Proof theory for higher pointclasses

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Strength of natural theories

Definition

For two theories S and T, define

$$S \leq_{\mathsf{Con}} T \iff (\mathsf{Con}(T) \to \mathsf{Con}(S))$$

and

$$S <_{\mathsf{Con}} T \iff T \vdash \mathsf{Con}(S).$$

Phenomenon

For two 'natural' theories S and T, either

$$S \leq_{\mathsf{Con}} T \text{ or } T \leq_{\mathsf{Con}} S.$$

Also, there is no sequence of 'natural' theories

$$T_0 >_{\mathsf{Con}} T_1 >_{\mathsf{Con}} T_2 >_{\mathsf{Con}} \cdots .$$

Various people pointed out that it holds (Steel, Koellner, Simpson, Montalban, etc.)

$\leq_{\mathsf{Con}}$ is ill-behaved

Theorem (Folklore)

There are theories T_0 and T_1 such that neither $T_0 \leq_{\mathsf{Con}} T_1$ nor $T_1 \leq_{\mathsf{Con}} T_0$.
Also, there are theories $\langle T_n \mid n < \omega \rangle$ such that $T_0 >_{\mathsf{Con}} T_1 >_{\mathsf{Con}} T_2 >_{\mathsf{Con}} \cdots$.

What is wrong?

In practice, when we prove Con(T) from T', we actually prove stronger statements. (For example, the existence of a transitive model of T.)

$\leq_{\mathsf{Con}}$ is too 'fine' to catch the behavior of the strength of natural theories.

Proof-theoretic ordinal

Proof theorists found a characteristic gauging the strength of a theory linearly.

Definition

For a theory T, let us define the proof-theoretic ordinal of T by

$$|T|_{\mathsf{WO}} = \sup\{|\alpha| : \alpha \text{ is a recursive linear order such that } T \vdash \mathsf{WO}(\alpha)\}.$$

Example

- $|\mathsf{PA}|_{\mathsf{WO}} = |\mathsf{ACA}_0|_{\mathsf{WO}} = \varepsilon_0$ (Gentzen)
- $|\mathsf{ACA}_0^+|_{\mathsf{WO}} = \varphi_\omega(0)$
- $|\mathsf{ATR}_0|_{\mathsf{WO}} = \Gamma_0$
- $|\mathsf{KP}|_{\mathsf{WO}} = \psi_\Omega(\varepsilon_{\Omega+1})$.

Example

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It does not precisely gauge the strength of a theory, e.g., $|T|_{\mathsf{WO}} = |T + \mathsf{Con}(T)|_{\mathsf{WO}}$

Then what does $|T|_{\mathsf{WO}}$ gauge?

Another phenomenon for natural theories

The only practical tools to get theories of the same strength are forcing and inner models, and they do not change Σ^1_2-consequences by Shoenfield absoluteness.

Phenomenon

For every 'sufficiently strong' set theory S and T, either

$$S \subseteq_{\Sigma^1_2} T \text{ or } T \subseteq_{\Sigma^1_2} S.$$

Also, the size of the Σ^1_2-consequences of T is determined by the strength of T:

$$S \subseteq_{\Sigma^1_2} T \iff S \leq_{\mathsf{Con}} T.$$

It fails for 'unnatural' examples.

Also, if we have large cardinals, then we cannot change Π^1_∞-consequences of a theory.

Phenomenon (Steel)

For every S and T including roughly $\mathsf{ZFC}+\mathsf{PD}$ (or ZFC with infinitely many Woodin cardinals),

$$S \subseteq_{\Pi^1_\infty} T \text{ or } T \subseteq_{\Pi^1_\infty} S.$$

Also, we have

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Question

Can we explain why they happen?

Kleene normal form theorem

Theorem (Kleene, ACA_0)

For every Π^1_1-formula $\phi(X)$ and a real A, we can find an A-recursive linear order α such that

$$\phi(A) \leftrightarrow \mathsf{WO}(\alpha).$$

Recall the definition of $|T|_{\mathsf{WO}}$: it is a supremum of all recursive well-orders whose well-orderedness is provable from T.

Then does $|T|_{\mathsf{WO}}$ say something about the Π^1_1-consequences of T?

Characterizing Ordinal Analysis

Theorem (Walsh 2023)

For Π^1_1-sound theories S, T extending ACA_0,

$$|S|_{\mathrm{WO}} \leq |T|_{\mathrm{WO}} \iff S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T.$$

Here

1. $S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T$ means $S \vdash^{\Sigma^1_1} \phi \implies T \vdash^{\Sigma^1_1} \phi$ for all $\phi \in \Pi^1_1$,
2. $T \vdash^{\Sigma^1_1} \phi$ is 'ϕ is provable from T with true Σ^1_1 sentences.'

Γ-reflection

To describe the well-foundedness of the strength of theories, we need an appropriate analogue of Con(T) for Π^1_1 sentences:

Definition

For a class of formulas Γ, Γ-RFN(T) is the assertion

$$\forall\phi \in \Gamma[T \vdash \phi \to \phi \text{ is true}].$$

Con(T) is equivalent to Π^0_1-RFN(T).

Theorem (Walsh 2023)

For arithmetically definable Π^1_1-sound theories S, T extending ACA_0,

$$|S|_{\mathsf{WO}} \le |T|_{\mathsf{WO}} \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_1} \Pi^1_1\text{-RFN}(T) \to \Pi^1_1\text{-RFN}(S).$$

Also,

Theorem

For arithmetically definable Π^1_1-sound theories S, T extending ACA_0,

$$|S|_{\mathsf{WO}} < |T|_{\mathsf{WO}} \iff T \vdash^{\Sigma^1_1} \Pi^1_1\text{-RFN}(S).$$

Why dilators?

Ordinals capture the strength of theories, but they only gauge the Π^1_1-consequences of a theory.

Girard developed the notion of dilators and ptykes to describe the Π^1_2- and Π^1_n-proof theory; i.e. the proof theory for Π^1_n-consequences of a theory.

An example: Class ordinals

Example

There is no transitive class isomorphic with $\mathrm{Ord}+\mathrm{Ord}$, but there is a way to represent it.

Let X be the class of pairs of the form $(0,\xi)$ or $(1,\xi)$ for an ordinal ξ, and impose an order over X as follows:

- $(i,\eta)<(i,\xi)$ iff $\eta<\xi$.
- $(0,\eta)<(1,\xi)$ always holds.

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- $(0, \eta) < (1, \xi)$ always holds.

Observation: The above construction is 'uniform.'

Dilators

Let F be a map sending α to the expression for $\alpha + \alpha$. Then

1. We can extend F to a functor from the category of linear orders to the same category.
2. F preserves direct limits and pullbacks.
3. If α is a well-order, then so is $F(\alpha)$.

Dilators

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Definition

A predilator is a functor from the category of linear orders LO to LO preserving direct limits and pullbacks.
A predilator F is a dilator if $F(\alpha)$ is a well-order when α is.

Dilators look too 'large,' but it turns out that we can recover a dilator from its small part:

Lemma

Every predilator is determined by its restriction to the category of finite ordinals.

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Definition

A predilator D is countable if $D(n)$ is countable for each $n \in \mathbb{N}$ (if viewed as objects of the category of finite ordinals.)
A countable predilator D is A-recursive if we can code D into an A-recursive set.

The higher Kleene normal form theorem

Dilators represent Π^1_2-sentences like ordinals represent Π^1_1-sentences.

Theorem (Girard, ACA_0)

For every Π^1_2-formula $\phi(X)$ and a real A, we can find an A-recursive predilator D such that

$$\phi(A) \iff D \text{ is a dilator.}$$

Proof-theoretic dilator

Definition

For a theory T, define

$$|T|_{\Pi^1_2} = \sum\{D \mid D \text{ is a recursive predilator such that } T \vdash D \text{ is a dilator}\}.$$

$|T|_{\Pi^1_2}$ is unique up to bi-embeddability.

Ptykes

Ptykes (sing. ptyx) are 'higher' versions of dilators.

Definition

Let FN^0 be the category of linear orders, and for two categories $\mathcal{C}$ and D, let $\mathcal{C} \to \mathcal{D}$ be the category of continuous[1] functors from $\mathcal{C}$ to $\mathcal{D}$.
Define the category of n-preptykes $\mathsf{FN}^n := \mathsf{FN}^{n-1} \to \mathsf{FN}^0$. An n-preptyx P is an n-ptyx if it safisfies

$$\forall\pi[\pi \text{ is an } (n-1)\text{-ptyx} \implies P(\pi) \text{ is well-ordered}].$$

[1] Preserving direct limits and pullbacks

We can also encode n-ptykes into a small set, and define countable n-ptykes and A-recursive ptykes.

Theorem (Girard, ACA_0)

For every Π^1_{n+1}-formula $\phi(X)$ and a real A, we can find an A-recursive n-preptyx P such that

$$\phi(X) \iff P \text{ is an } n\text{-ptyx.}$$

We can define $|T|_{\Pi^1_{n+1}}$ as the sum of all T-provably recursive n-ptykes.

Non-linearity

Unfortunately, proof-theoretic ptykes are not ordinals, and they are not linearly comparable.
Also, Π^1_2-consequences are never linearly comparable:

Theorem (Aguilera-Pakhomov)

There is no ordinal characteristic $o(T)$ for a theory T satisfying

$$o(S) \leq o(T) \iff S \subseteq^{\Sigma^1_2}_{\Pi^1_2} T.^2$$

[2] It is still well-founded.

Linearity: Case Σ^1_2

Recall the statement of Π^1_2-completeness of dilators: Every Π^1_2-statement $\phi(X)$ is equivalent to 'D is a dilator' for an X-recursive predilator D.

Corollary

For every Σ^1_2-statement $\phi(X)$ and a real A, we can find an A-recursive predilator D such that

$$\phi(A) \iff D \text{ is } \underline{\text{not}} \text{ a dilator.}$$

Pseudodilators

Let us call a prediltor D an pseudodilator if D is not a dilator.
Each pseudodilator D is associated with an ordinal:

Definition

For an pseudodilator D, the climax $\mathrm{Clim}(D)$ of D is the least ordinal α such that $D(\alpha)$ is ill-founded.

Pseudodilators express more ordinals in the following sense:

Example

The supremum of all ordertypes of recursive well-orders is ω_1^{CK}.
The supremum of all $\mathrm{Clim}(D)$ for a recursive pseudodilator D is δ^1_2.

Σ^1_2-proof-theoretic ordinal

Definition

For a theory T, define

$$s^1_2(T) = \sup\{\mathrm{Clim}(D) \mid D \text{ is a recursive predilator and } T \vdash D \text{ is not a dilator}\}$$

Example

- $s^1_2(\mathrm{ACA}_0) = s^1_2(\mathrm{KP}) = \omega^{\mathrm{CK}}_1$.
- $s^1_2(\Pi^1_1\text{-}\mathrm{CA}_0) = \omega^{\mathrm{CK}}_\omega$.
- (Aguilera) $s^1_2(\Pi^1_2\text{-}\mathrm{CA}_0) = \sup_{n<\omega} \sigma_n$.[3]

[3] σ_n is the least ordinal with elementary chains of length n.

$s^1_2(T)$ and the height of transitive models of T

Question

For every ‘reasonably small’ (e.g., recursive) α, do we have

$$s^1_2(\mathsf{ID}_{<1+\alpha}) = \omega^{\mathsf{CK}}_\alpha?$$

Conjecture

For every Σ^1_2-sound r.e. extension T of $\Pi^1_1\text{-}\mathsf{CA}_0$, we have

$$s^1_2(T) = \min\{M \cap \mathrm{Ord} \mid M \text{ transitive and } M \vDash \mathsf{ATR}_0 + \Sigma^1_2(T)\}.$$

Where $\Sigma^1_2(T)$ is the set of all Σ^1_2-consequences of T.

Comparing Σ^1_2-consequences

Theorem (J.)

For Σ^1_2-sound theories S, T extending ACA_0,

$$s^1_2(S) \le s^1_2(T) \iff S \subseteq^{\Pi^1_2}_{\Sigma^1_2} T.$$

Also for arithmetically definable Σ^1_2-sound theories S, T extending Σ^1_2-AC_0, we have

$$s^1_2(S) \le s^1_2(T) \iff \Sigma^1_2\text{-}\mathrm{AC}_0 \vdash^{\Pi^1_2} \Sigma^1_2\text{-}\mathrm{RFN}(T) \to \Sigma^1_2\text{-}\mathrm{RFN}(S).$$

Case Π^1_3

So far, we have the linearity of Π^1_1 and Σ^1_2 consequences of a theory. But they enjoy a descriptive set theoretic property: the prewellordering property.

Π^1_3 also has the prewellordering property, which hints at the linearity of Π^1_3-consequences. But it requires $\mathbf{\Delta}^1_2$-Determinacy.

Universal dilator

We need a special parameter to compare Π^1_3-consequences linearly:

Definition

A dilator D is universal if it embeds every countable dilator.

There is a natural way to define a universal dilator Ω^1 from the sharps of reals.

Measurable dilator

Measurable dilator is a special type of universal dilator admitting a system of measures:

Definition

A universal dilator Φ is measurable if there are measures $\{\mu_d \mid d \text{ finite dilator}\}$ over Φ^d such that

1. (Coherence) For $i\colon d \to d'$, let $i^*\colon \Phi^{d'} \to \Phi^d$ by $i^*(p) = p \circ i$. Then $X \in \mu_d \iff (i^*)^{-1}[X] \in \mu_{d'}$.
2. (σ-completeness) For each $X_d \in \Phi_d$ and a countable dilator D, we can find $e \in \Phi^D$ such that for every $p\colon d \to D$, we have $e \circ p \in X_d$.

If there is a measurable dilator, then $\mathbf{\Delta}^1_2$-Determinacy holds.

Case Π^1_3 (cont'd)

Let us work over ZFC with the existence of a measurable dilator Φ.

Theorem (J.)

For every Π^1_3*-sound theories* S, T *extending* $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2\text{-Det}$, *we have*

$$|S|_{\Pi^1_3}(\Phi) \le |T|_{\Pi^1_3}(\Phi) \iff S \subseteq^{\Sigma^1_3}_{\Pi^1_3} T.$$

Theorem (J.)

For arithmetically definable Π^1_3*-sound theories* S, T *extending* $\mathsf{ACA}_0 + \mathbf{\Delta}^1_2\text{-Det}$, *we have*

$$|S|_{\Pi^1_3}(\Phi) \le |T|_{\Pi^1_3}(\Phi) \iff \mathsf{ACA}_0 \vdash^{\Sigma^1_3} \Pi^1_3\text{-RFN}(T) \to \Pi^1_3\text{-RFN}(S).$$

Future directions

It looks like that my arguments for Σ^1_2 and Π^1_3 generalize to all Σ^1_{2n} and Π^1_{2n+1}, so the following should hold:

Guess (The odd case)

For arithmetically definable Π^1_{2n+1}-sound theories S, T extending $\mathsf{ACA}_0 + \mathbf{\Delta}^1_{2n}$-Det, the following are all equivalent:

1. $|S|_{\Pi^1_{2n+1}}(\Omega^{2n}) \le |T|_{\Pi^1_{2n+1}}(\Omega^{2n})$,
2. $S \subseteq^{\Sigma^1_{2n+1}}_{\Pi^1_{2n+1}} T$,
3. $\mathsf{ACA}_0 \vdash^{\Sigma^1_{2n}} \Pi^1_{2n+1}\text{-RFN}(T) \to \Pi^1_{2n+1}\text{-RFN}(S)$.

where Ω^{2n} is a measurable $2n$-ptyx.

Guess (The even case)

We can define $s^1_{2n}(T)$ with '$2n$-antiptykes' satisfying the following: For arithmetically definable Σ^1_{2n}-sound theories S, T extending $\Sigma^1_{2n}\text{-AC}_0 + \mathbf{\Delta}^1_{2n-2}\text{-Det}$, the following are all equivalent:

1. $s^1_{2n}(S) \le s^1_{2n}(T)$,
2. $S \subseteq^{\Pi^1_{2n}}_{\Sigma^1_{2n}} T$,
3. $\Sigma^1_{2n}\text{-AC}_0 \vdash^{\Pi^1_{2n}} \Sigma^1_{2n}\text{-RFN}(T) \to \Sigma^1_{2n}\text{-RFN}(S)$.

Going further?

Steel also stated the following observation in his paper:

Phenomenon (Steel)

For natural theories S, T extending ZFC plus 'there are infinitely many Woodin cardinals and a measurable above,' we have

$$(\mathrm{Th}(L(\mathbb{R}))_S \subseteq (\mathrm{Th}(L(\mathbb{R}))_T \text{ or } (\mathrm{Th}(L(\mathbb{R}))_T \subseteq (\mathrm{Th}(L(\mathbb{R}))_S.$$

Here $(\mathrm{Th}(L(\mathbb{R}))_T$ is the set of all statements over $L(\mathbb{R})$ that is T-provable.

Can we find a proof-theoretic characteristic and an ordinal characteristic capturing $(\mathrm{Th}(L(\mathbb{R}))_T$?

Questions

Thank you!