On a cofinal Reinhardt embedding without powerset

Hanul Jeon

Cornell University

2024-10-18

CUNY Set theory seminar

Table of Contents

Large cardinals

- Large cardinals are means to gauge the strength of extensions of ZFC.
- Since the beginning of set theory, set theorists defined stronger notion of large cardinals (Inaccessible, Mahlo, Weakly compact, Measurable, Woodin, Supercompact, etc.)
- Large cardinals stronger than measurable cardinals are usually defined in terms of elementary embedding.

Reinhardt embedding

Reinhardt defined an 'ultimate' form of large cardinal axiom:

Definition

A Reinhardt embedding is a non-trivial elementary embedding $j\colon V \to V$.

Reinhardt embedding

Reinhardt defined an 'ultimate' form of large cardinal axiom:

Definition

A Reinhardt embedding is a non-trivial elementary embedding $j\colon V \to V$.

This poor axiom destined an Icarian fate:

Theorem (Kunen 1971)

ZFC *proves there is no Reinhardt embedding.*
In fact, there is no elementary emebdding $j\colon V_{\lambda+2} \to V_{\lambda+2}$.

(Not in)consistent weakenings

Set theorists studied the non-inconsistent weakening of Reinhardt cardinals:

Definition

- $\mathrm{I}_3(\lambda)$: There is an elementary $j\colon V_\lambda \to V_\lambda$.
- $\mathrm{I}_2(\lambda)$: There is a Σ_1-elementary $j\colon V_{\lambda+1} \to V_{\lambda+1}$.
- $\mathrm{I}_1(\lambda)$: There is an elementary $j\colon V_{\lambda+1} \to V_{\lambda+1}$.
- $\mathrm{I}_0(\lambda)$: There is an elementary $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$.

They are not known to be inconsistent over ZFC.

What about other options?

We may have a consistent version of Reinhardt embedding over a weakening of ZFC.

What about other options?

We may have a consistent version of Reinhardt embedding over a weakening of ZFC.
We do not know the consistency of ZF with a Reinhardt embedding, but

Theorem (Schlutzenberg 2024)

If $\mathsf{ZFC} + \mathrm{I}_0$ *is consistent, then so is*

$$\mathsf{ZF} + \mathsf{DC}_\lambda + \exists j \colon V_{\lambda+2} \to V_{\lambda+2}.$$

ZFC without powerset

The option we will examine is when we drop the axiom of powerset.

Remark

In ZFC without Replacement, the following are equivalent:

1. Replacement
2. Collection: For every family of proper classes $\{C_x \mid x \in I\}$ indexed by a set I, we have a family of sets $\{\hat{C}_x \mid x \in I\}$ such that $\hat{C}_x \subseteq C_x$.
3. Reflection principle.

It is no longer valid if we drop the Axiom of Powerset.

ZFC without Powerset can be weird

Theorem (Gitman-Hamkins-Johnstone 2011)

Let ZFC− be ZFC without Powerset. Then each of the following is consistent with ZFC−:

1. ω_1 *is singular.*
2. *Every set of reals is countable but* ω_1 *exists.*
3. *There are sets of reals of size* ω_n *for* $n < \omega$*, but none of size* ω_ω*.*
4. *The failure of Łoś's theorem.*

However, ZFC$^-$, ZFC without Powerset but Collection, is free from these ill-behaviors.

Formulating a Reinhardt embedding

Let us formulate a set theory with Reinhardt embedding j.
j is a 'proper class,' but it cannot be definable:

Theorem (Suzuki 1999)

ZF *proves there is no definable elementary embedding* $j\colon V \to V$.

Hence we must introduce a new symbol for a Reinhardt embedding.

Definition

ZFC_j is a first-order theory over the language $\{\in, j\}$ with the following axioms:

1. Axioms of ZFC.
2. Axiom schemes over the new language $\{\in, j\}$.

ZFC_j^- is defined similarly. Also, $j\colon V \to V$ is the combination of the following assertions:

1. $\exists x(j(x) \neq x)$.
2. An axiom scheme for the elementarity of j for $\{\in\}$-formulas: If $\psi(\vec{x})$ is a formula without j, then

$$\forall x[\phi(\vec{x}) \leftrightarrow \phi(j(\vec{x}))].$$

Matthews' result

Richard Matthews proved that $\mathrm{ZFC}_j^- + j\colon V \to V$ is consistent:

Theorem (Matthews 2022)

$\mathrm{ZFC} + \mathrm{I}_1$ *proves there is a transitive model of* $\mathrm{ZFC}_j^- + j\colon V \to V$.

However, Matthews' model does not satisfy

Definition

An embedding $j\colon V \to V$ is cofinal if for every set a, there is b such that $a \in j(b)$.

In fact, Hayut proved that ZFC_j^- is inconsistent with a cofinal Reinhardt embedding.

A Cofinal Reinhardt embedding

Question

Is $\mathsf{ZF}_j^- + j\colon V \to V$ consistent with the cofinality of j?

Theorem (J.)

$\mathsf{ZFC} + \mathrm{I}_0$ *proves there is a transitive model of* ZF_j^- *with a cofinal* $j\colon V \to V$.

Matthews' proof

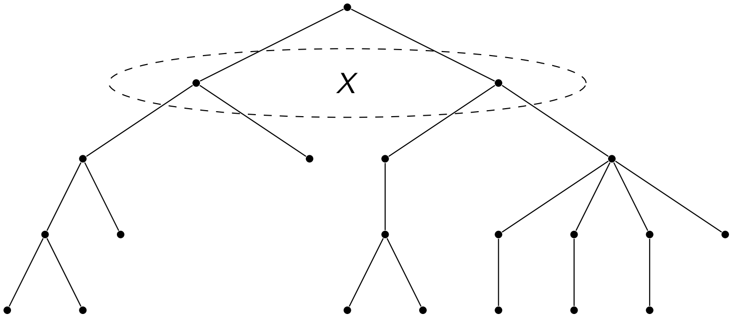
Let us sketch the main idea of (a variant of) a proof of Matthews' result.

Observation

Let λ be a strong limit cardinal, and let H_{λ^+} be the set of all hereditarily size $< \lambda^+$ sets:

$$H_{\lambda^+} = \{x : |\mathsf{TC}(x)| < \lambda^+\}.$$

Then we can code every member of H_{λ^+} into a tree of size λ.



$\mathrm{trcl}(X)$

The tree for X is: $\{\langle x_0, x_1, \cdots, x_n\rangle \mid X \ni x_0 \ni x_1 \ni \cdots \ni x_n\}$.

Tree coding

For every well-founded tree T over V_λ, we can associate a set $\mathfrak{t}(T)$.

Lemma

For a well-founded tree T, Let us define

$$\mathfrak{t}(T) = \{\mathfrak{t}(T \downarrow \langle x \rangle) \mid \langle x \rangle \in T\}.$$

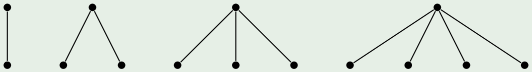
Then we have the following:

1. *If T is a well-founded tree over V_λ, then $\mathfrak{t}(T) \in H_{\lambda^+}$.*
2. *Every member of H_{λ^+} has a form $\mathfrak{t}(T)$.*

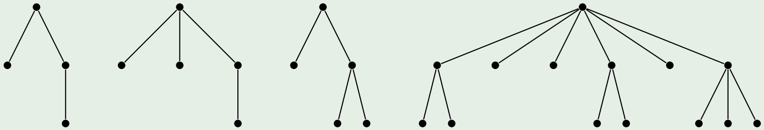
Note that even $1 = \{0\}$ has different ways for tree coding, even up to isomorphism.

Example

All of these code the same set $1 = \{0\}$:

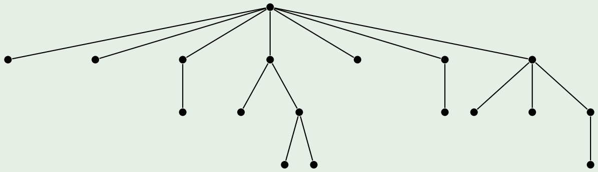


Also, all of these code the same set $2 = \{0, 1\}$:

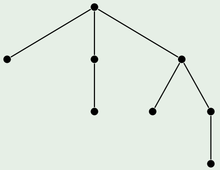


Example

The following tree codes $3 = \{0, 1, 2\}$:



But we also have a simpler tree coding 3:



Which trees are 'equal'

Definition

Let S and T be well-founded trees. Define $S =^* T$ if and only if there is a binary relation $R \subseteq S \times T$ such that $\langle\langle\rangle, \langle\rangle\rangle \in R$, and $\langle\sigma, \tau\rangle \in R$ iff

1. $\forall\langle u\rangle \in (S \downarrow \sigma)\exists\langle v\rangle \in (T \downarrow \tau)\big[(\sigma^\frown\langle u\rangle, \tau^\frown\langle v\rangle) \in R\big]$, and
2. and vice versa.

We say $S \in^* T$ iff there is $\langle u\rangle \in T$ such that $S = T \downarrow \langle u\rangle$.

Theorem

If S, T are well-founded, then $S =^* T$ iff $\mathfrak{t}(S) = (T)$. Also, $S \in^* T$ iff $\mathfrak{t}(S) \in (T)$.

Tree interpretation

We can pull a formula over H_{λ^+} into $V_{\lambda+1}$:

Definition

Let ϕ be a formula. Define ϕ^{t} as follows:

1. $(x \in y)^{\mathrm{t}} \equiv (x \in^* y)$. $(x = y)^{\mathrm{t}} \equiv (x =^* y)$.
2. $(\phi \circ \psi)^{\mathrm{t}} \equiv \phi^{\mathrm{t}} \circ \psi^{\mathrm{t}}$. $(\neg\phi)^{\mathrm{t}} \equiv \neg\phi^{\mathrm{t}}$. $(\circ = \wedge, \vee, \to.)$
3. For a quantifier Q,

$$(\mathrm{Q}x\phi(x))^{\mathrm{t}} \equiv \mathrm{Q}T[T \text{ is a well-founded tree over } V_\lambda \to \phi^{\mathrm{t}}(T)].$$

Lemma

For every formula ϕ and well-founded trees $T_0, \cdots, T_{m-1}$ over V_λ, we have

$$H_{\lambda^+} \vDash \phi(\mathfrak{t}(T_0), \cdots \mathfrak{t}(T_{m-1})) \iff V_{\lambda+1} \vDash \phi^{\mathfrak{t}}(T_0. \cdots T_{m-1}).$$

Pushing $j\colon V_{\lambda+1} \to V_{\lambda+1}$ into H_{λ^+}

Theorem

Let $j\colon V_{\lambda+1} \to V_{\lambda+1}$ be an I_1-embedding. For a well-founded tree T over V_λ, define

$$k(\mathfrak{t}(T)) = \mathfrak{t}(j(T)).$$

Then k is well-defined and an elementary embedding $H_{\lambda^+} \to H_{\lambda^+}$.

Corollary

(H_{λ^+}, k) is a model of $\mathsf{ZFC}_j^- + j\colon V \to V$.

Finding a cofinal embedding

The resulting embedding is not cofinal by a Kunen inconsistency-type argument.
To get a cofinal elementary embedding, we start from a base model with a stronger property.

Definition (Goldberg-Schlutzenberg 2021)

Let $j\colon V_{\lambda+n} \to V_{\lambda+n}$ be an elementary emebdding. We say j is cofinal if every $a \in V_{\lambda+n}$ is contained in $j(b)$ for some $b \in V_{\lambda+n}$...

Finding a cofinal embedding

The resulting embedding is not cofinal by a Kunen inconsistency-type argument.
To get a cofinal elementary embedding, we start from a base model with a stronger property.

Definition (Goldberg-Schlutzenberg 2021)

Let $j\colon V_{\lambda+n} \to V_{\lambda+n}$ be an elementary emebdding. We say j is cofinal if every $a \in V_{\lambda+n}$ is contained in $j(b)$ for some $b \in V_{\lambda+n}$...

... Is it a correct definition?

Cofinal embedding over $V_{\lambda+n}$

Such b may not exist when a has the largest rank. However, we can still state $a \in j(b)$ for a 'small' subset b of $V_{\lambda+n}$:

Definition

Let $a \in V_{\lambda+n}$ be a binary relation. For $i \in \operatorname{dom}(a)$, define

$$(a)_i = \{x \mid \langle i, x\rangle \in a\}.$$

Also, for $a, b \in V_{\lambda+n}$, define

$$(a : b) = \{(a)_i \mid i \in b\}.$$

The correct definition of a cofinality over $V_{\lambda+n}$

Definition (Goldberg-Schlutzenberg 2021)

$j\colon V_{\lambda+n} \to V_{\lambda+n}$ is cofinal if for every $a \in V_{\lambda+n}$ there is $b, c \in V_{\lambda+n}$ such that $a \in (j(b) : j(c))$.

Theorem (Goldberg-Schlutzenberg 2021)

$j\colon V_{\lambda+n} \to V_{\lambda+n}$ is cofinal iff n is even.
In particular, $j\colon V_{\lambda+2} \to V_{\lambda+2}$ is cofinal.

Flat pairing

The previous definitions of $(a : b)$ and $(a)_i$ also have a 'flaw' since the usual Kuratowski ordered pair $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$ raises the rank by $+2$.

Hence we have to use Quine-Rosser flat pairing instead of the usual pairing function.

Definition

Let

$$s(x) = \begin{cases} 2x+1 & x \in \omega, \\ x & \text{otherwise.} \end{cases}$$

Define $f_0(a) = s[a]$ and $f_1(a) = s[a] \cup \{0\}$, then

1. f_0, f_1 are one-to-one.
2. $\operatorname{ran} f_0 \cap \operatorname{ran} f_1 = \varnothing$.

Define $\langle a, b \rangle = f_0[a] \cup f_1[b]$.

We also need a flat tuple to define trees, whose definition is similar.

Where to find $V_{\lambda+2}$?

We turn $V_{\lambda+2}$ with an elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$ into a transitive model of ZF_j^-

$j\colon V_{\lambda+2} \to V_{\lambda+2}$ is inconsistent with ZFC. But...

Theorem (Schlutzenberg 2024)

Let $i\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an I_0-embedding. If $j = i \restriction V_{\lambda+2}^{L(V_{\lambda+1})}$, then $L(V_{\lambda+1}, j)$ satisfies

1. $\mathrm{ZF} + \mathrm{DC}_\lambda + \mathrm{I}_0(\lambda)$.
2. $j\colon V_{\lambda+2} \to V_{\lambda+2}$ *is elementary.*
3. $V_{\lambda+2} \subseteq L(V_{\lambda+1})$.

The model

Now let us work over the Schlutzenberg's model $L(V_{\lambda+1}, j)$, which is a choiceless model. H_{λ^+} or similar notions do not work well without Choice.

Definition

Let X be a set. $H(X)$ is the union of all transitive sets M such that M is a surjective image of a member of X.

$H(X)$ is a transitive set, and every non-empty set in $H(X)$ is a surjective image of a member of X.

The Collection Principle

We can prove that $H(V_{\lambda+2})$ satisfies all axioms of ZF^- except for Collection. For Collection, we need the Collection principle:

Definition (Goldberg)

We say $V_{\lambda+1}$ satisfies the Collection principle if every binary relation $R \subseteq V_\lambda \times V_{\lambda+1}$ has a subrelation $S \subseteq R$ of the same domain such that $\operatorname{ran} S$ is a surjective image of $V_{\lambda+1}$.

Theorem (Essentially by Goldberg)

$L(V_{\lambda+1}, j)$ thinks $V_{\lambda+2}$ satisfies the Collection principle.

Theorem

The Collection principle for $V_{\lambda+2}$ implies $H(V_{\lambda+2})$ satisfies Collection.

$L(V_{\lambda+1}, j)$ satisfies the Collection principle for $V_{\lambda+2}$, so $H(V_{\lambda+2}) \vDash \mathsf{ZF}^-$ in this model.

The main result

Again, we can define the tree interpretation $\mathfrak{t}$ satisfying

$$H(V_{\lambda+2}) \vDash \phi(\mathfrak{t}(T_0), \cdots, \mathfrak{t}(T_{m-1})) \iff V_{\lambda+2} \vDash \phi^{\mathfrak{t}}(T_0, \cdots, T_{m-1}).$$

Then we can push $j\colon V_{\lambda+2} \to V_{\lambda+2}$ to $k\colon H(V_{\lambda+2}) \to H(V_{\lambda+2})$ by letting $k(\mathfrak{t}(T)) = \mathfrak{t}(j(T))$.

Theorem

In $L(V_{\lambda+1}, j)$, $k\colon H(V_{\lambda+2}) \to H(V_{\lambda+2})$ is a cofinal elementary embedding.

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $\mathfrak{t}(T)$ for some well-founded tree over $V_{\lambda+1}$.

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $\mathfrak{t}(T)$ for some well-founded tree over $V_{\lambda+1}$.
Thus we prove: For every well-founded tree T we can find T' such that $T \in^* j(T')$.

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $\mathfrak{t}(T)$ for some well-founded tree over $V_{\lambda+1}$.
Thus we prove: For every well-founded tree T we can find T' such that $T \in^* j(T')$.
$T \in V_{\lambda+2}$, so by the cofinality of j, we can find sets $a, b \in V_{\lambda+2}$ such that $T \in (j(a) : j(b))$.

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $\mathfrak{t}(T)$ for some well-founded tree over $V_{\lambda+1}$.
Thus we prove: For every well-founded tree T we can find T' such that $T \in^* j(T')$.
$T \in V_{\lambda+2}$, so by the cofinality of j, we can find sets $a, b \in V_{\lambda+2}$ such that $T \in (j(a) : j(b))$.
Then define

$$T' = \{\langle x\rangle^\frown\sigma \mid x \in b \wedge \sigma \in (a)_x \wedge (a)_x \text{ is a well-founded tree}\}$$

Proof.

Every set in $H(V_{\lambda+2})$ is of the form $\mathfrak{t}(T)$ for some well-founded tree over $V_{\lambda+1}$.
Thus we prove: For every well-founded tree T we can find T' such that $T \in^* j(T')$.
$T \in V_{\lambda+2}$, so by the cofinality of j, we can find sets $a, b \in V_{\lambda+2}$ such that $T \in (j(a) : j(b))$.
Then define

$$T' = \{\langle x\rangle^\frown \sigma \mid x \in b \land \sigma \in (a)_x \land (a)_x \text{ is a well-founded tree}\}$$

$T \in (j(a) : j(b))$ implies there is $z \in j(b)$ such that $T = (j(a))_z$.
Hence $T \in^* j(T')$. □

Comparing the two proofs

Matthews' proof	My proof
Working over $\mathrm{ZFC} + \mathrm{I}_1$	Schlutzenberg's model
I_1 embedding $j\colon V_{\lambda+1} \to V_{\lambda+1}$.	An embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$
Turn $V_{\lambda+1}$ to H_{λ^+}	Turn $V_{\lambda+2}$ to $H(V_{\lambda+2})$
Collection holds by Choice	by Collection Principle
A model of ZFC^-_j	A model of ZF^-_j with a cofinal j

Questions

Question

How strong the theory ZF_j^- with a cofinal $j\colon V \to V$ is? For example, does it imply the consistency of $\mathsf{ZFC} + \mathrm{I}_1$?

Question

Does ZF_j^- with a cofinal $j\colon V \to V$ prove λ^+ or $V_{\lambda+1}$ exists, for $\lambda = \sup_{n<\omega} j^n(\operatorname{crit} j)$?

(Note: $V_{\lambda+1} \in H(V_{\lambda+2})$ in the Schlutzenberg's model.)

Any other Questions?

Thank you!