(f) \circ \mathsf{eaf}_{P,C}((a, t)(c)) &= P(f)(P(\mathrm{en}_c)(t)) = P(f \circ \mathrm{en}_c)(t) \\
&= P(\mathrm{en}_{f[c]})(t) = \mathsf{eaf}_{P,E}((a, t)(f[c])) \\
&= (\mathsf{eaf}_{P,C'} \circ \mathfrak{a}(\mathfrak{f}(P))(f))((a, t)(c)),
\end{aligned}$$

proving the naturality of eaf_P. However, we also need that $\mathsf{eaf}_{P,C}$ is natural also in P. To see this, we claim that the following diagram commutes for every natural transformation $\iota \colon P \to Q$ and a semidilator D:

$$\begin{array}{ccc}
\mathfrak{a}(\mathfrak{f}(P))(C) & \xrightarrow{\mathfrak{a}(\mathfrak{f}(\iota))_C} & \mathfrak{a}(\mathfrak{f}(Q))(C) \\
{\scriptstyle \mathsf{eaf}_{P,C}}\downarrow & & \downarrow{\scriptstyle \mathsf{eaf}_{Q,C}} \\
P(C) & \xrightarrow[\iota_C]{} & Q(C)
\end{array}$$

For each $(a, t)(c) \in \mathfrak{a}(\mathfrak{f}(P))(C)$, note that $a = \|d\|$, so

$$\begin{aligned}
\mathsf{eaf}_{Q,C}(\mathfrak{a}(\mathfrak{f}(\iota))_C((a, t)(c))) &= \mathsf{eaf}_{Q,C}(\mathfrak{f}(\iota)(a, t)(c)) = \mathsf{eaf}_{Q,D}\Big((a, \iota_{\|c\|}(t))(c)\Big) \\
&= Q(\mathrm{en}_c)(\iota_{\|c\|}(t)) = \iota_C(P(\mathrm{en}_d)(t)) \\
&= \iota_C\Big(\mathsf{eaf}_{P,C}((a, t)(c))\Big),
\end{aligned}$$

here the linebreaked equality follows from $Q(\mathrm{en}_c) \circ \iota_{\|a\|} = \iota_C \circ P(\mathrm{en}_c)$ since $\iota \colon P \to Q$ is a natural transformation. It finishes the proof. □

Theorem E.4.12. *There is a natural isomorphism* $\mathsf{efa} \colon \mathsf{Id}_{\mathsf{SPtyx}^2} \to \mathfrak{f} \circ \mathfrak{a}$.

Proof. Let us observe the structure of $\mathfrak{f}(\mathfrak{a}(P))$ for a 2-semiptyx P. Its field is

$$\begin{aligned}\mathrm{Tr}(\mathfrak{a}(P)) &= \{(a,t) \mid a \in \mathsf{SNat}^2 \wedge t \in P(a) \wedge \mathrm{supp}_a^{\mathfrak{a}(P)}(t) = a\} \\ &= \{(a, t(a)) \mid a \in \mathsf{SNat}^2 \wedge t \in \mathrm{field}(P) \wedge \mathrm{arity}^P(t) = a\}\end{aligned}$$

and $\mathrm{arity}^{\mathfrak{f}(\mathfrak{a}(P))}(a, t(\mathrm{Dec}(a))) = a$. For a 1-arity diagram Д of the form (B.5),

$$\begin{aligned}\mathfrak{f}(\mathfrak{a}(P)) \vDash (a_0, t_0(n_0)) <_{\text{Д}} (a_1, t_1(n_1)) &\iff P(a_\cup) \vDash P(e_0)(t_0(a_0)) < P(e_1)(t_1(a_1)) \\ &\iff P(a_\cup) \vDash t_0(e_0[a_0]) < t_1(e_1[a_1]),\end{aligned}$$

and observe that $\mathrm{Diag}_{a_\cup}(e_0[a_0], e_1[a_1]) = \text{Д}$, so

$$P(a_\cup) \vDash t_0(e_0[a_0]) < t_1(e_1[a_1]) \iff P \vDash t_0 <_{\text{Д}} t_1.$$

That is, $\mathfrak{f}(\mathfrak{a}(P)) \vDash (a_0, t_0(a_0)) <_{\text{Д}} (a_1, t_1(a_1)) \iff P \vDash t_0 <_{\text{Д}} t_1$. Hence if we let $\mathsf{efa}_P \colon P \to \mathfrak{f}(\mathfrak{a}(P))$ by $\mathsf{efa}_P(t) = (\mathrm{arity}\, t, t(\mathrm{arity}\, t))$, then efa_P is a 2-semiptyx isomorphism. The naturality of efa follows from an easy computation. □

Hence we have

Corollary E.4.13. SPtyx^2 *and* $\mathsf{SPtyx}^2_{\mathsf{F}}$ *are category equivalent.* □

Thus we can identify the two categories SPtyx^2 and $\mathsf{SPtyx}_{\mathsf{F}}$. Also, let us observe that, like semidilator cases, the construction of $\mathfrak{f}(P)$ does not require the whole P. In fact, we can construct $\mathfrak{f}(P)$ from the restriction of P to the category SNat^2 of natural semidilators, a full subcategory of the category of 2-semiptykes. Hence we can construct a 2-semiptyx not only from a whole F-2-semiptyx but also from its restriction to the SNat^2.

Definition E.4.14. An *F-coded 2-semiptyx* is a pair (P, supp), where $P \colon \mathsf{SNat}^2 \to \mathsf{LO}$ is a functor and supp is a natural transformation with the support condition (B.4). If P is positive and monotone, we say P is an *F-coded 2-preptyx*.

The previous proofs give the following corollary:

Corollary E.4.15. *If P is an F-coded 2-semiptyx, then we can construct a 2-semiptyx $\mathfrak{f}(P)$. If P is an F-2-semiptyx, then $\mathfrak{f}(P) = \mathfrak{f}(P \restriction \mathsf{SNat}^2)$ and $P \cong \mathfrak{a}(\mathfrak{f}(P \restriction \mathsf{SNat}^2))$.* □

If P is positive, then $P(a) \cong 0$ for every $a \in \mathsf{SNat}^2 \setminus \mathsf{Nat}^2$. Hence we also have

Corollary E.4.16. *If P is an F-2-preptyx, then $\mathfrak{f}(P) = \mathfrak{f}(P \restriction \mathsf{Nat}^2)$ and $P \cong \mathfrak{a}(\mathfrak{f}(P \restriction \mathsf{Nat}^2))$. That is, we can reconstruct the full 2-preptyx from its restriction to the category of natural dilators.* □

Like dilators, we can also discuss whether a given F-coded 2-semiptyx is recursive. We also have

Proposition E.4.17. *Suppose that P is an F-coded 2-semiptyx that is coded by a (primitive) recursive subset of $\mathbb{N}$. Then $\mathfrak{f}(P)$ is (primitive) recursive.* □

E.5 Prime 2-ptykes

In this section, we classify prime 2-ptykes, a 2-ptyx with a single term.

Definition E.5.1. A 2-ptyx P is *prime* if it has a unique term. The *degree of P* is the arity of the unique P-term.

Like prime dilators, we find a 'priority permutation' for a prime 2-ptyx, which is no longer a permutation. A priority permutation assigns which component must be compared first. The 2-ptyx analogue of the priority assignment assigns which node over a

dendrogram expression must be compared first. In the proof of Theorem C.2.4, we prepared subsidiary sequences c_i over $2n$, where n is the arity of the unique term in the prime dilator. Before characterizing prime 2-ptykes, we define a predendrogram analogue of $n \mapsto 2n$ and c_i:

Definition E.5.2. Let C be a predendrogram. We define the *duplication* $\mathrm{dup}(C)$ *of* C as a predendrogram defined as follows: The field of $\mathrm{dup}(C)$ is obatined from that of C by adding v^{+u} for each $u \in C$ and $v \in C$ extending u (including u itself.) Informally, the plus tree $\{v^{+u} \mid v \text{ extends } u\}$ with root u will form a duplicated copy of a subtree $\{v \mid v \text{ extends } u\}$ of root u in the dendrogram $\mathrm{dup}(C)$, and we put the plus tree of root u at the right side of the subtree of root u.

We define relations over $\mathrm{dup}(C)$ as follows:

1. C is a sub-predendrogram of $\mathrm{dup}(C)$, so $\mathrm{dup}(C)$ respects every relation over C.
2. If $C \vDash u \multimap v$, then $\mathrm{dup}(C) \vDash u \multimap v^{+v}$.
3. For $u, v, w \in C$ such that both v and w extends u and $C \vDash v \multimap w$, $\mathrm{dup}(C) \vDash v^{+u} \multimap w^{+u}$.
4. If $C \vDash u \lessdot v$, then $\mathrm{dup}(C) \vDash u \lessdot u^{+u} \lessdot v \lessdot v^{+v}$.
5. For $u, v, w \in C$ such that both v and w extends u and $C \vDash v \lessdot w$, $\mathrm{dup}(C) \vDash v^{+u} \lessdot w^{+u}$.
6. If v extends u, $\varpi^{\mathrm{dup}(C)}(v^{+u}) = \varpi^{C}(v)$.

We may understand the construction of $\mathrm{dup}(C)$ as follows: First, let us color the current term part black. In level m, copy each black-colored $u \in C$ and its successors and put the copy just right of the original term. We color the copied term parts white. For $u \in C$, let $u^{+u} \in \mathrm{dup}(C)$ be the copy of u placed just right of u. Also, we denote successors of u^{+u} corresponding to $v \supseteq u$ by v^{+u}.

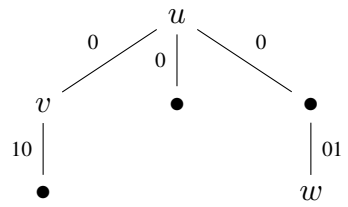


Figure E.2: Square dendrogram

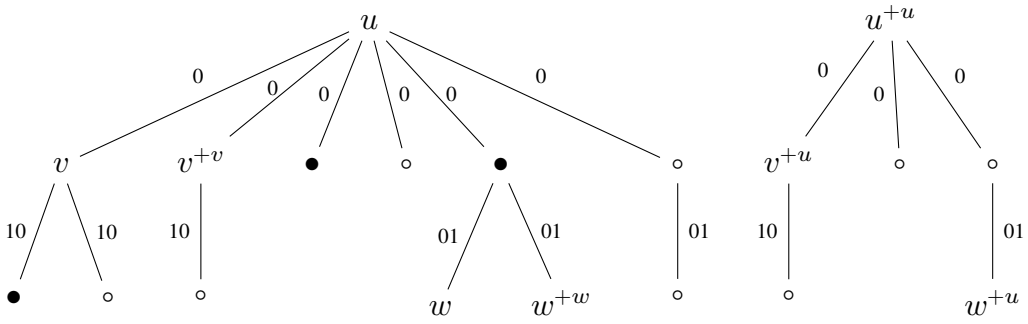


Figure E.3: Duplicated square dendrogram

Example E.5.3. Let us recall the natural dendrogram d isomorphic to the dilator $X \mapsto X \times X$ presented in Figure D.1. We reintroduce its figure as in Figure E.2 for convenience, with a different labeling on nodes for a later purpose. Figure E.3 represents the duplication of the square dendrogram.

Definition E.5.4. Let c be a natural predendrogram and $u \in c$. We define $\mathsf{c}_{c,u} \subseteq \mathrm{dup}(c)$ as a subdilator isomorphic to c given by

$$\mathsf{c}_{c,u} = \{w^{+u} \mid w \in c \wedge u \multimap^* w\} \cup \{w \mid w \in c \wedge w \not\multimap^* u\}.$$

Example E.5.5. Let c be the dendrogram representing $X \mapsto X \times X$ provided in Figure E.2. Recalling its duplication in Figure E.3, we can represent $\mathsf{c}_{c,u}$, $\mathsf{c}_{c,v}$, $\mathsf{c}_{c,w}$ as in Figure E.4.

Let us recall the details from Section C.2: We tracked the priority order (equivalently,

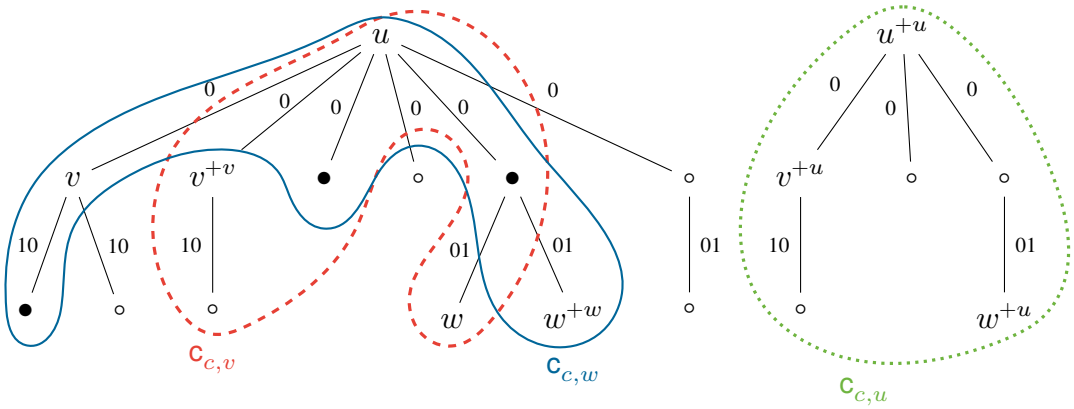


Figure E.4: Examples of subdendrograms of Figure E.3

the priority permutation) of a given term t of arity m by comparing $t(c_0), \cdots, t(c_{m-1})$. The priority permutation can be arbitrary. It corresponds to the fact that $\mathrm{Diag}_{2m}(c_i, c_j)$ is not monotone if $i \neq j$. (See the proof of Theorem C.2.4 for the definition of c_i.) However, unlike the previous case, $\mathrm{Diag}_{\mathrm{dup}(c)}(\mathsf{c}_{c,u}, \mathsf{c}_{c,v})$ can be monotone: We can see that $\mathrm{Diag}_{\mathrm{dup}(c)}(\mathsf{c}_{c,v}, \mathsf{c}_{c,u})$ in Figure E.4 is monotone. In general, $\mathrm{Diag}_{\mathrm{dup}(c)}(\mathsf{c}_{c,u}, \mathsf{c}_{c,v})$ is monotone when u is extended from v. It will come as a constraint on defining a prime 2-ptyx from a permutation since if $P = \{t\}$ is a prime 2-ptyx of degree c and u is extended from v, then we must have

$$P(c) \vDash t(\mathsf{c}_{c,u}) < t(\mathsf{c}_{c,v}).$$

It shows that 'priority permutations' for prime 2-ptykes cannot be as arbitrary as priority permutations for prime dilators. However, the following lemma says $u \circ\!\!-^* v$ is the only case when the diagram becomes monotone:

Lemma E.5.6. *Let c be a natural dendrogram and $u, v \in c$. Then* $\mathrm{Diag}_{\mathrm{dup}(c)}(\mathsf{c}_{c,u}, \mathsf{c}_{c,v})$ *is monotone if and only if $u \circ\!\!-^* v$.*

Proof. Suppose that $u \circ\!\!-^* v$. Then we have $\mathrm{en}_{\mathsf{c}_{c,u}}(w) = \mathrm{en}_{\mathsf{c}_{c,v}}(w)$ if $w \not\circ\!\!-^* v$ or $w \circ\!\!- u$.

If $w \circ\!\!-^* v$ but $w \not\circ\!\!-^* u$, then $\mathsf{en}_{\mathsf{c}_{c,u}}(w) = w \lessdot_{\mathsf{KB}} \mathsf{en}_{\mathsf{c}_{c,v}}(w)$. Hence for every $w \in c$, we have $\mathsf{en}_{\mathsf{c}_{c,u}}(w) \leq_{\mathsf{KB}} \mathsf{en}_{\mathsf{c}_{c,v}}(w)$, proving the monotonicity of the diagram. Conversely, if $u \not\circ\!\!-^* v$, then we have $\mathsf{en}_{\mathsf{c}_{c,u}}(u) = u^{+u} \gtrdot u = \mathsf{en}_{\mathsf{c}_{c,v}}(u)$, violating the monotonicity. □

We expect the following notion to characterize a prime 2-ptyx of given arity:

Definition E.5.7. A finite (pre)dendrogram c is a *priority (pre)dendrogram* if the field of c is equal to its cardinality $|c|$, and for every $i, j < |c|$, $c \vDash i \multimap j$ implies $i < j$ as natural numbers.

As in priority orders, nodes with lower indices have higher priority. It is easy to see that if c is a priority dendrogram and $c \vDash i \multimap^* j$, then $i \leq j$. It turns out that every priority (pre)dendrogram c induces a prime 2-ptyx of degree $\|c\|$:

Example E.5.8. Let c be a priority dendrogram. Let us define Prime_c as follows: For a given semidendrogram D, let us define $\mathsf{Prime}_c(D)$ be the set of every sub-predendrogram of D isomorphic to c, i.e.,

$$\mathsf{Prime}_c(D) = \{a \Subset D \mid a \cong c\}.$$

We equip $\mathsf{Prime}_c(D)$ with the order $<_c$ given as follows. We first define, for two isomorphic finite predendrograms a and c,

$$\mathsf{en}_{c,a} = \mathsf{en}_a \circ \mathsf{en}_c^{-1} \colon c \to a. \tag{E.9}$$

Then for two $a, b \in \mathsf{Prime}_C(D)$ isomorphic to c, $a <_c b$ if and only if $i < |c|$ is the least number such that $\mathsf{en}_{c,a}(i) \neq \mathsf{en}_{c,b}(i)$, then $D \vDash \mathsf{en}_{c,a}(i) \lessdot \mathsf{en}_{c,b}(i)$. Note that for the least i with $\mathsf{en}_{c,a}(i) \neq \mathsf{en}_{c,b}(i)$, either both $\mathsf{en}_{c,a}(i)$ or $\mathsf{en}_{c,b}(i)$ have length 0, or have the same immediate predecessor: The immediate predecessor of $f_a(i)$, if exists, is $\mathsf{en}_{c,a}(j)$ for some $j < i$, so $\mathsf{en}_{c,a}(j) = \mathsf{en}_{c,b}(j)$. From $c \vDash j \multimap i$, we also have $D \vDash \mathsf{en}_{c,b}(j) \multimap$

$\mathrm{en}_{c,b}(i)$. The support transformation is defined trivially, i.e., $\mathrm{supp}_D^{\mathsf{Prime}_c}(a) = a$. For a semidendrogram embedding $f \colon C \to D$, we define $\mathsf{Prime}_c(f)(a) = f[a]$. We can easily check the support condition of Prime_c.

We can see that Prime_c is a 2-ptyx. In fact, we can see that if D is a locally well-founded predendrogram, then $\mathsf{Prime}_c(D)$ is well-founded. Suppose the contradiction that $\mathsf{Prime}_c(D)$ is ill-founded, so we have $a_m \mathrel{\Subset} D$ $(m \in \mathbb{N})$ such that $a_0 >_c a_1 >_c a_2 >_c \cdots$. Let $h_m = \mathrm{en}_{c,a_m} \colon c \to a_m$ be the unique isomorphism from c to a_m for each m. We claim by induction on $i < |c|$ that $\langle h_m(i) \mid m \in \mathbb{N} \rangle$ is eventually constant: Suppose that either i is a root in c, or $c \vDash j \multimap i$ for some $j < i$, and we inductively have that $\langle h_m(j) \rangle \mid m \in \mathbb{N} \rangle$ is eventually constant. In the latter case, we may assume that $\langle h_m(j) \rangle \mid m \in \mathbb{N} \rangle$ is constant. In either case, we have

$$D \vDash h_0(i) \trianglerighteq h_1(i) \trianglerighteq h_2(i) \trianglerighteq \cdots,$$

and if $\langle h_m(i) \mid m \in \mathbb{N} \rangle$ is not eventually constant, then it contradicts the local well-foundedness of D.

From the remaining part of this section, we prove the following theorem:

Theorem E.5.9. *Let P be a prime 2-ptyx of degree d. Then there is a priority (pre)dendrogram c isomorphic to d such that $P \cong \mathsf{Prime}_c$.*

Proof. Suppose that t is the unique P-term of arity d. We define the priority predendrogram c by relabeling d as follows: First, let us enumerate d as $\{u_i \mid i < |d|\}$ in a way that

$$P(\mathrm{dup}(d)) \vDash t(\mathsf{c}_{d,u_0}) > t(\mathsf{c}_{d,u_1}) > t(\mathsf{c}_{d,u_2}) > \cdots.$$

Then define c in a way that the map $u_i \mapsto i$ is a predendrogram isomorphism from d to c. Lemma E.5.6 implies that c is a priority (pre)dendrogram.

We claim that for a semidendrogram D and $a, b \Subset D$ isomorphic to d,

$$P(D) \vDash t(a) < t(b) \iff \exists i < |c|\big[\forall j < i[\mathrm{en}_{c,a}(j) = \mathrm{en}_{c,b}(j)] \wedge \mathrm{en}_{c,a}(i) \lessdot \mathrm{en}_{c,b}(i)\big]. \tag{E.10}$$

Note that if $a \Subset D$ is isomorphic to d, then $a \Subset \mathrm{pos}(D)$. Hence, we may assume that D is a predendrogram. In the remaining part of the proof, we assume D is a predendrogram.

By linearity of P, it suffices to show the backward implication of (E.10). For a natural predendrogram d, let $\ell(d)$ be the height of d, or equivalently, the maximum length of nodes of d plus 1. For each $k < \ell(d)$ and $a, b \Subset D$ isomorphic to D, let $N_{a,b}(k)$ be the number of $u \in d$ of length k such that $\mathrm{en}_a(u) \gtrdot_{\mathsf{KB}} \mathrm{en}_b(u)$. Then define

$$N_{a,b} = |d|^{\ell(d)-1} \cdot N_{a,b}(0) + |d|^{\ell(d)-2} \cdot N_{a,b}(1) + \cdots + N_{a,b}(\ell(d) - 1). \tag{E.11}$$

Note that $|N_{a,b}(k)| < |d|$ for every k and $a, b \cong d$, so we can think of $N_{a,b}$ as a natural number coding a tuple $\langle N_{a,b}(k) \mid k < \ell(d)\rangle$ in a lexicographic-order preserving manner.

We prove (E.10) by induction on $N_{a,b}$. If $N_{a,b} = 0$, then $\mathrm{en}_a(u) \lesssim_{\mathsf{KB}} \mathrm{en}_b(u)$ for every $u \in d$, so $P(D) \vDash t(a) \leq t(b)$ since every 2-ptyx is monotone. For the induction step, let $l < |c|$ be the $\lesssim_{\mathsf{KB}}$-least node computed in c such that $\mathrm{en}_{c,a}(l) \gtrdot_{\mathsf{KB}} \mathrm{en}_{c,b}(l)$. If i is the natural number such that the right-hand side of (E.10) holds, then $i < l$: If $i \geq l$, then we have $\mathrm{en}_{c,a}(l) \gtrdot \mathrm{en}_{c,b}(l)$ witnesses the failure of the right-hand-side of (E.10).

Let us define $\overline{a} \Subset \mathrm{dup}(D)$ isomorphic to d by specifying $\mathrm{en}_{c,\overline{a}} \colon c \to a$, which is as follows:

$$\mathrm{en}_{c,\overline{a}}(j) = \begin{cases} \mathrm{en}_{c,a}(j)^{+\mathrm{en}_{c,a}(i)} & c \vDash j \circ\!\!-^* i, \\ \mathrm{en}_{c,b}(j) & c \vDash j \circ\!\!-^* l, \\ \mathrm{en}_{c,a}(j) & \text{otherwise.} \end{cases}$$

(See Figure E.5 for the illustration of $\overline{a}$.) Observe that in c, neither i extends to l or vice versa: If $c \vDash l \multimap^* i$, then $l < i$ since c is a priority predendrogram. If $c \vDash i \multimap^* l$, then

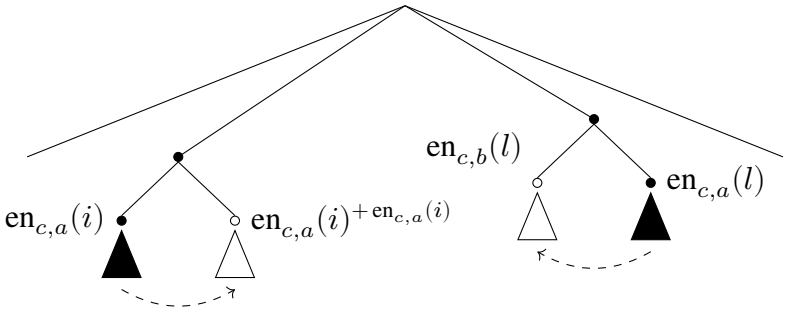


Figure E.5: The construction of $\overline{a}$ from a. We replace filled nodes and triangles with hollow nodes and triangles to get $\overline{a}$ from a, with all other parts unchanged.

either $i = l$ or $\mathrm{en}_{c,a}(i) = \mathrm{en}_{c,b}(i)$ by the choice of l. Also, if $c \vDash k \lessdot l \lessdot m$, then

$$\mathrm{dup}(D) \vDash \mathrm{en}_{c,a}(k) \leq \mathrm{en}_{c,b}(k) \lessdot \mathrm{en}_{c,b}(l) \lessdot \mathrm{en}_{c,a}(l) \lessdot \mathrm{en}_{c,a}(m).$$

This shows that $\overline{a}$ is isomorphic to c. We claim that

$$P(\mathrm{dup}(D)) \vDash t(a) < t(\overline{a}) < t(b).$$

First, we can see that $P(\mathrm{dup}(D)) \vDash t(\overline{a}) < t(b)$ holds: By the definition of $\overline{a}$, we have $N_{\overline{a},b}(\mathrm{lh}^c(l)) = N_{a,b}(\mathrm{lh}^c(l)) - 1$, so $N_{\overline{a},b} < N_{a,b}$. ($\mathrm{lh}^c(l)$ is the length of l computed in c.) Hence, the induction hypothesis implies the desired result.

It remains to prove $P(\mathrm{dup}(D)) \vDash t(a) < t(\overline{a})$. Let us consider a predendrogram morphism $g \colon \mathsf{c}_{c,i} \cup \mathsf{c}_{c,l} \to a \cup \overline{a}$ given by

- $g(j) = \mathrm{en}_{c,a}(j)$ for $c \vDash j \multimap^* i$.
- $g(j^{+i}) = \mathrm{en}_{c,a}(j)^{+\mathrm{en}_{c,a}(i)}$ for $c \vDash j \multimap^* i$.
- $g(j) = \mathrm{en}_{c,b}(j)$ for $c \vDash j \multimap^* l$.
- $g(j^{+l}) = \mathrm{en}_{c,a}(j)$ for $c \vDash j \multimap^* l$.
- $g(j) = \mathrm{en}_{c,a}(j)$ otherwise.

Then g is an isomorphism, and $g[\mathsf{c}_{c,i}] = \overline{a}$, $g[\mathsf{c}_{c,l}] = a$. Since $i < l$, we have

$$P(\mathrm{dup}(c)) \vDash t(\mathsf{c}_{c,i}) > t(\mathsf{c}_{c,l}).$$

Then applying $P(g)$ yields $P(\mathrm{dup}(D)) \vDash t(\overline{a}) > t(a)$, as desired. □

As we did in Section C.2, we have the following result:

Theorem E.5.10. *Let P be a 2-preptyx. For each $t \in \mathrm{field}(P)$ of arity d, we can find a priority (pre)dendrogram $c = \varpi^P(t)$ such that for every predendrogram D and $a, b \Subset D$ isomorphic to $\varpi^D(t)$, $P(D) \vDash t(a) < t(b)$ if and only if*

$$\exists i < |c| \big[\forall j < i [\mathrm{en}_{c,a}(j) = \mathrm{en}_{c,b}(j)] \wedge \mathrm{en}_{c,a}(i) \lessdot \mathrm{en}_{c,b}(i)\big].$$ □

E.6 Abstract construction of 2-preptykes

In this section, we further analyze the structure of 2-preptykes. We want to define $\mathbf{p}^P(s,t)$ for a 2-preptyx P and two P-terms s and t such that when we compare $s(a)$ and $t(b)$, we only compare the first $\mathbf{p}^P(s,t)$ parts from a and b under the priority predendrograms $\varpi^P(s)$ and $\varpi^P(t)$ calculated by the characterization of prime 2-ptykes. Here, we confront an issue we have not experienced in predilator cases due to the presence of 'hidden terms' in predendrograms. For example, it is possible that in comparing $s, t \in P$, the last compared argument of s is hidden while the last compared argument of t is not.

Example E.6.1. Let P be the evaluation 2-ptyx $P(D) = D(1)$. We can see that $\mathrm{Tr}(P)$ has only two elements, namely, $\langle \underline{1}, 0\rangle$ and $\langle \mathsf{Id}_{\mathsf{Dil}}, 1\rangle$. Let us denote each as τ and τ_0, and we understand them as

- τ takes a nullary dilator term, and acts identically.

- τ_0 takes a unary term t, and returns $t(0)$.

For the ease of the explanation, suppose we are already given Theorem E.6.10. $\mathbf{p}^P(\tau, \tau_0)$ cannot be 0 since $\mathbf{p}^P(\tau, \tau_0) = 0$ means comparing τ and τ_0 does not depend on their arguments, which is not. Since $\mathbf{p}^P(\tau, \tau_0) \leq |\text{arity } \tau| = 1$, we must have $\mathbf{p}^P(\tau, \tau_0) = 1$.

However, we cannot detect $\mathbf{p}^P(\tau, \tau_0) = 1$ by finding the least natural number p satisfying "two arguments agreeing by p gives the same comparison results independent of the arguments." Put differently, we cannot find a and b satisfying the first case of Theorem E.6.10: We can see that $\pi^P(\tau)$ has 0 as a terminal node, but $\pi^P(\tau_0)$ has 0 as a non-terminal node. Indeed, the comparison between $\tau(a)$ and $\tau_0(b)$ always happens by the second case of Theorem E.6.10. The arguments for the existence of $\mathbf{p}^D(s, t)$ in Section C.3 does not work for 2-preptykes *per se* since finding $\mathbf{p}^D(s, t)$ for two predilator term $s, t \in D$ relies on finding p such that the agreement of arguments by p gives the same comparison between s and t.

To overcome this issue, we pretend non-terminal nodes as valid arguments that 2-ptyx terms can take; For example, suppose that $D \vDash x_0 \multimap x_1$ and x_0 has length 0. We cannot form $\tau(x_0)$ in principle since x_0 is not a terminal node. However, if we ignore the terminal condition and compare $\tau(x_0)$ and $\tau_0(x_1)$, then we must have $\tau_0(x_1) < \tau(x_0)$ no matter what x_0 and x_1 are. To formulate this idea formally, we rely on the completion of a predendrogram, where all intermediate nodes are unraveled into terminal nodes.

We first define a broader notion of a 1-arity diagram:

Definition E.6.2. An *ostensible 1-arity diagram* is a diagram of the form

$$\text{Д} = \begin{array}{ccc} d_\cap & \dashrightarrow & d_1 \\ \downarrow & & \downarrow{\scriptstyle e_1} \\ d_0 & \xrightarrow[e_0]{} & d_\cup \end{array} \tag{E.12}$$

such that every predendrogram occurring Д is natural, and Д is a pullback and $\operatorname{ran} e_0 \cup \operatorname{ran} e_1 = d_\cup$, but both e_0 and e_1 can be quasimorphisms. (In particular, e_0 and e_1 may not respect terminal points. For two terms s, t of a 2-preptyx P, we say Д is an *ostensible 1-arity diagram for s and t* if $d_0 = \operatorname{arity} s$ and $d_1 = \operatorname{arity} t$.

Definition E.6.3. Let C be a predendrogram and $a, b \subseteq D$ be predecessor-closed subsets of D (That is, if $x \in a$ and $y \multimap x$, then $y \in a$, and the same for b.) We define $\operatorname{Diag}_D(a, b)$ as an ostensible 1-arity diagram isomorphic to the following commutative diagram:

$$\text{Д} = \begin{array}{ccc} a \cap b & \xrightarrow{\subseteq} & b \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ a & \xrightarrow[\subseteq]{} & a \cup b \end{array}$$

Then we unravel the hidden points in an ostensible 1-arity diagram by taking the completion:

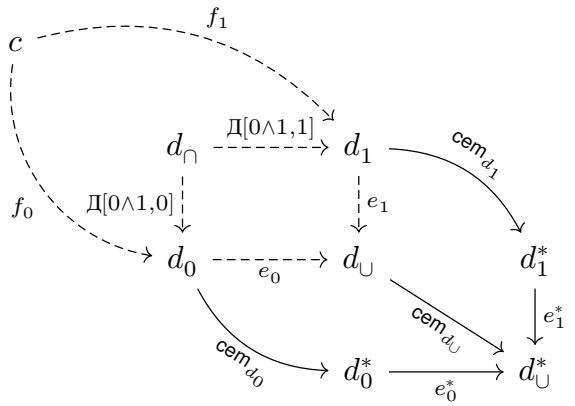
Proposition E.6.4. *For an ostensible 1-arity diagram* (E.12)*, let* Д* *be a new commutative diagram whose objects are all natural predendrograms and isomorphic to the following square diagram:*

$$\begin{array}{ccc} d_\cap & \xrightarrow{\text{Д}[0\wedge 1,1]} & d_1 \\ {\scriptstyle \text{Д}[0\wedge 1,0]}\downarrow & & \downarrow{\scriptstyle e_1^* \circ \mathsf{cem}_{d_1}} \\ d_0 & \xrightarrow[e_0^* \circ \mathsf{cem}_{d_0}]{} & \operatorname{ran}(e_0^* \circ \mathsf{cem}_{d_0}) \cup \operatorname{ran}(e_1^* \circ \mathsf{cem}_{d_1}) \end{array}$$

Then Д* *is a 1-arity diagram.*

Proof. For the pullback condition, let c be a predendrogram and $f_0 : c \to d_0$ and $f_1 : c \to d_1$ be quasimorphisms such that $e_0^* \circ \mathsf{cem}_{d_0} \circ f_0 = e_1^* \circ \mathsf{cem}_{d_1} \circ f_1$. The reader may be tempted to argue that $\mathsf{cem}_{d_\cup} \circ e_0 \circ f_0 = \mathsf{cem}_{d_\cup} \circ e_1 \circ f_1$, but this argument does not work since the squares having $\mathsf{cem}_{d_\cup}$ as an arrow are *not* commutative in the following

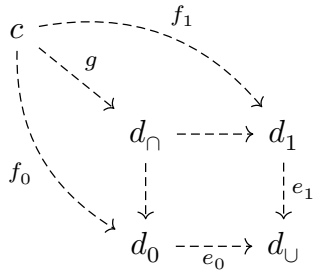
diagram:



Hence we directly prove that $e_0 \circ f_0 = e_1 \circ f_1$: For $x \in c$, we have

- $e_0^* \circ \mathsf{cem}_{d_0} \circ f_0(x) = e_0^*(f_0(x)^*) = (e_0 \circ f_0(x))^*$, and
- $e_1^* \circ \mathsf{cem}_{d_1} \circ f_1(x) = e_1^*(f_1(x)^*) = (e_1 \circ f_1(x))^*$.

Hence $e_0^* \circ \mathsf{cem}_{d_0} \circ f_0 = e_1^* \circ \mathsf{cem}_{d_1} \circ f_1$ implies $(e_0 \circ f_0(x))^* = (e_1 \circ f_1(x))^*$ for every $x \in c$, so $e_0 \circ f_0 = e_1 \circ f_1$. Since Д is a pullback, we get a quasimorphism $g \colon c \to d_\cap$ making the following diagram commute:

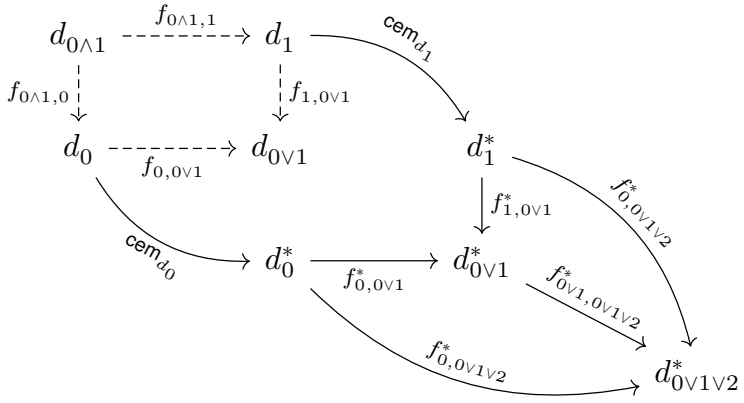


Such g witnesses Д* is also a pullback. We defined Д* in a way that the rightmost-bottommost is a union of the range of the two arrows towards the object. Moevoer, e_0^* and e_1^* respect terminal points by Proposition D.3.15, so Д* is a 1-arity diagram. □

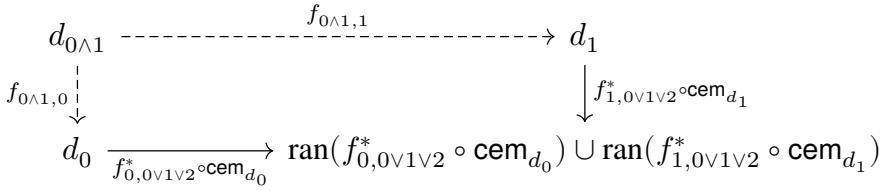
We can also define an ostensible 1-IU diagram by mimicking Definition E.1.3 but requiring each square to be just an ostensible 1-arity diagram. Then we have the following.

Proposition E.6.5. *Let P be a 2-preptyx, $s, t, u \in \mathrm{field}(P)$, and Д be an ostensible 1-IU diagram for s, t, and u. If $P \vDash s <_{\text{Д}[0,1]^*} t <_{\text{Д}[1,2]^*} u$, then $P \vDash s <_{\text{Д}[0,2]^*} u$.*

Proof. For $i, j \in \mathtt{L}_3$, let us write $d_i = \text{Д}(i)$ and $f_{i,j} = \text{Д}(i,j)$. Then we can see that the following diagram commutes:



This shows the following diagram is isomorphic to $\text{Д}[0,1]^*$:



In particular, this implies from $P \vDash s <_{\text{Д}[0,1]^*} t$ that

$$P(d^*_{0\vee 1\vee 2}) \vDash s(\mathrm{ran}\, f^*_{0,0\vee 1\vee 2} \circ \mathsf{cem}_{d_0}) < t(\mathrm{ran}\, f^*_{1,0\vee 1\vee 2} \circ \mathsf{cem}_{d_1}).$$

By a similar argument, we have

$$P(d^*_{0\vee 1\vee 2}) \vDash t(\mathrm{ran}\, f^*_{1,0\vee 1\vee 2} \circ \mathsf{cem}_{d_0}) < u(\mathrm{ran}\, f^*_{2,0\vee 1\vee 2} \circ \mathsf{cem}_{d_2}).$$

Hence we get

$$P(d^*_{0\vee 1\vee 2}) \vDash s(\mathrm{ran}\, f^*_{0,0\vee 1\vee 2} \circ \mathsf{cem}_{d_0}) < u(\mathrm{ran}\, f^*_{2,0\vee 1\vee 2} \circ \mathsf{cem}_{d_2}).$$

By the earlier argument in reverse, we get $P \vDash s <_{\text{Д}[0,2]^*} u$. □

Definition E.6.6. Let P be a 2-preptyx and $s, t \in \mathrm{field}(P)$. Let $\hat{p}_{s,t} \leq \min(|\pi^P(s)|, |\pi^P(t)|)$ be the largest natural number such that

1. For every $i, j < \hat{p}_{s,t}$, $\pi^P(s) \vDash i \multimap j$ iff $\pi^P(t) \vDash i \multimap j$. In particular, for every $i < \hat{p}_{s,t}$, $\mathrm{lh}^{\pi^P(s)}(i) = \mathrm{lh}^{\pi^P(t)}(i)$.
2. For every $i, j < \hat{p}_{s,t}$, $\pi^P(s) \vDash i \lessdot j$ iff $\pi^P(t) \vDash i \lessdot j$.
3. For every $i < \hat{p}_{s,t}$ and j, k such that $\pi^P(s) \vDash i \multimap j$ and $\pi^P(t) \vDash i \multimap k$, we have $\pi^{\pi^P(s)}(j) = \pi^{\pi^P(t)}(k)$. That is, for every $i < \hat{p}_{s,t}$, $\pi^P(s)$ and $\pi^P(t)$ agree with the priority order of immediate successors of i.

That is, $\hat{p}_{s,t}$ is the largest natural number p such that the restriction of $\pi^P(s)$ to p is equal to the restriction of $\pi^P(t)$ to p, possibly except the set of terminal points. We say an ostensible 1-arity diagram Д for s and t is *p-congruent (relative to s and t)* if $e_0 \circ \mathrm{en}^{-1}_{\pi^P(s)}(i) = e_1 \circ \mathrm{en}^{-1}_{\pi^P(t)}(i)$ holds for every $i < p$. (Note that $d_0 \cong \pi^P(s)$ and $d_1 \cong \pi^P(t)$ by the assumption on Д.) Then we say $p \leq \hat{p}_{s,t}$ is *secure (relative to s and t)* if $P \vDash s <_{\text{Д}^*} t$ holds for some p-congruent ostensible Д, then $P \vDash s <_{\text{Д}^*} t$ holds for every p-congruent ostensible Д. In other words, the validity of $P \vDash s <_{\text{Д}^*} t$ does not depend on the choice of a p-congruent ostensible Д.

As we did in Section C.3, we prove that a secure number relative to s and t exists by showing the following:

Theorem E.6.7. *$\hat{p}_{s,t}$ is secure.*

Proof. For notational convenience, let us write $\pi_s = \pi^P_s$ and $\pi_t = \pi^P_t$. If $\hat{p}_{s,t} = |\pi_s| \leq |\pi_t|$, then we have a predendrogram quasimorphism $h \colon d_0 \to d_1$ given by

$$h(\mathrm{en}^{-1}_{\pi_s}(i)) = \mathrm{en}^{-1}_{\pi_t}(i).$$

Furthermore, if Д is a $\hat{p}_{s,t}$-congruent ostensible 1-arity diagram for s and t of the form (E.12), then we have $e_0 = e_1 \circ h$ and $e_0^* = e_1^* \circ h^*$. Then we have

$$\begin{aligned} P \vDash s <_{Д^*} t &\iff P(d_\cup^*) \vDash s(\operatorname{ran} e_0^* \circ \mathsf{cem}_{d_0}) < t(\operatorname{ran} e_1^* \circ \mathsf{cem}_{d_1}) \\ &\iff P(d_1^*) \vDash s(\operatorname{ran} h^* \circ \mathsf{cem}_{d_0}) < t(\operatorname{ran} \mathsf{cem}_{d_1}), \end{aligned}$$

and the last sentence is independent of a $\hat{p}_{s,t}$-congruent Д. Hence $\hat{p}_{s,t}$ is secure relative to s and t. The case $\hat{p}_{s,t} = |\pi_t| \leq |\pi_s|$ is analogous, so let us consider the case $\hat{p}_{s,t} < \min(|\pi_s|, |\pi_t|)$. Note that $s \neq t$ in this case since $\hat{p}_{t,t} = |\pi_t|$.

For notational convenience, let us write $\hat{p} = \hat{p}_{s,t}$. Now suppose that

$$Д' = \begin{array}{ccc} d'_\cap & \dashrightarrow & d_1 \\ \vdots & & \vdots{\scriptstyle e'_1} \\ \downarrow & & \downarrow \\ d_0 & \underset{e'_0}{\dashrightarrow} & d'_\cup \end{array} \tag{E.13}$$

is another ostensible $\hat{p}$-congruent 1-arity diagram with respect to s and t. We will claim that $P \vDash s <_{Д^*} t$ implies $P \vDash s <_{(Д')^*} t$. Observe that the $\hat{p}$-congruence of Д implies

$$\forall i < \hat{p}[e_0 \circ \mathsf{en}_{\pi_s, d_0}(i) = e_1 \circ \mathsf{en}_{\pi_t, d_1}(i)].$$

The next lemma says we can join $d_\cup$ and $d'_\cup$ in a way that their intersection is isomorphic to a predendrogram isomorphic to $\pi_s \restriction \hat{p} = \pi_t \restriction \hat{p}$, and the $\hat{p}$th nodes differ in size.

Proposition E.6.8. *Let $C = \pi_s \restriction \hat{p} = \pi_t \restriction \hat{p}$. Then we can find a new predendrogram D and embeddings $h \colon d_\cup \to D$, $h' \colon d'_\cup \to D$ such that the following diagram commutes. (Here, the dashed arrows are quasimorphisms, and not morphisms in general.)*

$$\begin{array}{ccc} C & \overset{e'_0 \circ \mathsf{en}_{\pi_s, d_0}}{\dashrightarrow} & d'_\cup \\ {\scriptstyle e_0 \circ \mathsf{en}_{\pi_s, d_0}}\vdots & & \downarrow{\scriptstyle h'} \\ \downarrow & & \\ d_\cup & \underset{h}{\longrightarrow} & D \end{array}$$

Furthermore, h and h' satisfy

1. $D \vDash h'(e'_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p})) \lessdot h(e_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p}))$, *and*
2. $D \vDash h(e_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p})) \lessdot h'(e'_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p}))$.

Proof. We rely on $\mathbb{Q}_1$ constructed in Proposition D.3.13. We can find an embedding $h \colon d_\cup \to \mathbb{Q}_1$. Now we are going to construct $h' \colon d'_\cup \to \mathbb{Q}_1$ with the desired properties. First, we define a quasimorphism

$$h'_0 \colon d'_\cup \restriction (e'_0 \circ \mathsf{en}_{\pi_s,d_0}[C]) \to \mathbb{Q}_1$$

by $h'_0(e'_0 \circ \mathsf{en}_{\pi_s,d_0}(x)) = h(e_0 \circ \mathsf{en}_{\pi_s,d_0}(x))$ for $x \in C$. Then we extend h'_0 into

$$h'_1 \colon \operatorname{dom} h'_0 \cup \{e'_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p}), e'_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p})\} \to \mathbb{Q}_1$$

in the following way:

1. $\mathbb{Q}_1 \vDash h'_1(e'_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p})) \lessdot h(e_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p}))$,
2. $\mathbb{Q}_1 \vDash h(e_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p})) \lessdot h'_1(e'_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p}))$,
3. $h'_1(e'_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p}))$ is a terminal in $\mathbb{Q}_1$ iff $e'_0 \circ \mathsf{en}_{\pi_s,d_0}(\hat{p})$ is a terminal in $d'_\cup$.
4. $h'_1(e'_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p}))$ is a terminal in $\mathbb{Q}_1$ iff $e'_1 \circ \mathsf{en}_{\pi_t,d_1}(\hat{p})$ is a terminal in $d'_\cup$.

Now we are going to extend h'_1 to $h' \colon d'_\cup \to \mathbb{Q}_1$ under the constraint

$$\operatorname{ran} h' \subseteq F := \{x \in \mathbb{Q}_1 \mid x \text{ is not extended from some } y \in \operatorname{ran} h \setminus \operatorname{ran} h'_1\}.$$

Then take $D = \operatorname{ran} h \cup \operatorname{ran} h'$. □

We are ready to prove that $\hat{p}$ is secure. Given the amalgamation in Proposition E.6.8, we have

- $(h' \circ e'_0)(\mathsf{en}_{\pi_s,d_0}(i)) = (h \circ e_0)(\mathsf{en}_{\pi_s,d_0}(i))$ for every $i < \hat{p}$, and

- $(h' \circ e'_0)(\mathsf{en}_{\pi_s, d_0}(\hat{p})) \prec (h \circ e_0)(\mathsf{en}_{\pi_s, d_0}(\hat{p}))$.

This implies

- $(h' \circ e'_0)^* \circ \mathsf{cem}_{d'_0}(\mathsf{en}_{\pi_s, d'_0}(i)) = (h \circ e_0)^* \circ \mathsf{cem}_{d_0}(\mathsf{en}_{\pi_s, d_0}(i))$ for every $i < \hat{p}$, and
- $(h' \circ e'_0)^* \circ \mathsf{cem}_{d'_0}(\mathsf{en}_{\pi_s, d'_0}(\hat{p})) \prec (h \circ e_0)^* \circ \mathsf{cem}_{d_0}(\mathsf{en}_{\pi_s, d_0}(\hat{p}))$.

Hence by Theorem E.5.10, we have

$$P(D^*) \vDash s(\mathrm{ran}(h' \circ e'_0)^* \circ \mathsf{cem}_{d'_0}) < s(\mathrm{ran}(h \circ e_0)^* \circ \mathsf{cem}_{d_0}).$$

Similarly, we have

$$P(D^*) \vDash t(\mathrm{ran}(h \circ e_1)^* \circ \mathsf{cem}_{d_1}) < t(\mathrm{ran}(h' \circ e'_1)^* \circ \mathsf{cem}_{d_1})$$

Hence we have

$$\begin{aligned}
& P \vDash s <_{Д^*} t \\
\iff & P(d^*_{\cup}) \vDash s(\mathrm{ran}\, e^*_0 \circ \mathsf{cem}_{d_0}) < t(\mathrm{ran}\, e^*_1 \circ \mathsf{cem}_{d_1}) \\
\iff & P(D^*) \vDash s(\mathrm{ran}(h \circ e_0)^* \circ \mathsf{cem}_{d_0}) < t(\mathrm{ran}(h \circ e_1)^* \circ \mathsf{cem}_{d_1}) \\
\implies & P(D^*) \vDash s(\mathrm{ran}(h' \circ e'_0)^* \circ \mathsf{cem}_{d_0}) < t(\mathrm{ran}(h' \circ e'_1)^* \circ \mathsf{cem}_{d_1}) \\
\iff & P((d'_{\cup})^*) \vDash s(\mathrm{ran}(e'_0)^* \circ \mathsf{cem}_{d_0}) < t(\mathrm{ran}(e'_1)^* \circ \mathsf{cem}_{d_1}) \\
\iff & P \vDash s <_{(Д')^*} t.
\end{aligned}$$

Since Д and Д′ are arbitrary $\hat{p}$-congruent ostensible 1-arity diagrams, we have the desired result. □

Hence, we can define the following:

Definition E.6.9. For a 2-preptyx P and two different P-terms s and t, let $\mathbf{p}^P(s,t)$ be the least secure number relative to s and t. Then we define $\varepsilon^P_{s,t} = +1$ if $P \vDash s <_{Д^*} t$ holds for every (equivalently, some) $\mathbf{p}^P(s,t)$-congruent ostensible 1-arity diagram Д for s and t, and $\varepsilon^P_{s,t} = -1$ otherwise.

As like predilators, Σ^P, $\mathbf{p}^P$, and ε^P completely determine the structure of a 2-preptyx P:

Theorem E.6.10. *Let P be a 2-preptyx and D be a predendrogram. For $s, t \in \mathrm{field}(P)$ and $a, b \Subset D$ with $a \cong \mathrm{arity}\, s$, $b \cong \mathrm{arity}\, t$, we have $P(D) \vDash s(a) < t(b)$ if and only if either*

1. $\mathrm{en}_{\pi^P(s),a}(i) = \mathrm{en}_{\pi^P(t),b}(i)$ *for every* $i < \mathbf{p}^P(s,t)$ *and* $\varepsilon^P_{s,t} = +1$, *or*
2. *There is* $j < \mathbf{p}^P(s,t)$ *such that* $\mathrm{en}_{\pi^P(s),a}(i) = \mathrm{en}_{\pi^P(t),b}(i)$ *for every* $i < j$ *and* $\mathrm{en}_{\pi^P(s),a}(j) \lessdot \mathrm{en}_{\pi^P_t,b}(j)$.

Proof. It suffices to prove the backward implication of the theorem. Suppose that the first disjunct holds. In this case, $\mathrm{Diag}_D(a,b)$ is $\mathbf{p}^P(s,t)$-congruent, so $\varepsilon^P_{s,t} = +1$ implies $P \vDash s <_{\mathrm{Diag}_D(a,b)^*} t$, which implies $P(D^*) \vDash s(\mathrm{ran}\,\mathsf{cem}_a) < t(\mathrm{ran}\,\mathsf{cem}_b)$. We can see that $\mathrm{ran}\,\mathsf{cem}_a$ and $\mathrm{ran}\,\mathsf{cem}_b$ are images of a and b under $\mathsf{cem}_D \colon D \to D^*$, so we get $P(D) \vDash s(a) < t(b)$.

Now suppose that the second disjunct holds. Then the witnessing j is not secure, so there is a j-congruent ostensible 1-arity diagram

$$Д = \begin{array}{ccc} d_\cap & \dashrightarrow & d_1 \\ \vdots & & \vdots{\scriptstyle e_1} \\ d_0 & \xrightarrow[e_0]{} & d_\cup \end{array} \tag{E.14}$$

for s and t such that $P \vDash s <_{Д^*} t$. Note that $\|a\| = \|\pi^P(s)\| = d_0$ and $\|b\| = \|\pi^P(t)\| = d_1$.

Lemma E.6.11. *Let $C = \pi^P(s) \restriction j$. We can find a finite predendrogram E and embeddings $h \colon a \cup b \to E$, $h' \colon d_\cup \to E$ such that the following diagram commutes, where the dashed arrows are quasimorphisms:*

$$\begin{array}{ccc} & C & \overset{\mathrm{en}_{\pi^P(s),a} \restriction j}{\dashrightarrow} a \cup b \\ {\scriptstyle e_0 \circ \mathrm{en}_{\pi^P(s),d_0} \restriction j}\big\downarrow & & \big\downarrow{\scriptstyle h'} \\ d_\cup & \xrightarrow[h]{} & E \end{array}$$

Furthermore, we have

- $E \vDash h'(\mathrm{en}_{\pi^P(s),a}(j)) \lessdot h(e_0 \circ \mathrm{en}_{\pi^P(s),d_0}(j))$,
- $E \vDash h(e_1 \circ \mathrm{en}_{\pi^P(t),d_1}(j)) \lessdot h'(\mathrm{en}_{\pi^P(t),b}(j))$.

Proof. Its proof is similar to that of Proposition E.6.8. We first find an embedding $h \colon d_\cup \to \mathbb{Q}_1$. Then we define a quasimorphism

$$h'_0 \colon a \cup b \restriction \mathrm{en}_{\pi^P(s),a}[C] \to \mathbb{Q}_1$$

by $h'_0(\mathrm{en}_{\pi^P(s),a}(x)) = h(e_0 \circ \mathrm{en}_{\pi^P(s),d_0}(x))$ for $x \in C$. Then we extend h'_0 into

$$h'_1 \colon \operatorname{dom} h'_0 \cup \{\mathrm{en}_{\pi^P(s),a}(j), \mathrm{en}_{\pi^P(t),b}(j)\} \to \mathbb{Q}_1$$

in the following way:

1. $\mathbb{Q}_1 \vDash h'(\mathrm{en}_{\pi^P(s),a}(j)) \lessdot h(e_0 \circ \mathrm{en}_{\pi^P(s),d_0}(j))$,
2. $\mathbb{Q}_1 \vDash h(e_1 \circ \mathrm{en}_{\pi^P(t),d_1}(j)) \lessdot h'(\mathrm{en}_{\pi^P(t),b}(j))$,
3. $h'_1(\mathrm{en}_{\pi^P(s),a}(j))$ is a terminal in $\mathbb{Q}_1$ iff $\mathrm{en}_{\pi^P(s),a}(j)$ is a terminal in $a \cup b$.
4. $h'_1(\mathrm{en}_{\pi^P(t),b}(j))$ is a terminal in $\mathbb{Q}_1$ iff $\mathrm{en}_{\pi^P(t),b}(j)$ is a terminal in $a \cup b$.

Then extend h'_1 to $h' \colon a \cup b \to \mathbb{Q}_1$ under the constraint

$$\operatorname{ran} h' \subseteq F := \{x \in \mathbb{Q}_1 \mid x \text{ is not extended from some } y \in \operatorname{ran} h \setminus \operatorname{ran} h'_1\}.$$

Then take $E = \operatorname{ran} h \cup \operatorname{ran} h'$. □

Let us go back to the main proof. By the choice of j, h, and h', we have the following two:

1. For every $i < j$, we have $h(e_0 \circ \mathsf{en}_{\pi^P(s),d_0}(i)) = h'(\mathsf{en}_{\pi^P(s),a}(i))$.
2. $h(e_0 \circ \mathsf{en}_{\pi^P(s),d_0}(j)) \succcurlyeq h'(\mathsf{en}_{\pi^P(s),a}(j))$.

Then we can see that

$$(h \circ e_0)^*(\mathsf{cem}_{d_0} \circ \mathsf{en}_{\pi^P(s),d_0}(i)) = (h' \circ \mathsf{en}_{d_0,a})^*(\mathsf{cem}_{d_0} \circ \mathsf{en}_{\pi^P(s),d_0}(i))$$

for $i < j$ as follows: For notational convenience, let $x = \mathsf{en}_{\pi^P(s),d_0}(i)$. Then

$$(h \circ e_0)^*(\mathsf{cem}_{d_0}(x))$$
$$= (h_0 \circ e_0)^*(x) = \begin{cases} (h \circ e_0)(x), & x \in \mathsf{term}\, d_0 \leftrightarrow (h \circ e_0)(x) \in \mathsf{term}\, E \\ (h \circ e_0)(x)^*, & \text{otherwise.} \end{cases}$$

Similarly,

$$(h' \circ \mathsf{en}_{d_0,a})^*(\mathsf{cem}_{d_0}(x)) = (h' \circ \mathsf{en}_{d_0,a})^*(x)$$
$$= \begin{cases} (h' \circ \mathsf{en}_{d_0,a})(x), & x \in \mathsf{term}\, d_0 \leftrightarrow (h' \circ \mathsf{en}_{d_0,a})(x) \in \mathsf{term}\, E \\ (h' \circ \mathsf{en}_{d_0,a})(x)^*, & \text{otherwise.} \end{cases}$$

But we know that $(h \circ e_0)(x) = (h' \circ \mathsf{en}_{d_0,a})(x)$. Hence, the conditionals in each equality coincide, and we get the desired equality. We also have

$$E^* \vDash (h \circ e_0)^*(\mathsf{cem}_{d_0} \circ \mathsf{en}_{\pi^P(s),d_0}(j)) \succcurlyeq (h' \circ \mathsf{en}_{d_0,a})^*(\mathsf{cem}_{d_0} \circ \mathsf{en}_{\pi^P(s),a}(j)),$$

so Theorem E.5.10 implies

$$P(E^*) \vDash s(\mathsf{ran}((h')^* \circ \mathsf{en}^*_{d_0,a} \circ \mathsf{cem}_{d_0})) < s(\mathsf{ran}(h^* \circ e_0^* \circ \mathsf{cem}_{d_0})).$$

By a similar argument, we also have

$$P(E^*) \vDash t\big(\mathrm{ran}(h^* \circ e_1^* \circ \mathsf{cem}_{d_1})\big) < t\big(\mathrm{ran}((h')^* \circ \mathrm{en}^*_{d_1,b} \circ \mathsf{cem}_{d_1})\big).$$

From $P \vDash s <_{Д^*} t$, we have

$$P(E^*) \vDash s\big(h^*\big[\mathrm{ran}\, e_0^* \circ \mathsf{cem}_{d_0}\big]\big) < t\big(h^*\big[\mathrm{ran}\, e_1^* \circ \mathsf{cem}_{d_1}\big]\big).$$

In sum, we can deduce

$$P(E^*) \vDash s\big(\mathrm{ran}((h')^* \circ \mathrm{en}^*_{d_0,a} \circ \mathsf{cem}_{d_0})\big) < t\big(\mathrm{ran}((h')^* \circ \mathrm{en}^*_{d_1,b} \circ \mathsf{cem}_{d_1})\big).$$

This readily implies

$$P((a \cup b)^*) \vDash s\big(\mathrm{ran}(\mathrm{en}^*_{d_0,a} \circ \mathsf{cem}_{d_0})\big) < t\big(\mathrm{ran}(\mathrm{en}^*_{d_1,b} \circ \mathsf{cem}_{d_1})\big).$$

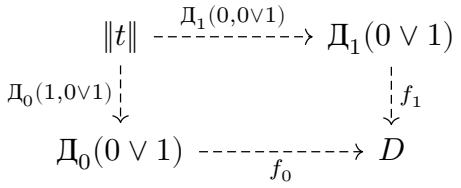
We have $\mathrm{ran}(\mathrm{en}^*_{d_0,a} \circ \mathsf{cem}_{d_0}) = \mathrm{ran}\, \mathsf{cem}_a = \mathsf{cem}_{a\cup b}[a]$ and $\mathrm{ran}(\mathrm{en}^*_{d_1,b} \circ \mathsf{cem}_{d_1}) = \mathrm{ran}\, \mathsf{cem}_b = \mathsf{cem}_{a\cup b}[b]$. Hence we have $P(a \cup b) \vDash s(a) < t(b)$, which is equivalent to $P(D) \vDash s(a) < t(b)$. □

Like the predilator case, $\mathbf{p}^P$ and ε^P also have similar constraints:

Lemma E.6.12. *Let P be a 2-preptyx. For $s, t, u \in \mathrm{field}(P)$, we have $\mathbf{p}^P(s,u) \geq \min(\mathbf{p}^P(s,t), \mathbf{p}^P(t,u))$.*

Proof. Let $p < \min(\mathbf{p}^P(s,t), \mathbf{p}^P(t,u))$. We claim that p is not secure relative to s and u, which implies $p < \mathbf{p}^D(s,u)$. From $p < \min(\mathbf{p}^P(s,t), \mathbf{p}^P(t,u))$, we can find p-congruent ostensible 1-arity diagrams $Д_0, Ж_0$ for s and t, and ostensible 1-arity diagrams $Д_1$ and $Ж_1$ for t and u, such that $P \vDash s <_{Д_0^*} t <_{Д_1^*} u$ and $P \vDash s >_{-Ж_0^*} t >_{-Ж_1^*} u$. Observe that $Д_0(1) = \|t\| = Д_1(0)$. By the amalgamation theorem Theorem D.3.11, we can find a new predendrogram D and quasimorphisms $f_0 \colon Д_0(0 \vee 1) \to D$ and

$f_1 : Д_1(0 \vee 1) \to D$ making the following diagram commute



and $\operatorname{ran} f_0 \cap \operatorname{ran} f_1 = \operatorname{ran} f_0 \circ Д_0(1, 0 \vee 1) = \operatorname{ran} f_1 \circ Д_1(0, 0 \vee 1)$. Now consider an ostensible 1-IU diagram $\mathbf{Д}$

$$\mathbf{Д} = \operatorname{Diag}_D\Big(\operatorname{ran}(f_0 \circ Д_0(0, 0 \vee 1)), \operatorname{ran}(f_0 \circ Д_0(1, 0 \vee 1)), \operatorname{ran}(f_1 \circ Д_1(1, 0 \vee 1))\Big).$$

Note that $\mathbf{Д}(0) = \|s\|$, $\mathbf{Д}(1) = \|t\|$, and $\mathbf{Д}(2) = \|u\|$. Now we claim that $\mathbf{Д}[0, 2]$ is p-congruent: To see this, it suffices to show that for $i < p$,

$$f_0 \circ \mathbf{Д}(0, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(s)}(i) = f_1 \circ \mathbf{Д}(1, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(u)}(i),$$

which follows from the following calculation:

$$\begin{aligned} f_0 \circ \mathbf{Д}(0, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(s)}(i) &= f_0 \circ \mathbf{Д}(1, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(t)}(i) \\ = f_1 \circ \mathbf{Д}(0, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(t)}(i) &= f_1 \circ \mathbf{Д}(1, 0 \vee 1) \circ \mathrm{en}^{-1}_{\pi^P(u)}(i). \end{aligned}$$

This shows $Д_2 := \mathbf{Д}[0, 2]$ is a p-congruent ostensible 1-arity diagram such that $P \vDash s <_{Д_2^*} u$. Similarly, we can find a p-congruent ostensible 1-arity diagram $Ж_2$ such that $P \vDash s >_{-Ж_2^*} u$. This shows $p < \mathbf{p}^P(s, u)$. □

Lemma E.6.13. *For $s, t, u \in \operatorname{field}(P)$, suppose that $\varepsilon^P_{s,t} = \varepsilon^P_{t,u} = +1$. Then $\varepsilon^P_{s,u} = +1$ and $\mathbf{p}^P(s, u) = \min(\mathbf{p}^P(s, t), \mathbf{p}^P(t, u))$.*

Proof. One direction of the inequality was shown in Lemma E.6.12. Hence we prove from the assumption that $\varepsilon^D_{s,u} = +1$ and $\mathbf{p}^P(s, u) \leq \min(\mathbf{p}^P(s, t), \mathbf{p}^P(t, u))$ hold.

Let $p = \min(\mathbf{p}^P(s, t), \mathbf{p}^P(t, u))$. For a p-congruent ostensible 1-arity diagram $Д$ for s and u, we claim that $P \vDash s <_{Д^*} u$, proving both the inequality and $\varepsilon^P_{s,u} = +1$. To ease

the proof, we move our attention from P to $P(\mathbb{Q}_1^*)$ as follows: Let Д be a p-congruent ostensible 1-arity diagram of the form (E.14), and $\iota\colon d_\cup \to \mathbb{Q}_1$ be a predendrogram morphism. Then $P \vDash s <_{Д^*} u$ is equivalent to

$$P(\mathbb{Q}_1^*) \vDash s(\mathrm{ran}(\iota \circ e_0)^* \circ \mathsf{cem}_{d_0}) < u(\mathrm{ran}(\iota \circ e_1)^* \circ \mathsf{cem}_{d_1}).$$

Now let $a = \mathrm{ran}(\iota \circ e_0)$ and $c = \mathrm{ran}(\iota \circ e_1)$. a and c may not be subpredendrograms of $\mathbb{Q}_1$ since their terminal points may not be terminal points of $\mathbb{Q}_1$. Still, they form subsets closed under predecessors. If we let $\iota_a\colon a \to \mathbb{Q}_1$ and $\iota_c\colon c \to \mathbb{Q}_1$ be the inclusion quasimorphims, then $P \vDash s <_{Д^*} u$ becomes equivalent to

$$P(\mathbb{Q}_1^*) \vDash s(\mathrm{ran}\, \iota_a^* \circ \mathsf{cem}_a) < u(\mathrm{ran}\, \iota_c^* \circ \mathsf{cem}_c).$$

Moreover, we also have $\mathrm{en}_{\pi^P(s),a}(i) = \mathrm{en}_{\pi^P(u),c}(i)$ for every $i < p \leq \mathbf{p}^P(s,u)$. Now, let us divide the cases:

1. Suppose that $p = \mathbf{p}^P(s,t) = \mathbf{p}^P(t,u)$. By the ω-homogeneity of $\mathbb{Q}_1$, we can find a predecessor-closed finite subset $b \subseteq \mathbb{Q}_1$ such that $b \cong$ arity t and $\mathrm{en}_{\pi^P(s),a}(i) = \mathrm{en}_{\pi^P(t),b}(i)$ for every $i < p$. Hence $\mathrm{Diag}_{\mathbb{Q}_1}(a,b)$ and $\mathrm{Diag}_{\mathbb{Q}_1}(b,c)$ become p-congruent ostensible 1-arity diagram. Hence Theorem E.6.10 with the assumption $\varepsilon^P_{s,t} = \varepsilon^P_{t,u} = +1$ implies

$$P(\mathbb{Q}_1^*) \vDash s(\mathrm{ran}\, \iota_a^* \circ \mathsf{cem}_a) < t(\mathrm{ran}\, \iota_b^* \circ \mathsf{cem}_b) < u(\mathrm{ran}\, \iota_c^* \circ \mathsf{cem}_c). \tag{E.15}$$

2. Now suppose that $p = \mathbf{p}^P(s,t) < \mathbf{p}^P(t,u)$. By the ω-homogeneity argument, let us find a predecessor-closed finite subset $b \subseteq \mathbb{Q}_1$ isomorphic to arity t such that $\mathrm{en}_{\pi^P(s),a}(i) = \mathrm{en}_{\pi^P(t),b}(i)$ for every $i < p$ and $\mathrm{en}_{\pi^P(t),b}(p) \lessdot \mathrm{en}_{\pi^P(u),c}(p)$ over $\mathbb{Q}_1$. (E.15) still holds in this case; The first inequality is witnessed by that $\mathrm{Diag}_{\mathbb{Q}_1}(a,b)$ is a $\mathbf{p}^P(s,t)$-congruent ostensible 1-arity diagram and $\varepsilon^P_{s,t} = +1$. The second inequality is witnessed by the second case of Theorem E.6.10.

3. Now suppose that $\mathbf{p}^P(s,t) > \mathbf{p}^P(t,u) = p$. We can find a predecessor-closed finite subset $b \subseteq \mathbb{Q}_1$ isomorphic to arity t such that $\mathrm{en}_{\pi^P(t),b}(i) = \mathrm{en}_{\pi^P(u),c}(i)$ for every $i < p$ and $\mathrm{en}_{\pi^P(s),a}(p) \lessdot \mathrm{en}_{\pi^P(t),b}(p)$. We again have (E.15).

Hence, in all possible cases, we get the inequality. □

Hence we have the following abstract construction theorem:

Theorem E.6.14. *Every 2-preptyx is characterized by the following data:*

1. *The domain set X with a linear order $<^X$.*
2. *A function $\mathbf{p} \colon X \times X \to \mathbb{N}$ such that*
   - $\mathbf{p}(x,y) = \mathbf{p}(y,x)$ *for* $x, y \in X$.
   - $\mathbf{p}(x,z) = \min(\mathbf{p}(x,y), \mathbf{p}(y,z))$ *for* $x, y, z \in X$ *such that* $x \leq^X y \leq^X z$.
3. *For each $x \in X$, an associated natural dendrogram* arity x *and a priority pre-dendrogram $\pi(x)$ isomorphic to* arity x *such that for every $i, j < \mathbf{p}(x,y)$,*
   - $\mathrm{lh}^{\pi(x)}(i) = \mathrm{lh}^{\pi(y)}(i)$,
   - $\pi(x) \vDash i \multimap j$ *iff* $\pi(y) \vDash i \multimap j$,
   - $\pi(x) \vDash i \lessdot j$ *iff* $\pi(y) \vDash i \lessdot j$,
   - $\pi^{\pi(x)}(i) = \pi^{\pi(y)}(i)$,
   - *If* $\pi(x) \vDash i \multimap k$ *and* $\pi(y) \vDash i \multimap k'$ *(k and k' need not be less than $\mathbf{p}(x,y)$) then* $\pi^{\pi(x)}(k) = \pi^{\pi(y)}(k')$.

Proof. For a given 2-preptyx P, $(\mathrm{field}(P), <^D, \mathbf{p}^D, \mathrm{arity}^P, \Sigma^P)$ satisfies the conditions. Conversely, we can construct a 2-preptyx from the given data as follows: Define P by

$\mathrm{field}(P) = X$, and for $t \in \mathrm{field}(P)$, $\mathrm{arity}^P\, t = \mathrm{arity}(t)$. For $s, t \in \mathrm{field}(P)$ and an arity diagram Д for s and t of the form (E.14), let us define $s <_{\text{Д}} t$ iff either

1. $s \leq^X t$ and $e_0 \circ \mathrm{en}_{\pi(s),d_0}(i) = e_1 \circ \mathrm{en}_{\pi(t),d_1}(i)$ for $i < \mathbf{p}(s,t)$, or
2. There is $i < \mathbf{p}(s,t)$ such that $e_0 \circ \mathrm{en}_{\pi(s),d_0}(i) = e_1 \circ \mathrm{en}_{\pi(t),d_1}(i)$ for $i < \mathbf{p}(s,t)$ and $e_0 \circ \mathrm{en}_{\pi(s),d_0}(j) = e_1 \circ \mathrm{en}_{\pi(t),d_1}(j)$ for every $j < i$.

From the given conditions of $\leq$ and $\mathbf{p}$, we can prove the inequality

$$\min(\mathbf{p}(s,t), \mathbf{p}(t,u)) \leq \mathbf{p}(s,u)$$

unconditionally. Then we can check that P is a positive 2-semiptyx, and P being a 2-preptyx easily follows. Furthermore, Theorem E.6.10 implies $\leq^P = \leq$, $\mathbf{p}^P = \mathbf{p}$, $\mathrm{arity}^P = \mathrm{arity}$, and $\pi^P = \pi$. □

Example E.6.15. Let us describe an evaluation 2-ptyx $\mathrm{ev}_\alpha \colon D \mapsto D(\alpha)$ for a fixed linear order α as follows: For each finite predendrogram d, $\{\sigma \in d(\alpha) \mid \mathrm{supp}_d^{\mathrm{ev}_\alpha}(\sigma) = d\}$ is not empty iff d is prime, i.e., the predendrogram representation of d has only one finite branch (with an obvious single terminal point.) For a prime $d = \{t\}$,

$$\{\sigma \in d(\alpha) \mid \mathrm{supp}_d^{\mathrm{ev}_\alpha}(\sigma) = d\} = \{t(a) \mid a \in [\alpha]^{\mathrm{arity}\, t}\}.$$

Suppose that t has priority order $\prec$. For each $t(a)$ in the set above, let us associate an enumeration $\vec{a}$ of a given by

$$\vec{a} = \langle \mathrm{en}_{\prec,a}(i) \mid i < \mathrm{arity}\, t\rangle = \langle \mathrm{en}_a(\Sigma_t^d(i)) \mid i < \mathrm{arity}\, t\rangle.$$

We can think that an evaluation $a \mapsto t(a)$ corresponds to an ev_α-term $\tau_{\vec{a}}$. Then the following holds:

1. $\pi^{\mathrm{ev}_\alpha}(\tau_{\vec{a}})$ is a unique priority predendrogram with a unique terminal point of length $\mathrm{lh}\,\vec{a}$ such that the priority order $\prec$ of the unique terminal point satisfies

$$\forall i, j < \mathrm{lh}\,\vec{a}\big[i \prec j \iff \alpha \vDash \vec{a}(i) < \vec{a}(j)\big].$$

2. $\mathbf{p}^{\mathrm{ev}_\alpha}(\tau_{\vec{a}}, \tau_{\vec{b}})$ is the least i such that the ith component of $\vec{a}$ and that of $\vec{b}$ differs.

3. $\tau_{\vec{a}} <^{\mathrm{ev}_\alpha} \tau_{\vec{b}}$ iff the $i = \mathbf{p}^{\mathrm{ev}_\alpha}(\tau_{\vec{a}}, \tau_{\vec{b}})$th component of $\vec{a}$ is less than that of $\vec{b}$.

E.7 2-Predendrograms

We finish this chapter with the 2-predendrograms, which must have formed a separate chapter.

Definition E.7.1. A *2-predendrogram* is a structure $(P, \lessdot, \multimap, \pi)$ satisfying the following:

1. $(P, \multimap)$ is a forest with the immediate successor relation $\multimap$. Moreover, if $\multimap^*$ is the transitive closure of $\multimap$, then for each $x \in P$, $\{y \in P \mid y \multimap^* x\}$ is finite and well-ordered by $\multimap^*$. We call the size of $\{y \in P \mid y \multimap^* x\}$ the *length of* x and denote it by $\mathrm{lh}(x)$.

2. For every $x \in P$, either x is a terminal point (i.e., x has no immediate successor) or there is y such that $x \multimap^* y$ and y is a terminal point.

3. The *local comparison order* $\lessdot$ is a partial order over P. Moreover, the following sets are linearly ordered by $\lessdot$:

    - The set of *roots* of P, i.e. elements with no immediate predecessor.
    - The set immediate successors of x for each $x \in P$.

4. The *priority (1-)predendrogram* ϖ is a function of domain P. For each $x \in P$, $\varpi(x) = (\mathrm{lh}(x), <_{\varpi(x)})$ is a priority predendrogram of field $\mathrm{lh}(x)$. Moreover,

   - $x \multimap y$ implies $\varpi(x) \subseteq \varpi(y)$.
   - If x and y have the same immediate predecessor, then $\varpi(x) = \varpi(y)$.

Like the 1-predendrogram case, $\mathrm{term}(P)$ denotes the set of terminal points of P. We also define $\mathrm{pred}(x)$ as the $\multimap$-increasing enumeration of $\{y \in P \mid y \multimap^* x\} \cup \{x\}$. $\mathrm{pred}(x)$ is a sequence of length $\mathrm{lh}(x) + 1$. If $\langle x_0, \cdots, x_{m-1}, x_m \rangle$ is the sequence of predecessors of x and $l \leq m$, we write $x \restriction l$ for x_l.

We can turn every 2-predendrogram into a 2-preptyx as follows:

Definition E.7.2. For a 2-predendrogram P, let us define its *functorization* $\mathrm{Fun}(P)$ as follows: For a 1-predendrogram C, we define $\mathrm{Fun}(P)(C)$ by the set of

$$\langle x_0, c_0, x_1, \cdots, x_{m-1}, c_{m-1}, x_m \rangle$$

such that $x_0, \cdots, x_m \in P$ is a $\multimap$-increasing chain with $\mathrm{lh}^P(x_i) = i$, x_m is a terminal point of P, and $c_0, \cdots, c_{m-1} \in D$ be such that the map $i \mapsto c_i$ is a 1-predendrogram morphism from $\varpi^P(x_m)$ to D. We compare elements of $\mathrm{Fun}(P)(D)$ by the Kleene-Brouwer order. For a 1-predendrogram morphism $f \colon C \to D$, we define $\mathrm{Fun}(P)(f)$ by

$$\langle x_0, c_0, x_1, \cdots, x_{m-1}, c_{m-1}, x_m \rangle \mapsto \langle x_0, f(c_0), x_1, \cdots, x_{m-1}, f(c_{m-1}), x_m \rangle.$$

Also, we define $\mathrm{supp}_C^{\mathrm{Fun}(P)}$ by

$$\langle x_0, c_0, x_1, \cdots, x_{m-1}, c_{m-1}, x_m \rangle \mapsto \{c_0, c_1, \cdots, c_{m-1}\} \Subset D.$$

One can see that $\mathrm{Fun}(P)$ is a Freund-style 2-preptyx. We can also define a cell decomposition of a 2-preptyx as follows: By mimicking Definition D.1.10, we can define

$\equiv_n^P$ over a denotation system for a 2-preptyx P. For given information from the abstract construction of 2-preptykes, we can formulate the following:

Proposition E.7.3. *For a 2-preptyx P, let us define its cell decomposition* $\mathrm{Cell}(P)$ *as follows:*

1. $\mathrm{Cell}(P) = \{[t]_{\equiv_n^P} \mid t \in \mathrm{field}\, P \wedge n \leq |\mathrm{arity}\, t|\}$.
2. $\mathrm{Cell}(P) \vDash [s]_{\equiv_m^P} \multimap [t]_{\equiv_n^P}$ *iff* $n = m + 1$ *and* $s \equiv_m^P t$.
3. $\mathrm{Cell}(P) \vDash [s]_{\equiv_m^P} \lessdot [t]_{\equiv_n^P}$ *iff* $m = n$, $s \equiv_{m-1}^P t$, *and* $s <^P t$.
4. $\pi^{\mathrm{Cell}(P)}([s]_{\equiv_m^P}) = \pi^P(s) \restriction m$.

Then $\mathrm{Cell}(P)$ *is a 2-predendrogram. Also, for a predilator morphism* $\iota \colon P \to Q$*, if we define* $\mathrm{Cell}(\iota)$ *by*

$$\mathrm{Cell}(\iota)([s]_{\equiv_m^P}) = [\iota(s)]_{\equiv_m^Q},$$

then $\mathrm{Cell}(\iota)$ *is a well-defined 2-predendrogram morphism.* □

Cell and Fun are naturally isomorphic in the sense that a version of Proposition D.1.14 and Proposition D.1.16 holds. By mimicking the arguments in Section D.2, we also have the following:

Theorem E.7.4. *For a 2-predendrogram P, we say a* thread predendrogram associated with an infinite branch $b = \langle x_i \mid i \in \mathbb{N} \rangle$ *is a predendrogram* $\bigcup_{i \in \mathbb{N}} \pi^P(x_i)$*. A* thread predendrogram of P *is a thread predendrogram associated with some infinite branch b. A* thread predilator of P *is a predilator of the form* $\mathrm{Fun}(C)$ *for some thread predendrogram C of P.*

For a 2-predendrogram P, $\mathrm{Fun}(P)$ *is a 2-ptyx iff P is locally well-founded, and the closure of every thread predilator of P is not a dilator.* □

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LIST OF SYMBOLS

$\trianglelefteq_{\mathsf{Grow}^3}$, 325

$<^{T_0}_{\mathsf{trans}}$, 103

$<^{T_0}_{\beta}$, 102

$<_{\mathsf{Con}}$, 3

$<_{\mathsf{KB}}$ Kleene-Brouwer order, 338

$<_{\mathsf{LV}}$ Level-then-value order, 274

$[\Pi^0_1, \Pi^0_0]$, 203

$\mathsf{ATR}^{\mathsf{set}}_0$, 345

$\beta\text{-rank}_{T_0}(M)$ β-rank of a transitive model of T_0, 105

$\mathbf{\Delta}(D)$, 397

$\mathfrak{e}$, 227

$\mathcal{G}_{\mathsf{Appl}^2}(P, Q)$, 308

$\mathcal{G}_{\vartheta^2}(P, Q)$, 311

$\mathcal{L}(\mathfrak{M})$, 140

$\mathcal{L}[\alpha]$, 148

$\mathcal{P}^2$, 307

$\check{\Gamma}$ Dual pointclass of Γ, 351

$\mathsf{Clim}(D)$ The climax of D, 19

$\delta(R)$ The supremum of R-recursive ordinals, 135

δ^1_n Projective ordinal, 343

$\delta^{\mathfrak{M}}$, 141

$\mathsf{Dil}(D)$ "D is a dilator", 42

eaf , 382

efa , 384

en_a , 338

$\forall(\mu)x$ Measure quantifier, 222

$\mathfrak{a}$, 381

$\mathfrak{f}$, 379

$\mathfrak{i}$, 348

$\mathfrak{t}$, 347

$\Gamma[R]$, 131

$\mathsf{Grow}(D, E)$, 57

$\mathsf{Grow}_{\Pi^1_2}$, 300

$\mathsf{Grow}_{\Sigma^1_2}$, 300

$\widehat{\mu}_s^{d,\beta}$, 265

$\widehat{\nu}_s^{d,\beta}$, 264

$\widehat{\nu}_s^{d,\beta}[\vec{k}]$, 289

$\mathsf{HJ}^{\alpha}(\emptyset)$ Iterated hyperjump of $\emptyset$, 207

$\int^2 P$, 306

$\kappa_{\mathfrak{M}}$, 143

KP Kripke-Platek set theory, 341

KPℓ , 342

KPi , 342

$\leq_{\mathsf{Con}}$, 3

$\mathsf{Alt}_{\Sigma^1_2}(R)$ Σ^1_2-altitude of R, 177

Emb^k_γ , 220

$\mathsf{Imp}_{(D,\varrho),a,b}$ Implicational predilator, 187

LK_α , 148

opp , 400

Prime_σ , 398

$\mathsf{SP}^\xi_{\mathfrak{M}}$ ξth least Spector class over $\mathfrak{M}$, 144

$\mathsf{SP}_{\mathfrak{M}}$ The least Spector class over $\mathfrak{M}$, 143

μ^k_γ , 220

$\mu^{d,\beta}_s$, 290

$\nu^{d,\beta}_s$ A measure family over Ω^1_{M}, 265

Ω^1_{M} Martin flower, 249

$\mathsf{Den}(\pi)$, 156

$\mathsf{Occ}(\Gamma)$, 151

$\pi^l_{\gamma_0,\gamma_1}$, 220

$\prec^{\check{\Gamma}}_{\Gamma}$, 53

$\prec_\Gamma$, 51

PreSep Pre-separation functor, 440

PRS Primitive Recursive Set theory, 342

RCA^β_0 , 147

Γ-RFN(T) Γ-reflection of T, 351

$\mathsf{SDil}_{\mathsf{F}}$, 377

$\mathsf{in}_i(x)$, 344

$\mathsf{m}_f(\phi)$, 151

$\subseteq^{\check{\Gamma}}_{\Gamma}$, 8

$\subseteq_\Gamma$, 8

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