Very large set axioms over Constructive set theories

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3 Main results
Measurable cardinals

- Defined in terms of ultrafilters
- Scott (1961) associated measurable cardinals with **elementary embeddings**:

**Definition**

Let $M \subseteq V$ be a transitive class.

- $j: V \rightarrow M$ is an **elementary embedding** if $j$ respects all first-order formulas, that is,

\[ V \models \phi(\vec{a}) \iff M \models \phi(j(\vec{a})). \]

- The **critical point** $\text{crit } j$ of an elementary embedding $j: V \rightarrow M$ is the least ordinal moved by $j$. 

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Very large set axioms over Constructive set theories
Theorem (Scott 1961)

A cardinal $\kappa$ is measurable if and only if it is a critical point of some elementary embedding $j : V \rightarrow M$.

Theorem (Scott 1961)

Measurable cardinals do not exist in $L$. 
Climbing to the large cardinal hierarchy

Generalizations of measurable cardinals were extensively studied by various people. (For example, Solovay-Reinhardt-Kanamori 1978).

One of these examples include:

Definition

A cardinal $\kappa$ is extendible if for each $\alpha$ we can find an elementary embedding $j: V_{\kappa+\alpha} \rightarrow V_\zeta$ for some $\zeta$ with $\text{crit } j = \kappa$. 
Definition (Rank-into-rank embeddings)

A cardinal $\kappa$ is

1. $I_3$ if $\kappa$ is a critical point of $j : V_\lambda \rightarrow V_\lambda$,
2. $I_2$ if $\kappa$ is a critical point of $j : V \rightarrow M$ such that $V_\lambda \subseteq M$,
3. $I_1$ if $\kappa$ is a critical point of $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$,
4. $I_0$ if $\kappa$ is a critical point of $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$,

where $\lambda = \sup_{n<\omega} j^n(\kappa)$ is the first ordinal above $\kappa$ fixed by $j$. 
Very large set axioms over Constructive set theories
Reinhardt’s dream

The attempt to find a stronger notion of large cardinal bore the notion now known as a Reinhardt cardinal.

**Definition**

A Reinhardt cardinal is a critical point of an elementary embedding $j : V \rightarrow V$. 
Kunen (1971) proved that Reinhardt’s notion cannot be realized over ZFC:

**Theorem (Kunen Inconsistency theorem, 1971)**

*Work over ZFC, there is no Reinhardt cardinals. In fact, there is no elementary embedding $j: V_{\lambda+2} \rightarrow V_{\lambda+2}$*

It is still open whether a Reinhardt cardinal is compatible with ZF, that is, in the choiceless context.
Night Flying: Choiceless large cardinals

Some difficulties in working without the axiom of choice:

- Not all cardinal correspond to an ordinal.
- Not all successor cardinal are regular.
- Constructions and proofs become harder.
- Needs to consider more before applying the known results.
Super Reinhardtness

**Definition**

A cardinal $\kappa$ is

- **Super Reinhardt** if for each ordinal $\alpha$ we can find an elementary embedding $j : V \to V$ such that $\text{crit } j = \kappa$ and $j(\kappa) > \alpha$.

- $A$-super Reinhardt for a class $A$ if for each ordinal $\alpha$ we can find an elementary embedding for $A$-formulas $j : V \to V$ such that $\text{crit } j = \kappa$ and $j(\kappa) > \alpha$.

Super Reinhardtness was defined by Woodin in 1983, and extensively studied in the mid-2010s in the context of Woodin’s HOD-dichotomy.
Total Reinhardtness

**Definition**

- We call $\text{Ord}$ is total Reinhardt if for every class $A$ we can find a cardinal $\kappa$ which is $A$-super Reinhardt.

- $\kappa$ is total Reinhardt if $(V_\kappa, V_{\kappa+1}) \models \text{Ord}$ is total Reinhardt.
Theorem (Goldberg)

The following two theories are equiconsistent over ZF + DC:

1. For some ordinal $\lambda$, there is an elementary embedding $V_{\lambda+2} \rightarrow V_{\lambda+2}$
2. ZFC + I$_0$.

Theorem (Goldberg)

Work over ZF + DC, if there is an elementary embedding $j: V_{\lambda+3} \rightarrow V_{\lambda+3}$, then we have the consistency of ZFC + I$_0$. 
The Wholeness axiom

Before defining the Wholeness axiom, let us analyze the definition of a Reinhardt cardinal:

**Definition**

ZFC with a Reinhardt cardinal is a theory comprising:

- Language: $\in$ and a unary function symbol $j$,
- Axioms: Usual axioms of ZFC, with the elementarity of $j$, and Separation and Replacement for $j$-formulas.

**Definition (Corazza 2000)**

The Wholeness Axiom WA is obtained by restricting Replacement to formulas with no $j$. 
We can further weaken WA as follows:

**Definition**

The theory Basic Theory of Elementary Embedding (BTEE) is a claim that $j: V \rightarrow V$ is an elementary embedding. BTEE does not subsume Separation and Replacement for $j$-formulas. That is, we obtain BTEE by dropping Separation for $j$-formulas from WA.

We can strengthen BTEE by adding TI$_j$, the transfinite induction for $j$-formulas.
Theorem (Corazza)

Work over ZFC,

1. $\mathbf{I}_3 \implies \text{Con}(\text{ZFC} + \text{WA})$ and $\text{WA} \implies$ a proper class of extendibles.

2. $0^\# \implies L \models \text{BTEE}$ and $\text{BTEE} \implies n$-ineffable cardinal for each (meta-)natural $n$.

(A cardinal $\kappa$ is $n$-effable if for every $f : [\kappa]^n \to 2$ there is a stationary $S$ subset of $\kappa$ such that $f \upharpoonright [S]^n$ is constant.)
Weakening set theory: the theory $\text{ZFC}^-$

**Definition**

$\text{ZFC}^-$ is obtained by dropping Powerset and replacing Replacement to Collection from ZFC.

Collection is the following statement:

$$\forall a[\forall x \in a \exists y \phi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \phi(x, y)].$$

- Collection is stronger than Replacement, and they are equivalent if we assume Powerset.
- $\text{ZFC}^-$, a mere ZFC without Powerset, is ill-behaved. (Gitman-Hamkins-Johnstone 2016)
Reinhardt embeddings over $\text{ZFC}^-$

Work over $\text{ZFC}_j^-$, the theory obtained by adding $j$ and allowing $j$ to the axiom schemes of $\text{ZFC}^-$. The following result shows a Reinhardt embedding is compatible with $\text{ZFC}^-$:

**Theorem (Matthews)**

$\text{ZFC}$ proves the followings are equivalent:

- There is an elementary embedding $j : H_{\lambda^+} \to H_{\lambda^+}$, and
- There is an elementary embedding $k : V_{\lambda+1} \to V_{\lambda+1}$.

Especially, if $\lambda$ is $I_1$, then $(H_{\lambda^+}, j)$ is a model of $\text{ZFC}_j^-$ with a non-trivial elementary embedding $j : V \to V$ and $V_{\text{crit}, j}$ exists.
However, a Reinhardt embedding cannot be cofinal:

**Definition**

An elementary embedding $j : V \rightarrow V$ is **cofinal** if for each $x$ we can find $y$ such that $x \in j(y)$.

**Theorem (Matthews)**

*Work in $\text{ZFC}^-$, if $j : V \rightarrow V$ is a non-trivial $\Sigma_0$-elementary embedding and $V_{\text{crit}_j}$ exists, then $j$ cannot be cofinal.*
IZF and CZF: A brief history

- (H. Friedman, 1973) Intuitionistic ZF (IZF) with the double-negation translation between IZF and ZF.
- Various attempts to formalize the foundation for Bishop-styled constructive mathematics.
- Myhill’s constructive set theory CST.
Axioms of ZF

Definition

1. Extensionality: \( a = b \iff \forall x(x \in a \iff x \in b) \).
2. Pairing: \( \{a, b\} \) exists.
3. Union: \( \bigcup a \) exists.
4. Separation: \( \{x \in a \mid \phi(x)\} \) exists,
5. Replacement: \( \{F(x) \mid x \in a\} \) exists if \( F \) is a class function.
6. Power set: \( \mathcal{P}(a) = \{x \mid x \subseteq a\} \) exists.
7. Regularity: Every set has a \( \in \)-minimal element.
8. Infinity: \( \mathbb{N} \) exists.
Axioms of IZF

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>1. Extensionality: ( a = b \iff \forall x(x \in a \iff x \in b) ).</td>
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<tr>
<td>2. Pairing: ( {a, b} ) exists.</td>
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<tr>
<td>3. Union: ( \bigcup a ) exists.</td>
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<td>4. Separation: ( {x \in a \mid \phi(x)} ) exists.</td>
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<tr>
<td>5. Collection: if ( \forall x \in a \exists y \phi(x, y) ), then there is ( b ) such that ( \forall x \in a \exists y \in b \phi(x, y) ).</td>
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<td>6. Power set: ( \mathcal{P}(a) = {x \mid x \subseteq a} ) exists.</td>
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<td>7. Set Induction: ( \forall a[\forall x \in a \phi(x) \rightarrow \phi(a)] \rightarrow \forall a \phi(a) ).</td>
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<tr>
<td>8. Infinity: ( \mathbb{N} ) exists.</td>
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# Axioms of CZF

## Definition

1. **Extensionality:** \( a = b \iff \forall x(x \in a \iff x \in b) \).  
2. **Pairing:** \( \{a, b\} \) exists.  
3. **Union:** \( \bigcup a \) exists.  
4. **Bounded Separation:** \( \{x \in a \mid \phi(x)\} \) exists if \( \phi \) is bounded.  
5. **Strong Collection:** if \( \forall x \in a \exists y \phi(x, y) \), then there is \( b \) such that \( \forall x \in a \exists y \in b \phi(x, y) \) and \( \forall y \in b \exists x \in a \phi(x, y) \).  
6. **Subset Collection:** There is a full subset of \( \text{mv}(a, b) \).  
7. **Set Induction:** \( \forall a[\forall x \in a \phi(x) \rightarrow \phi(a)] \rightarrow \forall a \phi(a) \)  
8. **Infinity:** \( \mathbb{N} \) exists.
Differences between IZF and CZF

IZF

1. Full separation
2. Powerset
3. Impredicative
4. Equiconsistent with ZF

and more...

CZF

1. Bounded separation
2. Subset collection
3. Allows type-theoretic interpretation
4. Far more weaker than ZF
Large set axioms

- Ordinals over constructive set theories are not well-behaved. (e.g., every ordinal is well-ordered $\implies$ Excluded Middle for $\Delta_0$-formulas, $\alpha \subseteq \beta$ does not imply $\alpha \in \beta$ or $\alpha = \beta$.)

- We define large cardinal properties over constructive set theories by mimicking the structural properties of $H_\kappa$ and $V_\kappa$. 
Multi-valued functions

The notion of multi-valued function provides a syntactic sugar for Collection.

**Definition**

Let $A$ and $B$ be classes. We call a relation $R$ of domain $A$ and codomain $B$ a multi-valued function from $A$ to $B$. (Notation: $R: A \rightrightarrows B$)

If $R: A \rightrightarrows B$ and $R^{-1}: A \rightrightarrows B$, we write $R: A \iff B$, and we call $B$ a subimage of $R$.

Multi-valued functions replace functions in CZF-context.
Definition (Collection, restatement)

If $R: a \Rightarrow V$ is a first-order definable class multi-valued function with parameters, then we can find a set codomain $b$ of $R$. (That is, $R: a \Rightarrow b$.)

Definition (Strong Collection, restatement)

If $R: a \Rightarrow V$ is a first-order definable class multi-valued function with parameters, then we can find a set subimage $b$ of $R$. (That is, $R: a \Leftrightarrow b$.)

Definition (Subset Collection, equivalent formulation)

For given sets $a$, $b$, we can find a set $c$ such that if $r: a \Rightarrow b$, then $c$ contains a subimage of $r$. (i.e., there is $d \in c$ such that $r: a \Leftrightarrow d$.)
Defining large sets

With some pain, we can prove

**Lemma (ZFC)**

A cardinal $\kappa$ is regular if and only if for $a \in H_\kappa$ and $f : a \to H_\kappa$, we have $\text{Im} f \subseteq H_\kappa$.

By mimicking the above result, we have

**Definition**

A transitive set $K$ is regular if $a \in K$ and $r : a \Vdash K$, then we can find some subimage $b \in K$ of $r$ (that is, $r : a \Vdash b$.)
Similarly, we can see

**Lemma (ZFC)**

A cardinal $\kappa$ is inaccessible if and only if $\kappa$ is regular and $H_\kappa$ satisfies:

1. $\omega \in H_\kappa$, $H_\kappa$ is closed under union and intersection, and
2. if $a, b \in H_\kappa$, then $c := \{ \text{Im} f \mid f : a \to b \} \in H_\kappa$.

**Definition**

A set $K$ is inaccessible if $K$ is regular and

1. $\omega \in K$, $K$ is closed under union and intersection, and
2. if $a, b \in K$, then we can find $c \in K$ such that we can always find a subimage of $r : a \supseteq b$, $r \in K$ from $c$. 
Consistency hierarchy

- PA
- KP
- ZF
- ZFC
- ZF \text{inacc.}
- RTEE
- V=L
- 0# measurable
- super-
  - extensible
- WA
- I_3
- I_1
- R,
  - SR.
  - Ord is
  - TR
- ZFC^+ + R
- (Kunen
  - Inconsistency)

\text{CZF}^+ \text{ regular,}
\text{CZF}^+ \text{ inacc,} \ldots
= KP^+ \text{ recursively}
large ord.s.
Large sets and elementary embeddings

Let us consider elementary embeddings over CZF:

Definition

Let \( j: V \rightarrow M \) be an elementary embedding. A set \( K \) is a critical point of \( j \) if \( K \) is the ‘least’ set lifted by \( j \) in the sense that \( j(x) = x \) for all \( x \in K \) and \( K \in j(K) \).

Definition

A set \( K \) is critical if \( K \) is inaccessible and a critical point of an elementary embedding \( j: V \rightarrow M \).

Question: the consistency strength of CZF with a critical cardinal.
Two scenarios

Scenario 1

Scenario 2

Very large set axioms over Constructive set theories
A Lower bound

Theorem (J., Matthews, CZF)

Let $K$ be a critical point of a $\Sigma_0$-elementary embedding $j: V \rightarrow M$ such that $K$ satisfies $\Delta_0$-separation. Then $K \models IZF$.

(Note: the above theorem does not require Separation, Strong Collection or Set Induction for $j$-formulas.)

Theorem

CZF with a critical set proves the consistency of $ZFC + BTEE$
### Reinhardt embeddings

**Definition**

An inaccessible set $K$ is a Reinhardt set if $K$ is a critical point of $j: V \rightarrow V$.

**Theorem**

*CZF with a Reinhardt cardinal proves $\text{Con}(\text{ZF} + \text{WA})$.*
Go beyond the Reinhardtness

**Definition**

- An inaccessible set $K$ is **super Reinhardt** if for every set $a$ we can find an elementary embedding $j : V \to V$ such that $K$ is a critical point of $j$ and $a \in j(K)$.

- An inaccessible set $K$ is **$K$-super Reinhardt** if for every set $a$ we can find an $A$-elementary embedding $j : V \to V$ such that $K$ is a critical point of $j$ and $a \in j(K)$.

**Theorem**

*CZF with a super Reinhardt set proves the consistency of ZF with a Reinhardt cardinal.*
Definition

$V$ is total Reinhardt (abbr. $V$ is TR) is the following claim: for every class $A$, there is an $A$-super Reinhardt set $K$.

Theorem

CZF with ‘$V$ is TR’ proves all axioms of IZF. Furthermore, the following two theories are equiconsistent:

- CZF + ‘$V$ is TR,’ and
- ZF + ‘$V$ is TR.’

(The exact definition for super/total Reinhardtness require the formulation of constructive second-order set theory.)
Review: Large cardinals

Constructive set theories

Main results

Q&A

Very large set axioms over Constructive set theories
A rough sketch for the proofs

The proof divides into 2-3 main steps:

1. Internal analysis of the given large set axiom. Usually produces a model of IZF + X.

2. Double-negation translation: Friedman-styled translation, Gambino’s Heyting-valued model, or their combinations. The resulting lower bound is of the form Con(ZF + X)

3. If possible, derive the consistency strength in terms of ZFC with large cardinal axioms.
Open problems

1. Non-trivial upper bounds for the consistency strength.
2. Better lower bounds. (For example, can we derive \( \text{Con}(\text{ZFC} + \text{WA}) \) from \( \text{Con}(\text{ZF} + \text{WA}) \)?)
3. Defining other large set notions (e.g., supercompactness and extendibles) and analyzing their consistency strength.
4. Questions regarding machinery in the paper, e.g., second-order constructive set theory.
Questions

Very large set axioms over Constructive set theories
Thank you!
Constructive second-order set theory

Definition (Constructive Gödel-Bernays set theory, CGB)

CGB is defined over the two-sorted languages (sets and classes) with the following axioms:

- Axioms of CZF for sets.
- Every set is a class, and every element of a class is a set.
- Class Extensionality: two classes are equal if they have the same set members.
- Elementary Comprehension: if $\phi(x, p, C)$ is a first-order formula with a class parameter $C$, then there is a class $A$ such that $A = \{x \mid \phi(x, p, C)\}$. 
Definition (CGB, Continued)

- **Class Set Induction:**
  \[ \forall^1 A\left[\forall^0 x (\forall^0 y \in x (y \in A) \to x \in A)\right] \to \forall^0 x (x \in A) \].

- **Class Strong Collection:**
  \[ \forall^1 R \forall^0 a [R : a \Rightarrow V \to \exists^0 b (R : a \Leftrightarrow b)] \].

Definition (Intuitionistic Gödel-Bernays set theory, IGB)

IGB is obtained by adding the following axioms to CGB:

- Axioms of IZF for sets.
- **Class Separation:** if \( A \) is a class and \( a \) is a set, then \( A \cap a \) is a set.

Note that CGB and IGB are conservative extensions of CZF and IZF respectively.
The definition of an elementary embedding $j: V \to M$ requires quantifying over formulas $\phi$:

$$\phi(\vec{a}) \iff \phi^M(j(\vec{a})).$$

We resolve this problem by introducing the infinite conjunction $\land$. 

**Definition (CGB with the infinite connectives, $\text{CGB}_\infty$)**

$\text{CGB}_\infty$ has the same axiom with CGB, but defined over the first-order intuitionistic logic with the infinite connectives $\land$ and $\lor$.

Super Reinhardtness and ‘$V$ is TR’ is defined over $\text{CGB}_\infty$. Also, $\text{CGB}_\infty$ is a conservative extension of CGB.

(Back to main.)