The Axiom of Double Complement and its opposites

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Abstract

Powell introduced the Axiom of Double Complement (DCom) to give his double-negation interpretation of ZF into $\mathsf{IZF}_{\mathsf{Rep}}$. However, the consistency strength and compatibility of DCom had not been known. This article aims to survey the compatibility and consistency strength of DCom, its consequence and opposites which will be named NDCom and ADCom. We will also develop Lubarsky's Kripke models over CZF to derive these results.

We will show that DCom proves the Axiom of Power Set over CZF and is not provable from the intuitionistic Zermelo set theory IZ. We will also show that ADCom does not add the consistency strength over CZF, by modify the construction of the Lubarsky's model for $\mathsf{CZF} + \neg\mathsf{Pow}$. We will also show that DCom, ADCom and NDCom are persistent under realizability under modest conditions.

1 Introduction

The Axiom of Double Complement (DCom), which is introduced by Powell in [21], states that every set x has a *double complement*

$$x^{\mathsf{CC}} = \{z : \neg\neg(z \in x)\} \tag{1}$$

Powell uses DCom to construct an inner model of ZF under $\mathsf{IZF}_{\mathsf{Rep}} + \mathsf{DCom}$, intuitionistic set theory with Replacement in place of Collection and DCom. However, Powell's consistency proof does not seem to be widely accepted. It could be because the relation between IZF and DCom is unclear, although Powell's method possesses attractive features. For example, it does not involve any non-existential set theories appearing in double-negation translation à la Friedman [9].

There are some articles that studied DCom. For example, Grayson [13] showed that Powell's inner model is isomorphic to the forcing extension $V^{\mathcal{P}^{\neg\neg}(1)}$ under $\mathsf{IZF} + \mathsf{DCom}$. Vladimirov [29] investigated absoluteness of certain arithmetic formulas over $\mathsf{IZF} + \mathsf{DCom}$ and its extensions. Hahanyan studied on DCom comprehensively: he research shows the relationship between DCom, other axioms of IZF and non-classical axioms. For example, he proved in [14] that $\mathsf{IZF} + \mathsf{DCom}$ is consistent with some Brouwnian principles and Church's thesis.[1]

In 1992, Hahanyan [8] claimed that he proved DCom is independent of IZF, but the author fails to find full proof of this result. Six years later, Hahanyan [32] published a proof of a weakening of the previous independence result: he proved that DCom is not provable over $\mathsf{IZF} - \mathsf{Pow}$, Intuitionistic set theory without the Axiom of Power Set. Unfortunately, Hahayan's method does not seem to be adequate to derive the independence of DCom over IZF. His proof employes functional realizability (See [26] for functional realizability), and functional realizability satisfies the negation of Pow. (In fact, it satisfies the axiom of anti-double complement ADCom which will be introduced. It implies the negation of the axiom of power set.)

The purpose of this paper is to clarify the relation between DCom and constructive set theories like CZF and IZF. Here is a brief description of each section:

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[1] We need to notice that Hahanyan uses formal systems that are different from what is usually called IZF. His IZF takes natural numbers and real numbers as urelements, not sets. However, methods in Chapter VII, Section 1 of [5] seems to provide a way to translate Hahanyan's result to the familiar standard form.

- In Section 2, we will introduce some preliminaries, including IZF and CZF, some variations of power set operations and progressive relation, a stronger version of well-founded relation.
- In Section 3, we will examine basic properties of DCom and stronger forms of its negation called NDCom and ADCom. We will also examine double complements of some simple sets like 2 and ω. Interestingly, the double complement of 2 is $\mathcal{P}(1)$. Hence DCom implies the Axiom of Power Set Pow over CZF. We also discuss the consistency strength of DCom and NDCom over CZF.
- In Section 4, we will develop the theory of Kripke sets inspired by Lubarsky (see [18] and [15]). We want to work over CZF, so we will make his Kripke models fit on CZF.
- In Section 5, we will derive some results on DCom by employing Kripke models. We will show that the Kripke universe $V^{\mathbb{P}}$ satisfies DCom if V is a model of ZF and $\mathbb{P}$ is linear. We will also see that our proof could not work well when $\mathbb{P}$ is not linear. We will see that ω need not be stable, so IZF cannot prove the double complement of ω is ω itself.

  The main result of this section is that there is a model of $\mathsf{IZ} + \mathsf{NDCom}$, the intuitionistic Zermelo set theory with NDCom.
- In Section 6, we will construct a Kripke model of $\mathsf{CZF} + \mathsf{ADCom}$ by modifying the construction of Lubarsky's first model for $\mathsf{CZF} + \neg\mathsf{Pow}$ in [18]. The whole construction will be performed over CZF, so it also establishes the equiconsistency between CZF and $\mathsf{CZF} + \mathsf{ADCom}$.
- In Section 7, we show that DCom, NDCom ADCom is persistent under realizability (see [19] or [25] for details of realizability). We need a Σ_1-separation or the regular extension axiom to establish the persistence of DCom, and Pow for the persistence of NDCom.NDCom

2 Preliminaries

2.1 Intuitionistic and Construcive Set theories

ZF provides a satisfactory formulation of classical set theory. However, there are nonequivalent formulations of a set theory under constructive mathematics. For example, Intuitionistic ZF (IZF), which is obtained by replacing Replacement to Collection, Regularity to $\in$-induction and uses intuitionistic logic instead of classical logic, is an example of set theory in constructive mathematics.

Another example is Constructive ZF (CZF) introduced by Aczel (see [1], [2] and [3]). CZF is obtained by replacing Collection to Strong Collection, and Power set to Subset Collection from axioms of IZF. Unlike IZF, CZF allows type-theoretic interpretation. Moreover, CZF is equiconsistent with Kripke-Platek set theory KP and its intuitionistic version IKP, which are dramatically weaker than IZF. On the other hand, IZF is equiconsistent with ZF. Despite their different consistency strength, both theories coincide if we add the law of excluded middle (LEM).

We will see some variations of IZF and CZF in this article. IZ and CZ are theories is obtained by dropping Collection or Strong Collection from IZF and CZF respectively. Similarly, IZF^- and CZF^- are theories that obtained by forgetting the Axiom of Power Set or Subset Collection from IZF and CZF respectively.

We do not make an effort to examine details of intuitionistic and constructive set theory. Readers could consult with [4] or Chapter 2 of [31] for basic facts on constructive set theory if needed. However, we will formulate some notations and basic facts for future use.

We will use $R : A \rightrightarrows B$ to denote that R is a multi-valued function from A to B, which is a synonym of a relation of the domain A and codomain B. The 'reversed' symbol $R : A \leftleftarrows B$ means $R \subseteq A \times B$ is a relation whose range is B. If both conditions hold, we write $R : A \leftrightarrows B$. The notion of multi-valued function is useful to describe the Axiom of Subset Collection and fullness, which are predicative substitutes of the Axiom of Power Set over CZF.

Definition 2.1. The *Axiom of Subset Collection* is the following scheme: for a class family of classes $\langle R_u \mid u \in V\rangle$, we have

$$\forall A, B \exists \mathcal{C} \forall u : (R_u : A \rightrightarrows B) \implies \exists D \in \mathcal{C} : R_u : A \leftrightarrows D. \tag{2}$$

The *Axiom of Fullness* is the following statement:

$$\forall A, B \exists \mathcal{C} \forall R : (R : a \rightrightarrows b) \implies \exists S \in \mathcal{C} : S \subseteq R \wedge S : A \leftleftarrows B. \tag{3}$$

In other words, the Axiom of Fullness states there is a collection of multi-valued functions from A to B, which is a support of any given multi-valued function. We call $\mathcal{C} \subseteq \mathrm{mv}(A, B)$ is *full in* $\mathrm{mv}(A, B)$ if $\mathcal{C}$ satisfies the condition of (3).

It is well-known that Subset collection implies Fullness. Moreover, CZF^- can prove Subset collction and Fullnness are equivalent. We introduce its proof for later reference. The following lemma is useful to prove the equivalence:

Lemma 2.2. Let $R : A \rightrightarrows B$ be a multi-valued function. Define $\mathcal{A}(R) : A \rightrightarrows A \times B$ by

$$\mathcal{A}(R) = \{\langle a, \langle a, b\rangle\rangle \mid \langle a, b\rangle \in R\}, \tag{4}$$

then the following holds:

1. $\mathcal{A}(R) : A \rightrightarrows S \iff R \cap S : A \rightrightarrows B$,
2. $\mathcal{A}(R) : A \leftleftarrows S \iff S \subseteq R$.

Proof. For the first statement, observe that $\mathcal{A}(R) : A \rightrightarrows S$ is equivalent to

$$\forall a \in A \exists s \in S : \langle a, s\rangle \in \mathcal{A}(R). \tag{5}$$

By the definition of $\mathcal{A}$, this is equivalent to

$$\forall a \in A \exists s \in S [\exists b \in B : s = \langle a, b\rangle \wedge \langle a, b\rangle \in R]. \tag{6}$$

We can see that the above statement is in fact equivalent to $\forall a \in A \exists b \in B : \langle a, b\rangle \in R \cap S$, which means $R \cap S : A \rightrightarrows B$.

The proof of the second statement is also analogous: $\mathcal{A}(R) : A \leftleftarrows S$ is equivalent to

$$\forall s \in S \exists a \in A : \langle a, s\rangle \in \mathcal{A}(R). \tag{7}$$

By unpacking $\mathcal{A}$, we have

$$\forall s \in S \exists a \in A : [\exists b \in B : s = \langle a, b\rangle \in R]. \tag{8}$$

We can see that it readily implies and equivalent to $S \subseteq R$. $\square$

Proposition 2.3. (CZF^-)

1. Subset Collection implies Fullness.
2. Fullness implies Subset Collection.

Proof. 1. Let A and B be sets. Apply Subset collection to A, $A \times B$ and $R_u = \mathcal{A}(u)$. Then we can find $\mathcal{C}$ such that

$$\forall R : (\mathcal{A}(R) : A \rightrightarrows A \times B) \implies \exists S \in \mathcal{C} : (\mathcal{A}(R) : A \leftrightarrows S). \tag{9}$$

Lemma 2.2 ensures $\mathcal{A}(R) : A \rightrightarrows A \times B$ is equivalent to $R : A \rightrightarrows B$ and $\mathcal{A}(R) : A \leftrightarrows S$ is equivalent to $S \subseteq R \wedge R \cap S : A \rightrightarrows B$. By Lemma 2.2, we have

$$\forall R : (R : A \rightrightarrows B) \implies \exists S \in \mathcal{C} : (S \subseteq R \wedge S : A \rightrightarrows B), \tag{10}$$

Therefore, $\mathcal{C}$ witnesses Fullness.

2. Take $\mathcal{C}$ which is full in $\mathrm{mv}(A, B)$. Let $\langle R_u \mid u \in V\rangle$ be a class family and $R_u : A \rightrightarrows B$. Then $\mathcal{A}(R_u) : A \rightrightarrows A \times B$. By Strong Collection (see (2.2.15) of [31]), there is S such that $A(\mathcal{R}_u) : A \leftrightarrows S$, which is equivalent to $S \subseteq R_u$ and $S : A \rightrightarrows B$.

   Since $\mathcal{C}$ is full in $\mathrm{mv}(A, B)$, there is $S' \in \mathcal{C}$ such that $S' \subseteq S$ and $S' : A \rightrightarrows B$. From $S' \subseteq R_u$ and $S' : A \rightrightarrows B$, we can conclude $R_u : A \rightrightarrows \operatorname{ran} S'$ and $R_u : \operatorname{ran} S' \rightrightarrows A$. Therefore, the set $\{\operatorname{ran} S \mid S \in \mathcal{C}\}$ witnesses Subset Collection. $\square$

2.2 Variations of Power sets

In this subsection, we discuss some special subclasses of the powerclass of a given set. We will see later that DCom for certain types of sets implies the existence of these subclasses. Conversely, we will also assert that the existence of a certain subclass of a powerclass implies DCom for some sets.

Definition 2.4. Let a be a set.

- The *powerclass of* a, $\mathcal{P}(a)$ is the collection of all subsets of a. That is, $\mathcal{P}(a) = \{b \mid b \subseteq a\}$.
- The *negative powerclass of* a, $\mathcal{P}^{\neg\neg}(a)$ is the class $\{b \mid \forall x \in a : \neg\neg(b \in a) \to b \in a\}$.
- The *class of decidable subsets* $\mathrm{Dec}(a)$ is the class $\mathrm{Dec}(a) = \{b \mid \forall x \in a : x \in b \vee \neg(x \in b)\}$.
- The *class of weakly decidable subsets* $\mathrm{WDec}(a)$ is the class $\mathrm{WDec}(a) = \{b \mid \forall x \in a : \neg(x \in b) \vee \neg\neg(x \in b)\}$.

Note that the notion of negative powerclass was introduced by Gambino [12].

If each class is a set, we call them *power set*, *negative power set* and *set of (weakly) decidable sets* respectively.

Under the presence of function sets, we can reduce the existence of a variation of power sets to the existence of a variation of the power set of 1. Especially, we can prove $\mathrm{Dec}(a)$ is always a set since $\mathrm{Dec}(1) = 2$ is a set:

Proposition 2.5. ($\mathsf{CZF}^- + \mathsf{Exp}$) Let F be the one of $\mathcal{P}$, $\mathcal{P}^{\neg\neg}$, Dec and WDec. If $F(1)$ exists, then $F(a)$ exists for all set a.

Proof. The outline of the proof does not depend on F, so we will not concentrate details of F until it is needed. We can see that if $F(a)$ is a subclass of a set, then we can define $F(a)$ by using Δ_0-separation. Therefore, it suffices to prove

$$F(a) \subseteq \{f^{-1}\{1\} \mid f : a \to F(1)\}. \tag{11}$$

For each $b \in F(a)$, consider its *characteristic function* $\chi_b : a \to \mathcal{P}(1)$ defined by $\chi_b(x) = \{0 \mid x \in b\}$. We can see that $b = \chi_b^{-1}\{1\}$. It remains to show that $\operatorname{ran}\chi_b \subseteq F(1)$. Its proof depends on F, but all proofs employ the following equivalence:

$$x \in b \iff 0 \in \chi_b(x). \tag{12}$$

The proof for remaining cases are shown by Proposition 5.2.4, 10.1 of [4] and Lemma 4.3.2 of [12]. Hence, we only concentrate on the detail for $F = \mathrm{WDec}$. We will show that the following holds: $\operatorname{ran}\chi_b \subseteq \mathrm{WDec}(1)$ if $b \in \mathrm{WDec}(a)$.

By the assumption b, we have

$$\neg(x \in b) \vee \neg\neg(x \in b), \tag{13}$$

which turns out to be equivalent to

$$\neg(0 \in \chi_b(x)) \vee \neg\neg(0 \in \chi_b(x)) \tag{14}$$

by (12). Therefore, $\chi_b(x) \in \mathrm{WDec}(1)$. □

Note that Proposition 2.5 still holds even if we replace $F(1)$ to $F(b)$ for any singleton b. Moreover, if $a \subseteq b$ and $F(b)$ exists, then $F(a)$ also exists by Δ_0-separation. Hence the assertion '$F(a)$ exists for all a' is equivalent to '$F(a)$ exists for some *inhabited* a'.

We will describe the appearance of WDec(1) for future use. Suppose that $x \in \mathrm{WDec}(1)$. Then either $0 \notin x$ or $\neg\neg(0 \in x)$. In the former case, x must be empty since 0 is the only possible element of x. If $\neg\neg(0 \in x)$, we have $\neg\neg(x = 1)$ since $\neg\neg(x \subseteq 1)$ holds.

Conversely, it is easy to check that the empty set 0 and a set $x \subseteq 1$ satisfying $\neg\neg(x = 1)$ are members of WDec(1). Therefore, we have the following equation:

Lemma 2.6.

$$\mathrm{WDec}(1) = \{x \subseteq 1 \mid x = 0 \vee \neg\neg(x = 1)\}. \tag{15}$$

□

2.3 Progressive relations

In ZF, well-founded relations are useful because it allows recursive definition. However, this is not true if our background theory does not allow Full separation. For example, CZF cannot prove the transitive collapsing function

$$\pi(x) := \{\pi(y) \mid y \prec x\} \tag{16}$$

on a well-founded set $\langle A, \prec\rangle$ is not a set function. In fact, π is just a Σ_1-definable class function. Hence we cannot define subsets of A like

$$\{x \in A \mid \exists f \in {}^x\pi(x)(f \text{ is bijective})\}, \tag{17}$$

due to the lack of Full separation. It also means we cannot apply $\prec$-induction scheme to prove the defining formula of (17). It motivates to define a stronger notion of a well-founded set:

Definition 2.7. Let A be a class and $\prec$ be a binary relation over A. We call $\prec$ is *progressive* if it is *uniformly extensional* in the sense that $\operatorname{ext}_\prec(x) := \{y \in A : y \prec x\}$ is a set which is uniformly definable on x and satisfies the following $\prec$-induction scheme for formulas ϕ:

$$[(\forall y \in A : y \prec x \to \phi(y)) \to \phi(x)] \to \forall x \in A\phi(x) \tag{18}$$

We call $\prec$ is Δ_0*-progressive* if $\prec$ is moreover Δ_0-definable.

We can show that $\prec$ is Δ_0-progressive if it satisfies $\prec$-induction scheme and $\operatorname{ext}_\prec(x)$ exists for all x. In other words, we can forget the uniform definability of $\operatorname{ext}_\prec$ for Δ_0-progressive relations. We concentrate on Δ_0-progressive relations in this article, since every progressive relation that will be appear is Δ_0.

The Axiom of $\in$-induction states that the membership relation $\in$ is progressive. Progressive relations behave desirably even if our background theory is weak, unlike well-founded relations. For example, we can prove that progressive relations satisfy well-founded recursion for arbitrary class functions:

Theorem 2.8 (Well-founded recursion)**.** (CZF) If $\prec$ is a progressive relation over a class A and $G : A \to A$ be a class function, then there is a class function $F : A \to V$ such that $F(x) = G(F \restriction \operatorname{ext}_\prec(x))$ for all $x \in A$.

Note that $F \restriction u = \{\langle y, F(y)\rangle : y \in u\}$, which exists by the Axiom of Replacement.

Proof. Let $\Phi(f)$ be the following formula:

$$[f \text{ is a function}] \wedge [\forall x \in \operatorname{dom} f : \operatorname{ext}_\prec(x) \subseteq \operatorname{dom} f] \wedge [\forall z \in \operatorname{dom} f : f(z) = G(f\restriction \operatorname{ext}_\prec(z))] \tag{19}$$

We will prove the following two lemmas before to construct the desired F.

Lemma 2.9. If $\Phi(f)$ and $\Phi(g)$ then

$$\forall x \in A : x \in \operatorname{dom} f \cap \operatorname{dom} g \to f(x) = g(x). \qquad \square \tag{20}$$

Proof. We will use the $\prec$-induction on x, which is possible as $\prec$ is a progressive relation on A. Assume that our theorem holds for all $y \prec x$. Now assume that $x \in \operatorname{dom} f \cap \operatorname{dom} g$. By the inductive assumption, we have $f \restriction \operatorname{ext}_\prec(x) = g \restriction \operatorname{ext}_\prec(x)$. Since $\Phi(f)$ and $\Phi(g)$ holds, we have $f(x) = G(f \restriction \operatorname{ext}_\prec(x)) = G(g \restriction \operatorname{ext}_\prec(x)) = g(x)$. $\square$

Lemma 2.10. For each $x \in A$ there is a function f such that $\Phi(f)$ and $x \in \operatorname{dom} f$.

Proof. We will prove it by $\prec$-induction on x. Suppose that for each $y \prec x$ there is a function f such that $\Phi(f)$. By Strong Collection, we can find a set A such that

$$[\forall y \prec x \exists f \in A : \Phi(f) \wedge y \in \operatorname{dom} f] \wedge [\forall f \in A \exists y \prec x : \Phi(f) \wedge y \in \operatorname{dom} f]. \tag{21}$$

Especially, every $f \in A$ satisfies $\Phi(f)$. Let $f_0 := \bigcup A$. We will check that $\Phi(f_0)$ holds. By Lemma 2.9, f_0 is a function. Moreover, it is easy to check the remaining conditions of $\Phi(f_0)$. Therefore $\Phi(f_0)$ holds.

Now take

$$f = f_0 \cup \{\langle x, G(f_0 \restriction \mathrm{ext}_\prec(x))\rangle\}. \tag{22}$$

The only non-trivial part to prove $\Phi(f)$ is to verify f is a function, especially the well-definedness of $f(x)$. If $\langle x, y\rangle, \langle x, z\rangle \in f$, then either both of them are in f_0, or one of them is in $\{\langle x, G(f_0 \restriction \mathrm{ext}_\prec(x))\rangle\}$. In the latter case, we can see $y = z = G(f_0 \restriction \mathrm{ext}_\prec(x))$. Hence f is a function. □

The main theorem follows by letting $F = \bigcup\{f : \Phi(f)\}$.

Is there an example of a progressive relation? The Axiom of $\in$-induction states $\in$ over V is progressive. It is also known that $\in$ over the class of ordinals Ord is progressive. (See Lemma 9.4.3. of [4].) We will see further examples later.

2.4 Inductive definitions

Some object in mathematics has an impredicative definition. An impredicative description justifies the following definition for a property P: an object H is the smallest object that satisfies P. Predicative constructive systems like CZF do not allow impredicative definitions. Fortunately, we can define 'smallest objects' without impredicative definition.

Let C be a class, and Γ be a monotone operation. A collection of classes $\langle C^\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is a Γ-*hierarchy for* C if it satisfies $C = \bigcup_{\alpha\in\mathrm{Ord}}$ and $C^\alpha = \Gamma(C^{\in\alpha})$, where $C^{\in\alpha} = \bigcup_{\beta\in\alpha} C^\beta$.

The proof of the theorem is available in [4], so we omit it:

Theorem 2.11 (Class Inductive Definition Theorem)**.** Let Φ be an inductive definition. Then we can find a smallest Φ-closed class I_Φ. Moreover, there is a Γ_Φ-hierarchy $\langle I_\Phi^\alpha \mid \alpha \in \mathrm{Ord}\rangle$ for I_Φ. In addition, the hierarchy is a family of sets if $\Gamma_\Phi(X)$ is a set for any set X.

3 Introducing Double Complement

3.1 Basics on Double Complement

Definition 3.1.

- DCom, *the Axiom of Double Complement* states every set has a double complement. Its formal statement is

$$\forall x \exists y \forall z : \neg\neg(z \in x) \to z \in y. \tag{23}$$

- NDCom states there is a set whose double complement does not exist.[2] Formally,

$$\exists x \forall y : \neg[\forall z : \neg\neg(z \in x) \to z \in y] \tag{24}$$

- ADCom, *the Axiom of Anti-Double-Complement* states every set which has a double complement is a subset of 1. Its formal statement is

$$\forall x (\exists y : \forall z : \neg\neg z \in x \to z \in y) \to x \subseteq 1. \tag{25}$$

We will use the following basic properties of double complement frequently:

Proposition 3.2. (CZF^-)

1. If $x \subseteq y$ then $x^{\mathsf{CC}} \subseteq y^{\mathsf{CC}}$.
2. $x \subseteq x^{\mathsf{CC}}$.
3. $\mathcal{P}(x)^{\mathsf{CC}} \subseteq \mathcal{P}(x^{\mathsf{CC}})$. Especially, if x is stable then $\mathcal{P}(x)$ is also stable.

Proof. We only give a proof for the last statement. Let $y \in \mathcal{P}(x)^{\mathsf{CC}}$, that is, $\neg\neg(\forall z : z \in y \to z \in x)$. Since $\neg\neg\forall z \phi(z) \to \forall z : \neg\neg\phi(z)$ and $\neg\neg(p \to q) \to (\neg\neg p \to \neg\neg q)$ hold, we have $\forall z \in y : \neg\neg(z \in x)$. That is, we have $y \subseteq x^{\mathsf{CC}}$. □

[2] NDCom is an abbreviation of 'Not-DCom'. NDCom implies the negation of DCom, but itself is not equivalent to DCom.

It is natural to ask there is a set that has a double complement. Moreover, it would be better if we can find a concrete representation of a double complement of a given set. Unfortunately, evaluating the double complement for arbitrary sets is usually hard, unless we have an additional axiom like Δ_0-LEM.

Despite that, we can find a concrete representation of the double complement for some simple sets. For example, it is easy to see that $0^{\mathsf{CC}} = 0$. In general, we can evaluate the double complement of subsets of 1:

Proposition 3.3. (CZF^-) If $x \subseteq 1$ then $x^{\mathsf{CC}} = \{0 \mid \neg\neg(0 \in x)\}$.

Proof. If $y \in x^{\mathsf{CC}}$, then $\neg\neg(y = 0 \land 0 \in x)$ holds, which is equivalent to

$$\neg\neg(y = 0) \land \neg\neg(0 \in x). \tag{26}$$

Since $y = 0$ is a negation of the statement 'y is inhabited', $\neg\neg(y = 0)$ is just $(y = 0)$. Therefore $x^{\mathsf{CC}} \subseteq \{0 \mid \neg\neg(0 \in x)\}$. The reverse inclusion can be shown easily. □

Especially, we can see that $1^{\mathsf{CC}} = 1$. We will see in Section 6 that $\mathsf{CZF} + \mathsf{ADCom}$ is consistent. As a consequence, subsets of 1 are only sets whose presence of a double complement is provable in CZF. If we allow the Axiom of Power Set, however, then we can find more sets that have a double complement:

Proposition 3.4. 1. (CZF^-) For each $n \in \omega$, $V_n^{\mathsf{CC}} = V_n$.

2. ($\mathsf{CZF} + \mathsf{Pow}$) If $x \in V_\omega$ then x^{CC} exists.

Proof. We will prove $V_{n+1} = \mathcal{P}(V_n)$ by induction on n: this is obvious if $n = 0$. If $V_{k+1} = \mathcal{P}(V_k)$ for all $k < n$, then

$$\begin{array}{lll} V_{n+1} & = \bigcup_{k<n} \mathcal{P}(V_k) \cup \mathcal{P}(V_n) & \\ & = \bigcup_{k<n} V_{k+1} \cup \mathcal{P}(V_n) & (\because \text{Inductive assumptions.}) \\ & = V_n \cup \mathcal{P}(V_n) & (\because \langle V_k \mid k \in n\rangle \text{ is increasing under } \subseteq.) \\ & = \mathcal{P}(V_n) & (\because V_n \text{ is transitive so } V_n \subseteq \mathcal{P}(V_n).) \end{array} \tag{27}$$

Now we prove the stability of V_n by induction on n. Furthermore, we have $V_\omega = \bigcup_{n\in\omega} V_n$, which proves the second assertion. □

As a corollary, we have

Corollary 3.5. (CZF^-) ADCom is not compatible with the Axiom of Power Set.

Proof. V_2 is a double complement of itself, but not a subset of 1. □

Proposition 3.4 says V_n is stable for each n, so we have examples of stable sets under Pow. Moreover, stable sets, like V_n under Pow, have a concrete representation of its double complement: itself.

We may ask we can find a concrete representation of a double complement for simple sets, especially for natural numbers and ω. The bad news is that even expressing double complement of 3 under simple terms is unlikely. Despite that, we can find a precise description of the double complement for 2 and $\{1\}$. A bit surprisingly, it is related to power sets:

Theorem 3.6. (CZF^-) The double complement of 2 is $\mathcal{P}(1)$. Moreover, the double complement of $\{1\}$ is $\{x \subseteq 1 \mid \neg\neg(x = 1)\}$.

Proof. We only give proof of the former statement, as the proof of the latter statement is analogous. Since $2 \subseteq \mathcal{P}(1)$, we have $2 \subseteq \mathcal{P}(1)^{\mathsf{CC}} = \mathcal{P}(1)$. It remains to show the reverse inclusion. Let $x \subseteq 1$. We must show that

$$\neg\neg(x = 0 \lor x = 1). \tag{28}$$

This follows from $\neg\neg(0 \in x \lor 0 \notin x)$ and $\neg\neg(x \subseteq 1)$. □

Especially, we have

Corollary 3.7. (CZF) DCom implies Pow.

Proof. Theorem 3.6 implies $\mathcal{P}(1)$ exists. Moreover, Proposition 2.5 states that the existence of $\mathcal{P}(1)$ implies Pow. $\square$

It is natural to ask the Axiom of Power Set can prove the Axiom of Double Complement. However, the situation is not easy. We will show that DCom is not provable from Intuitionistic Zermelo set theory in Section 5, although it is still not known whether DCom is a theorem of IZF.

Even a local version of $\mathsf{Pow} \to \mathsf{DCom}$ is not obvious. However, under a semi-classical setting, namely $\mathsf{CZF} + \mathsf{Pow}$ with *Markov's principle* (MP) and *weak excluded middle for bounded formulas* (Δ_0-WLEM), we have a positive result:

Theorem 3.8. ($\mathsf{CZF} + \mathsf{Pow} + \Delta_0\text{-}\mathsf{WLEM} + \mathsf{MP}$) V_ω is stable. Formally, $V_\omega^{\mathsf{CC}} = V_\omega$

Proof. Note that each V_n is stable. Therefore, Δ_0-WLEM proves $x \in V_n$ is decidable for each x. In other words, $x \in V_n \lor x \notin V_n$ holds. Now let $x \in V_\omega^{\mathsf{CC}}$, which is equivalent to

$$\neg\neg(\exists n \in \omega : x \in V_n). \tag{29}$$

Since the formula $x \in V_n$ is decidable, we have

$$\exists n \in \omega : x \in V_n \tag{30}$$

by MP. Therefore $x \in V_\omega$. $\square$

As a corollary, we can see that ω^{CC} exists under the mentioned conditions.

Some readers might think the mentioned conditions are too excessive. However, we will show later that Δ_0-WLEM is necessary for the proof. In other words, we will construct models of $\mathsf{IZF} + \mathsf{MP}$ that break the stability of V_ω. (See Theorem 5.5.)

We will discuss a possible size of the double complement of sets before to conclude this subsection. A double complement of a given set could not exist and could be a proper class. We know that all proper classes are quite large classically in the sense that they are *inexhaustible*.

Definition 3.9. A class A is *inexhaustible* if we can find $y \in A$ such that $y \notin x$ for any set x.

We can show that the class of all sets V and the class of all ordinals Ord is inexhaustible (cf. Example 6.24 of [31].) We may ask the double complement of a set can be inexhaustible, and the answer is negative:

Proposition 3.10. (CZF^-) If A is a set, then A^{CC} is not inexhaustible.

Proof. No y can satisfy both $y \in x^{\mathsf{CC}}$ and $y \notin x$, that have to exist if x^{CC} is inexhaustible. $\square$

Note that it proves an open question stated in [31], which asks the consistency of existence of two sets A and B such that $\mathcal{P}(A)$ is inexhaustible while $\mathcal{P}(B)$ is not:

Corollary 3.11. Working over CZF with the Axiom of Subcountability, which states every set is an image of a subset of ω, $\mathcal{P}(\omega)$ is inexhaustible but $\mathcal{P}(1)$ is not.

Proof. The Axiom of Subcountability proves $\mathcal{P}(\omega)$ is inexhaustible (cf. Example 6.24. of [31].) However, Proposition 3.10 proves $\mathcal{P}(1) = 2^{\mathsf{CC}}$ is never inexhaustible. $\square$

3.2 Further analysis on DCom over CZF

Corollary 3.7 shows that DCom implies Pow. From this, we have that the consistency of $\mathsf{CZF} + \mathsf{DCom}$ implies that of $\mathsf{CZF} + \mathsf{Pow}$. We may further ask whether they are equiconsistent or not. The following proposition shows the consistency strength of DCom and NDCom over CZF:

Proposition 3.12.

1. $\mathsf{CZF} + \mathsf{DCom}$ is equiconsistent with $\mathsf{CZF} + \mathsf{Pow}$.
2. CZF is equiconsistent with $\mathsf{CZF} + \mathsf{NDCom}$.

Proof.

1. One direction directly follows from Corollary 3.7. The other direction follows from the equiconsistency of $\mathsf{CZF} + \Delta_0\text{-}\mathsf{LEM}$ and $\mathsf{CZF} + \mathsf{Pow}$ which was proven by [23].

2. In fact, the Axiom of Subcountability proves $\neg\mathsf{Pow}$. In CZF, the Pow iff $\mathcal{P}(1)$ exists. Since $2^{\mathsf{CC}} = \mathcal{P}(1)$, 2 is an instance of NDCom. Moreover, [22] proves CZF and CZF + Axiom of Subcountability are equiconsistent. Hence the result follows. $\square$

How about the case CZF^-? We may expect a different result since Subset Collection has a critical role in Proposition 2.5. The following proposition settles the question:

Proposition 3.13. CZF^-, $\mathsf{CZF}^- + \mathsf{DCom}$, $\mathsf{CZF}^- + \mathsf{NDCom}$ are all equiconsistent.

Proof. By Theorem 4.2 of [24] and Lemma 2.4, CZF^- and CZF are equiconsistent. Therefore CZF^- + NDCom and CZF^- are equiconsistent. Moreover, Theorem 4.2 of [11] proves CZF^- and CZF^- + Δ_0-LEM are equiconsistent. From the fact that Δ_0-LEM proves every set is stable, we have the equiconsistency between CZF^- and $\mathsf{CZF}^- + \mathsf{DCom}$. $\square$

In conclusion, DCom does not increase the consistency strength, unless the background theory has the Axiom of Subset Collection (more precisely, the Axiom of Exponentiation.)

We have not established any consistency result about ADCom. We will show in Section 6 that $\mathsf{CZF} + \mathsf{ADCom}$ and CZF are equiconsistent. If the Axiom of Subcountablity proves ADCom, then the proof of Proposition 3.12 establishes the consistency strength of $\mathsf{CZF} + \mathsf{ADCom}$. However, the auther does not know whether the Axiom of Subcountability proves ADCom or not:

Question 3.14. Does the Axiom of Subcountability prove ADCom?

Proposition 3.12 shows if the double complement of 2 exists, then Pow holds. What happens if we assume the existence of the double complement of another set? The next proposition says the answer is related to variations of power sets.

Proposition 3.15. (CZF^-) Let a be a set which have a double complement.

1. If a has two *apart* elements, that is, if there are $x, y \in A$ such that
$$\exists z : (z \in x \wedge z \notin x) \vee (z \notin x \wedge z \in y) \tag{31}$$
then $\mathcal{P}(1)$ exists.

2. If a has two *nonequal* elements, that is, if a contains x and y such that $x \neq y$, then $\mathcal{P}^{\neg\neg}(1)$ exists.

3. If a has an inhabited element, then $\mathrm{WDec}(1)$ exists.

Proof. 1. Without loss of generality, assume that there is t such that $t \in x$ but $t \notin y$. Define w_c by
$$w_c = \{z \in x \mid 0 \in c\} \cup \{z \in y \mid 0 \notin c\}. \tag{32}$$
Then $\neg\neg(w_c \in a)$. Now we will prove that $c = \{0 \mid w_c = x\}$ for all $c \subseteq 1$: one inclusion is obvious. For the reverse inclusion, let $w_c = x$. Then $t \in w_c$, which is equivalent to
$$(t \in x \wedge 0 \in c) \vee (t \in y \wedge 0 \notin c). \tag{33}$$
We can exclude the latter case as $t \notin y$. Hence $0 \in c$ and we have $\{0 \mid w_c = x\} \subseteq c$. Therefore we have $\mathcal{P}(1) \subseteq \{\{0 \mid z = x\} \mid z \in a^{\mathsf{CC}}\}$.

2. For each $c \subseteq 1$ define w_c as before. We can deduce $\neg\neg(w_c \in a)$ from $\neg\neg(0 \in c \vee 0 \notin c)$. We claim that $\mathcal{P}^{\neg\neg}(1) \subseteq \{\{0 \mid z = x\} \mid z \in a^{\mathsf{CC}}\}$. This would follow from $c = \{0 \mid w_c = x\}$ for stable subset c of 1. $c \subseteq \{0 \mid w_c = z\}$ is obvious. For the reverse inclusion, suppose that $w_c = x$. Since $x \neq y$, $w_c \neq y$ holds so we have $\neg\neg(0 \in c)$. Therefore $0 \in c$ by stability of c.

3. Suppose that $x \in a$ and $y \in x$. For each $c \in \mathrm{WDec}(1)$, define $w_c = (x \setminus \{y\}) \cup \{y \mid 0 \in c\}$. We can show that $\mathrm{WDec}(1) \subseteq \{\{0 \mid z = x\} : z \in a^{\mathsf{CC}}\}$ by verifying $c = \{0 \mid w_c = x\}$ for each $c \in \mathrm{WDec}(1)$. The proof is similar to previous cases, so we omit the details. $\square$

We may regard the first part of Proposition 3.15 as a generalization of Theorem 3.6. As a corollary of 3.15, we have

Corollary 3.16. The following theories are equiconsistent.

1. CZF+'There is a set a such that a has two apart elements, and a^{CC} exists,'
2. CZF+'There is a set a such that a has two nonequal elements, and a^{CC} exists,'
3. $\mathsf{CZF} + \mathsf{Pow}$
4. $\mathsf{CZF} + \mathcal{P}^{\neg\neg}(1)$ exists.

Proof. 1 obviously implies 2. 2 implies 4 by Corollary 3.15. By Theorem 1.1. of [23], 4 and 3 have the same consistency strength. Moreover, 3 implies 1 by Theorem 3.6. $\square$

It is unclear that the last part of Proposition 3.15 implies any consistency strength results, since we do not know the consistency strength of CZF + 'WDec(1) exists'. However, it gives a hint to characterize ADCom in terms of weakly decidable sets, which will turn out to be useful:

Theorem 3.17. (CZF) ADCom is equivalent to the non-existence of WDec(1).

Proof. By Proposition 3.15, the absence of WDec(1) implies the absence of the double complement of sets that contains an inhabited element. Hence if a has a double complement, then every element of a must be empty. That is, we have $a \subseteq \{0\} = 1$. This is the very definition of ADCom.

Conversely, assume that ADCom holds. Then $\{1\}$ does not have a double complement. Therefore, $\{x \subseteq 1 \mid \neg\neg(x = 1)\}$ is not a set. Hence $\mathrm{WDec}(1) = \{0\} \cup \{x \subseteq 1 \mid \neg\neg(x = 1)\}$ is also not a set. $\square$

4 Lubarsky's Kripke models

In this section, we will develop basic facts on Kripke models over CZF and IZF. There are many possible definitions of Kripke models we can take: For example, [10] and [16] uses their own Kripke model to get their desired metamathematical results. We follow Lubarsky's Kripke models that appear in [18] and [15]. Our Kripke models may be a special case of Heyting-valued models that introduced by [11] or [6], but we prefer to develop the theory of Kripke models separately. In this section, we work over CZF^- unless noticed.

4.1 Kripke models over set frames

We need a *frame*, which is just a partial ordered set with the least element $\bot$, to construct a Kripke model. We will denote frames as $(\mathbb{P}, \leq)$, or just $\mathbb{P}$ if the order relation is clear. We use variables p, q, r, s, $\cdots$ for elements of $\mathbb{P}$. For each $p \in \mathbb{P}$, $\uparrow p$ denotes the upper set $\{q \in \mathbb{P} \mid q \geq p\}$ of p.

Let V be a model of CZF (or IZF) and $\mathbb{P} \in V$ be a frame.

Definition 4.1 (Informal definition of Kripke models)**.** The domain of the Kripke model $V^{\mathbb{P}} = \bigcup_{p\in\mathbb{P}} V^{\mathbb{P}}(p)$ on $\mathbb{P}$ with transition functions $\tau_{pq} : V^{\mathbb{P}}(p) \to V^{\mathbb{P}}(q)$ for $p \leq q$ satisfy the following conditions: for ordinals $\beta \subseteq \alpha$ and $p \leq q \leq r$,

(a) $V^{\mathbb{P}}_\beta(p) \subseteq V^{\mathbb{P}}_\alpha(p)$ for all $p \in \mathbb{P}$.

(b) $\tau_{pq}\restriction V^{\mathbb{P}}_\alpha(p)$ is a function from $V^{\mathbb{P}}_\alpha(p)$ to $V^{\mathbb{P}}_\alpha(q)$.

(c) $\tau_{pr} = \tau_{qr} \circ \tau_{pq}$ and τ_{pp} is the identity function.

(d) $V^{\mathbb{P}}_\alpha(p)$ is a class of all functions x of domain $\uparrow p$ such that

- $x(q) \subseteq \bigcup_{\beta\in\alpha} V^{\mathbb{P}}_\beta(q)$ for all $q \geq p$ and
- $\tau''_{qr}[x(q)] \subseteq x(r)$ for all $p \leq q \leq r$.

(We will call the last condition for x the *monotonocity* condition.)

If we have the Axiom of Power Set, we can take the above 'informal definition' as a formal definition of hierarchies $\langle V^{\mathbb{P}}_\alpha(p) \mid \alpha \in \mathrm{Ord} \land p \in \mathbb{P}\rangle$, which is recursive on α and simultaneous on p. This is because Pow proves $V^{\mathbb{P}}_\alpha(p)$ is a set for each p and α. Thus we can form a sequence of $V^{\mathbb{P}}_\alpha(p)$s. However, we work over CZF^-, and each stage of the hierarchy need not be a set. Thus we need to use inductive definitions.

The following lemma states Definition 4.1 of $V^{\mathbb{P}}$ actually defines a hierarchy of classes:

Lemma 4.2. There is a sequence of classes $\langle V_\alpha^{\mathbb{P}}(p) \mid \alpha \in \mathrm{Ord} \wedge p \in \mathbb{P}\rangle$ and a class function τ satisfying the mentioned conditions. Furthermore, if we have Pow, then $V_\alpha^{\mathbb{P}}(p)$ is a set for each α and p.

Proof. Let Φ be an inductive definition defined as follows: $\langle a, x\rangle \in \Phi$ iff x is a function of domain $\uparrow p$ for some $p \in \mathbb{P}$ which satisfies the following conditions:

(i) $x(q) \subseteq a$ for all $q \geq p$,

(ii) For each $q \geq p$ and $y \in x(q)$, y is a function of domain $\uparrow q$ and

(iii) If $p \leq q \leq r$ then $y \restriction \uparrow r \in x(r)$ for all $y \in x(q)$.

By Class Inductive definition theorem, there is a smallest Φ-closed class I and a Γ_Φ-hierarchy $\langle I^\alpha \mid \alpha \in \mathrm{Ord}\rangle$. Now define $V^{\mathbb{P}}(p) = \{x \in I \mid \operatorname{dom} x = \uparrow p\}$, $V_\alpha^{\mathbb{P}}(p) = \{x \in I^\alpha \mid \operatorname{dom} x = \uparrow p\}$ and $\tau_{pq}(x) = x\restriction(\uparrow q)$ for $x \in V^{\mathbb{P}}(p)$.

We will show that the conditions in Definition 4.1 hold for $V_\alpha^{\mathbb{P}}(p)$ and τ_{pq}. Condition (a) and (c) are easy to check. For (b), observe that the only difference between members of $V_\alpha^{\mathbb{P}}(p)$ and that of $V_\alpha^{\mathbb{P}}(q)$ are their domains.

It remains to show that the condition (d) holds. Assume inductively that (d) holds for all $\beta \in \alpha$. For the one direction, let $x \in V_\alpha^{\mathbb{P}}(p)$, which is equivalent to $\operatorname{dom} x = \uparrow p$ and $x \in I^\alpha = \Gamma_\Phi(I^{\in\alpha})$. Hence, there is a such that $a \subseteq I^{\in\alpha}$ and $\langle a, x\rangle \in \Phi$. Condition (i) and (ii) of the defining formula of Φ, combining with the inductive definition, implies $x(q) \subseteq \bigcup_{\beta\in\alpha} V_\beta^{\mathbb{P}}(q)$ for all $q \geq p$. Moreover, (iii) is just another way to state $\tau''_{qr}[x(q)] \subseteq x(r)$ for all $p \leq q \leq r$.

For the remaining direction, assume that x is a function of domain $\uparrow p$ which satisfies conditions in (d). We can see that the first condition implies (i) and (ii). We have observed that (iii) is equivalent to the second condition of (d). This completes the proof. $\square$

We call elements of $V^{\mathbb{P}}$ a *Kripke sets over* $\mathbb{P}$ or $\mathbb{P}$-*names*. The following lemma is useful to show a given function is a $\mathbb{P}$-name:

Lemma 4.3 (The Closure lemma)**.** Let x be a function of domain $\uparrow p$ which satisfies $x(q) \subseteq V^{\mathbb{P}}(q)$ for all $q \geq p$ and monotonocity. Then $x \in V^{\mathbb{P}}(p)$.

Proof. Direct from the inductive definition of the universe of Kripke sets $V^{\mathbb{P}}$. $\square$

The forcing relation $\Vdash$ over $V^{\mathbb{P}}$ is defined inductively for formulas. In the following definition, we assume $x, y \in V^{\mathbb{P}}(p)$.

- $p \Vdash x \in y \iff x \in y(p)$,
- $p \Vdash x = y \iff \forall q \geq p : x(q) = y(q)$,
- $p \Vdash \phi \wedge \psi \iff p \Vdash \phi$ and $p \Vdash \psi$,
- $p \Vdash \phi \vee \psi \iff p \Vdash \phi$ or $p \Vdash \psi$,
- $p \Vdash \phi \to \psi \iff$ For each $q \geq p$, if $q \Vdash \phi$ then $q \Vdash \psi$,
- $p \Vdash \forall x\phi(x) \iff$ For each $q \geq p$ and $x \in V^{\mathbb{P}}(q)$, $q \Vdash \phi(x)$ and
- $p \Vdash \exists x\phi(x) \iff$ There is $x \in V^{\mathbb{P}}(p)$ such that $p \Vdash \phi(x)$.

We assume that every parameter in this definition is a member of $V^{\mathbb{P}}(p)$. In many cases, however, we can face parameters appears in a given formula which is not a member of $V^{\mathbb{P}}(p)$, but a member of $V^{\mathbb{P}}(r)$ for some $r \leq p$. In this case, we take an appropriate transition function to the parameters, so the parameters belong to $V^{\mathbb{P}}(p)$. For example, if $r, s \leq p$, $x \in V^{\mathbb{P}}(r)$ and $x \in V^{\mathbb{P}}(s)$ then $p \Vdash \phi(x, y)$ is *defined* by $p \Vdash \phi(\tau_{rp}(x), \tau_{sp}(y))$.

The following lemma states $\in$ and $=$ are persistent up to the transition map. It also justifies the mentioned convention for parameters:

Lemma 4.4. The transition function respects $=$ and $\in$. Formally, for each $x, y \in V^{\mathbb{P}}(p)$, if $p \Vdash x = y$ or $p \Vdash x \in y$ then $q \Vdash \tau_{pq}(x) = \tau_{pq}(y)$ or $q \Vdash \tau_{pq}(x) \in \tau_{pq}(y)$ respectively.

Proof. Direct from the definition of $=$, $\in$ of the Kripke language and monotonocity condition for Kripke sets. $\square$

Observe that interpreting $\to$ and $\forall$ introduces a new frame variable. We will handle a series of universal quantifiers and a combination of $\forall$ and $\to$, so there are possibilities for introducing lots of frame variables. Fortunately, we can prove that introducing new frame variables are unnecessary in these special cases:

Lemma 4.5. Let $p \in \mathbb{P}$ and $y_0, \cdots, y_m \in V^{\mathbb{P}}(p)$ be parameters of a formula ϕ.

1. $p \Vdash \forall x_0 \cdots \forall x_n \phi(x_0, \cdots, x_n, y_0, \cdots y_m)$ iff for each $q \geq p$ and $x_0, \cdots, x_n \in V^{\mathbb{P}}(q)$ we have $q \Vdash \phi(x_0, \cdots, x_n, \tau_{pq}(y_0), \cdots, \tau_{pq}(y_m))$.
2. $p \Vdash \forall x \phi(x, y_0, \cdots, y_m) \to \psi(x, y_0, \cdots, y_m)$ iff for each $q \geq p$ and $x \in V^{\mathbb{P}}(q)$, we have

$$q \Vdash \phi(x, \tau_{pq}(y_0), \cdots, \tau_{pq}(y_m)) \implies q \Vdash \psi(x, \tau_{pq}(y_0), \cdots, \tau_{pq}(y_m)). \tag{34}$$

A proof of the previous lemma is just a direct calculation, so we omit it.

The following lemma will be useful to verify $V^{\mathbb{P}}$ satisfies Δ_0-separation if our V satisfies merely Δ_0-separation, not the full separation:

Lemma 4.6. If $q \in \mathbb{P}$ and $x \in V^{\mathbb{P}}(q)$ then

- $q \Vdash \forall y \in x \phi(y) \iff$ For each $r \geq q$ and $y \in x(r)$, $r \Vdash \phi(y)$ and
- $q \Vdash \exists y \in x \phi(x) \iff$ For some $y \in x(q)$, $q \Vdash \phi(y)$.

Especially, if ϕ is a Δ_0-formula then $p \Vdash \phi$ is also Δ_0. $\square$

The proof of this lemma is also straightforward, so we also omit its proof.

We will show that $V^{\mathbb{P}}$ is a model of CZF. Before to check that, we have to verify $V^{\mathbb{P}}$ models intuitionistic logic. In addition, we also need to verify equality axioms is valid in $V^{\mathbb{P}}$:

Proposition 4.7. $V^{\mathbb{P}}$ satisfies intuitionistic logic and equality axioms.

Proof. The proof for intuitionistic logic is similar to Chapter 2, Section 5, Theorem 5.10. of [28]. Hence we omit a proof. Equality axioms directly follow from the definition of $=$ and $\in$, with Lemma 4.5. (We can find a list of equality axioms from Lemma 2.0.2 of [18].) $\square$

We can construct some canonical names from given names. We will use the following names to show that the axioms of CZF are valid in $V^{\mathbb{P}}$:

Definition 4.8. Let $p \in \mathbb{P}$ and $x, y, y_0, \cdots, y_n \in V^{\mathbb{P}}(p)$. Define $\mathbb{P}$-names $\mathrm{up}(x,y)$, $\mathrm{op}(x,y)$, $\mathrm{Union}(x)$, $\mathrm{Power}(x)$, $\mathrm{Sep}_\phi(x; y_0, \cdots, y_n)$ with domain $\uparrow p$ as follows:

- $\mathrm{up}(x,y)(q) = \{\tau_{pq}(x), \tau_{pq}(y)\}$,
- $\mathrm{op}(x,y)(q) = \mathrm{up}(\mathrm{up}(x,x)(q), \mathrm{up}(x,y)(q))(q)$,
- $\mathrm{Union}(x)(q) = \bigcup\{z(q) \mid z \in x(q)\}$,
- $\mathrm{Power}(x)(q) = \{z \in V^{\mathbb{P}}(q) \mid \forall r \geq q : z(r) \subseteq y(q)\}$ and
- $\mathrm{Sep}_\phi(x; y_0, \cdots, y_n)(q) = \{z \in x(q) \mid q \Vdash \phi(z, \tau_{pq}(y_0), \cdots, \tau_{pq}(y_n))\}$

Note that $\mathrm{Power}(x)$ is a set only if the Pow is valid in V. Hence we always assume that Pow holds when referring $\mathrm{Power}(x)$.

Like canonical names for sets in V under classical forcing, we can define canonical $\mathbb{P}$-names for sets in V:

Definition 4.9. Let x be a set. Define its canonical name $\check{x}$ with domain $\mathbb{P}$ as follows:

$$\check{x}(p) = \{\tau_{\perp p}(\check{y}) \mid y \in x\}. \tag{35}$$

We can see that $\Vdash \check{\alpha}$ is an ordinal if α is. Especially, we will see that $\check{\omega}$ witnesses the Axiom of Infinity.

Theorem 4.10. Every axiom of CZF^- is valid in $V^{\mathbb{P}}$.

Moreover, if V satisfies either Subset Collection, Sep or Pow then $V^{\mathbb{P}}$ also satisfies Subset collection, Sep or Pow respectively.

Proof. 1. Extensionality: This directly follows from the definition of $\in$ and $=$ of Kripke models.

2. Pairing: We will see that $\operatorname{up}(x,y)$ witnesses Pairing for each $x, y \in V^{\mathbb{P}}(p)$. We have to check the following holds:
$$p \Vdash \forall z : z \in \operatorname{up}(x,y) \leftrightarrow (z = x \vee z = y). \tag{36}$$
In one direction, take $q \geq p$, $z \in V^{\mathbb{P}}(q)$ and suppose that $r \Vdash z \in \operatorname{up}(x,y)$ holds for $r \geq q$. Then $\tau_{qr}(z) \in \{\tau_{pr}(x), \tau_{pr}(y)\}$, so $\tau_{qr}(z) = \tau_{pr}(x)$ or $\tau_{qr}(z) = \tau_{pr}(y)$. By the definition of τ, it implies
   - $z(s) = x(s)$ for all $s \geq r$ or
   - $z(s) = y(s)$ for all $s \geq r$.

   Therefore $r \Vdash x = z \vee y = z$. The other direction follows from reversing our previous proof.

3. Union: We can prove that $\operatorname{Union}(x)$ witnesses Union. The detail is left to the readers.

4. $\in$-Induction: We have to show that
$$\bot \Vdash \forall x(\forall y \in x\phi(y) \to \phi(x)) \to \phi(x). \tag{37}$$
Suppose that
$$p \Vdash \forall x(\forall y \in x\phi(y) \to \phi(x)) \tag{38}$$
holds. We shall verify $p \Vdash \forall x\phi(x)$, which is equivalent to
$$\forall q \geq p \forall x \in V^{\mathbb{P}}_\beta(q) : q \Vdash \phi(x). \tag{39}$$
holds for all $\beta \in \mathrm{Ord}$. Note that the equivalence follows from the tautology $[(\exists x\phi(x)) \to \psi] \leftrightarrow [\forall x(\phi(x) \to \psi)]$ of intuitionistic logic, where x does not occur free in ψ.

   Now assume inductively that (39) holds for all $\beta \in \alpha$. Fix $q \geq p$ and let $x \in V^{\mathbb{P}}_\alpha(q)$. If $y \in x(r)$ for $r \geq q$, then $y \in V^{\mathbb{P}}_\beta(r)$ for some $\beta \in \alpha$. Therefore $r \Vdash \phi(y)$ for all $r \geq q$ and $y \in x(r)$. This implies $q \Vdash \forall y \in x\phi(y)$. By (38), $q \Vdash \forall y \in x\phi(y)$ implies $q \Vdash \phi(x)$. By induction for ordinals over the ground model, we have (39) for all $\beta \in \mathrm{Ord}$.

5. Infinity: We shall prove that $\check{\omega}$ witnesses the Axiom of Infinity. Formally, $\check{\omega}$ is closed under successor operator $x \mapsto x \cup \{x\}$ internally. By induction on n, we can prove
$$\bot \Vdash \widecheck{n+1} = \check{n} \cup \{\check{n}\}. \tag{40}$$
Moreover, $\check{\omega}(p) = \{\tau_{\bot p}(\check{n}) \mid n \in \omega\}$. Therefore, if $p \Vdash x \in \check{\omega}$, which implies $x = \tau_{\bot p}(\check{n})$ for some $n \in \omega$, then we have $p \Vdash x \cup \{x\} \in \check{\omega}$.

6. (Δ_0-)Separation: Let $x, y \in V^{\mathbb{P}}(p)$. If our ground model satisfies Δ_0-separation, then $\operatorname{Sep}_\phi(x; y_0, \cdots, y_n)$ is a set for a bounded formula $\phi(x, y_0, \cdots, y_n)$. We can easily see that
$$p \Vdash \forall z : z \in \operatorname{Sep}_\phi(x; y_0, \cdots y_n) \leftrightarrow z \in x \wedge \phi(z, y_0, \cdots y_n) \tag{41}$$
by the definition of $\operatorname{Sep}_\phi(x; y_0, \cdots y_n)$. The same proof can be applied to show $V^{\mathbb{P}}$ satisfies Full separation in the case when V satisfies Full separation.

7. Strong Collection: Fix $p \in \mathbb{P}$ and $a \in V^{\mathbb{P}}(p)$. Suppose that $q \geq p$ satisfies $q \Vdash \forall x \in a\exists y\phi(x,y)$, which is equivalent to
$$\forall r \geq q \forall x \in a(r) \exists y : y \in V^{\mathbb{P}}(r) \wedge r \Vdash \phi(x,y). \tag{42}$$

Define $A = \bigcup_{r \geq q} \{r\} \times a(r)$. Then (42) can be restated as

$$\forall \pi \in A \exists y : y \in V^{\mathbb{P}}(\jmath_0 \pi) \wedge \jmath_0 \pi \Vdash \phi(\jmath_1 \pi, y). \tag{43}$$

(where $\jmath_0$ and $\jmath_1$ are canonical projections of pairing.) By Strong Collection on the ground universe, we can find a set C such that

$$\forall \pi \in A \exists y \in C : y \in V^{\mathbb{P}}(\jmath_0 \pi) \wedge \jmath_0 \pi \Vdash \phi(\jmath_1 \pi, y) \tag{44}$$

and

$$\forall y \in C \exists \pi \in A : y \in V^{\mathbb{P}}(\jmath_0 \pi) \wedge \jmath_0 \pi \Vdash \phi(\jmath_1 \pi, y). \tag{45}$$

C itself is not a $\mathbb{P}$-name, but a collection of $\mathbb{P}$-names. Despite that, we can make use of C to define an instance of Strong Collection. Take $b \in V^{\mathbb{P}}(q)$ as follows:

$$b(r) = \{\tau_{sr}(y) \mid y \in C \wedge q \leq s \leq r \wedge \operatorname{dom} y =\uparrow s\}.\} \tag{46}$$

b is well-defined by Replacement and Δ_0-separation. We will show that $b \in V^{\mathbb{P}}(q)$ and b witnesses Strong Collection. Note that we do not need full $b(r)$ as an instance of a collection set. In fact, a subset $\{y \in C \mid \operatorname{dom} y =\uparrow s\}$ suffices to witness Strong Collection. However, other $\mathbb{P}$-names are essential for monotonicity of b.

We can show $b \in V^{\mathbb{P}}(q)$ by checking conditions in Lemma 4.3. Verifying them is trivial, so we omit it.

It remains to show that b witnesses Strong Collection. We shall prove

$$q \Vdash \forall x \in a \exists y \in b \phi(x, y) \wedge \forall y \in b \exists x \in a \phi(x, y). \tag{47}$$

This is a direct corollary of (44), (45) and the definition of b.

8. Subset Collection: We will show that $V^{\mathbb{P}}$ validates Fullness if V satisfies Subset Collection. We shall prove

$$\bot \Vdash \forall a, b \exists c \forall x : a \rightrightarrows b \exists y \in c : y : a \rightrightarrows b \wedge y \subseteq x. \tag{48}$$

Fix $p \in \mathbb{P}$ and $a, b \in V^{\mathbb{P}}(p)$. Notice that for $q \geq p$ and $x \in V^{\mathbb{P}}(q)$, we have

$$\begin{aligned} q \Vdash x : a \rightrightarrows b &\iff q \Vdash \forall u \in a \exists v \in b : \operatorname{op}(u, v) \in x & (49)\\ &\iff \forall r \geq q \forall u \in a(r) \exists v \in b(r) : \operatorname{op}(u, v) \in x(r). & (50)\end{aligned}$$

For each $q \geq p$, let $A_q = \bigcup_{r \geq q} \{r\} \times a(r)$ and $B = \bigcup_{r \geq p} b(r)$. Now consider the following relation with the parameter x:

$$R_x = \{\langle \langle r, u \rangle, v \rangle \in A_q \times B \mid \operatorname{dom} v =\uparrow r \wedge \operatorname{op}(u, v) \in x(r).\} \tag{51}$$

We can see that $R_x \subseteq A_q \times B$ and $R_x : A_q \rightrightarrows B$.

Now apply Subset Collection to $\mathcal{A}(R_x)$, where $\mathcal{A}$ is defined in Lemma 2.2, so we have a family $\mathcal{F}$ of sets, depending on $q \geq p$, such that

$$\forall x : [\mathcal{A}(R_x) : A_q \rightrightarrows A_q \times B] \to [\exists C \in \mathcal{F} : \mathcal{A}(R_x) : A_q \leftrightarrows C]. \tag{52}$$

By Lemma 2.2, this is equivalent to

$$\forall x : (R_x : A_q \rightrightarrows B) \to \exists C \in \mathcal{F} : C \subseteq R_x \wedge C : A \rightrightarrows B. \tag{53}$$

We will apply Collection to get a set $\mathbb{C}$ such that for each $q \geq p$ we can find $\mathcal{F} \in \mathbb{C}$ which satisfies (53).

We can start a construction of a $\mathbb{P}$-name which witnesses Fullness. We first define a classification of relations given as follows:

$$\mathcal{Z}_q = \{\mathcal{F} \cap \operatorname{mv}(A_q, B) \mid \mathcal{F} \in \mathbb{C}\}. \tag{54}$$

Since the formula '$R \in \operatorname{mv}(A_q, B)$' is bounded, $\mathcal{Z}_q$ is a set for each $q \geq p$.

For each $C \in \mathcal{Z}_q$, let

$$z_{q,C}(r) = \{\tau_{sr}(\mathrm{op}(u,v)) \mid q \le s \le r \wedge \langle\langle s,u\rangle,v\rangle \in C \wedge \operatorname{dom} v = \uparrow s\}. \tag{55}$$

$z_{q,C}$ itself does *not* witness Fullness, but is an element of a name witnessing Fullness. Note that $z_{q,C} \in V^{\mathbb{P}}(q)$ by Lemma 4.3.

Now we define a name c with domain $\uparrow p$ as follows:

$$c(q) = \{\tau_{sq}(z_{s,C}) \mid p \le s \le q \wedge C \in \mathcal{Z}_q\}. \tag{56}$$

We can see that c is a name by Lemma 4.3.

It remains to show that c witnesses Fullness: For given $x \in V^{\mathbb{P}}(q)$ satisfying $q \Vdash x : a \rightrightarrows b$, we can find $\mathcal{F} \in \mathbb{C}$ and $C \in \mathcal{F}$ such that the consequent of (53) holds. By definition of $z_{q,C}$ and $C \subseteq R_x$, we have $q \Vdash z_{q,C} \subseteq x$. Since $z_{q,C} \in c(q)$, c witnesses Fullness.

9. Powerset: We assume the Axiom of Power Set in this part. We can show that $\mathrm{Power}(x)$ exists for $x \in V^{\mathbb{P}}$. Moreover, we can prove that $\mathrm{Power}(x)$ witnesses Power set. The full proof is left to the readers, but note that the following equivalence is useful:

$$p \Vdash y \subseteq x \iff \forall q \ge p : y(q) \subseteq x(q). \tag{57}$$

□

4.2 Kripke rank

The rank on usual sets and $\mathcal{A}$-names defined in [19] is useful to develop to theory on sets and $\mathcal{A}$-names respectively. The usefulness is a motivation to define the rank on Kripke sets. Consider the following relation:

$$y \lhd x \iff \exists q \in \operatorname{dom} x : y \in x(q). \tag{58}$$

We want to use an recursive definition on $\lhd$, which requires $\lhd$ be progressive:

Proposition 4.11. $\lhd$ is progressive.

Proof. $\lhd$ has a Δ_0-definition. Moreover, $\mathrm{ext}_\lhd(x) = \bigcup \operatorname{ran} x$. Hence, it is sufficient to show that $\lhd$-induction scheme is valid. The idea of our proof – showing the induction schema is valid by applying induction on the level of hierarchy – will be akin to that of showing $\in$-induction over $V^{\mathbb{P}}$.

Suppose that

$$\forall x \in V^{\mathbb{P}} : (\forall y \in V^{\mathbb{P}} : y \lhd x \to \phi(y)) \to \phi(x) \tag{59}$$

holds. Furthermore, assume inductively that $\forall x \in V^{\mathbb{P}}_\beta \phi(x)$ holds for all $\beta \in \alpha$.

If $x \in V^{\mathbb{P}}_\alpha$ and $y \lhd x$, then $y \in V^{\mathbb{P}}_\beta$ for some $\beta \in \alpha$. Hence, we have $\phi(y)$ by the inductive hypothesis. Therefore, we can derive $\phi(x)$ from (59). □

We are ready to define a rank for Kripke sets: take recursively that

$$\operatorname{krk} x = \sup\{\operatorname{krk} y + 1 \mid y \lhd x\}. \tag{60}$$

Note that $\operatorname{krk} x$ is the same with $\sup_{q \in \operatorname{dom} x} \sup\{\operatorname{krk} y + 1 \mid y \in x(q)\}$.

The following proposition shows our Kripke rank behaves like the rank for actual sets:

Proposition 4.12. Let $x \in V^{\mathbb{P}}(p)$. Then we have $x(q) \subseteq V^{\mathbb{P}}_{\operatorname{krk} x}(q)$ for all $q \ge p$.

Proof. The proof uses induction on $\lhd$. Assume that the proposition holds for all $y \lhd x$, so we have

$$\forall q \ge p \forall y \in x(q) \forall r \ge q : y(r) \subseteq V^{\mathbb{P}}_{\operatorname{krk} y}(r). \tag{61}$$

By the previous sentence and the conditions on $V^{\mathbb{P}}$ of (d), we have $y \in V^{\mathbb{P}}_{\operatorname{krk} y+1}(q) \subseteq V^{\mathbb{P}}_{\operatorname{krk} x}(q)$ for all $y \in x(q)$. Hence $x(q) \subseteq V^{\mathbb{P}}_{\operatorname{krk} x}(q)$. □

Kripke rank can be changed if we alter the domain of a given Kripke set by applying the transition functions. The following lemma shows that the Kripke rank decreases under the transition functions. If $\mathbb{P}$ is linear, moreover, then the Kripke rank is invariant under the transition functions.

Lemma 4.13. Let $x \in V^{\mathbb{P}}(p)$ and $p \geq q$. Then $\operatorname{krk} x \geq \operatorname{krk} \tau_{pq}(x)$. If $\mathbb{P}$ is linear then $\operatorname{krk} x = \operatorname{krk} \tau_{pq}(x)$.

Proof. $\operatorname{krk} x \geq \operatorname{krk} \tau_{pq}(x)$ is easy to verify. Now suppose that $\mathbb{P}$ is linear. We will use the induction on $\lhd$: assume that $\operatorname{krk} y = \operatorname{krk} \tau_{pq}(y)$ holds for all $y \lhd x$. If $r \geq q$, then

$$\begin{array}{rcll} \sup \operatorname{krk}''[x(q)] & \geq & \sup \operatorname{krk}''[\tau''_{rq}[x(r)]] & \because \text{monotonicity of Kripke sets} \\ & = & \sup \operatorname{krk}''[x(r)] & \because \text{if } y \in x(r) \text{ then } y \lhd x. \end{array} \tag{62}$$

Note that we have applied the inductive hypothesis to derive the last equality. Hence

$$\begin{array}{rcl} \operatorname{krk} \tau_{pq}(x) & = & \sup_{r \geq q} \sup \operatorname{krk}''[x(r)] \\ & = & \sup_{r \geq p} \sup \operatorname{krk}''[x(r)]. \end{array} \tag{63}$$

We need to explain why the last inequality holds. By linearity of $\mathbb{P}$, we have either $r \geq q$ or $q \geq r$ for all r. If $r \leq q$, then there is nothing to prove. If $q \geq r$, then $\sup \operatorname{krk}''[x(q)] \geq \sup \operatorname{krk}''[x(r)]$ so $\sup_{r \geq q} \sup \operatorname{krk}''[x(r)]$ already bounds $\operatorname{krk}''[x(r)]$. □

4.3 Slicing function

The construction of a model of NDCom, which will appear, needs examining the structure of Krikpe sets at a fixed stage. We can achieve it by considering $\in_p$ given by

$$x \in_p y \qquad \text{iff} \qquad x \in y(p) \tag{64}$$

as a membership relation at the given stage. The idea yields the following definition of *slicing function at stage* p:

Definition 4.14. Let $x \in V^{\mathbb{P}}$ and $p \in \operatorname{dom} x$. Define a *slicing function at stage* p $\mathbb{s}_p(x)$, is defined by

$$\mathbb{s}_p(x) = \{\mathbb{s}_p(y) \mid y \in x(p)\}. \tag{65}$$

The definition of $\mathbb{s}_p$ uses recursion on $\in_p$. If we work over IZF, we only need to check $\in_p$ is just well-founded to justify our definition. We work over CZF^- in general, however, and it requires us to check $\in_p$ is not just well-founded, but progressive:

Lemma 4.15. (CZF^-) $\in_p$ is progressive.

Proof. The main idea of our proof is the same with that of showing induction scheme for $\lhd$. It suffices to show that $\in_p$-induction schema is valid. Suppose that

$$\forall y \in V^{\mathbb{P}}(p) : [y \in_p x \to \phi(y)] \to \phi(x) \tag{66}$$

holds for all $x \in V^{\mathbb{P}}(p)$. Now we will show that

$$\forall \alpha \in \mathrm{Ord} \forall x \in V^{\mathbb{P}}_\alpha(p) : \phi(x) \tag{67}$$

by induction on α.

Suppose that $\forall x \in V^{\mathbb{P}}_\beta(p) : \phi(x)$ holds for all $\beta \in \alpha$. If $x \in V^{\mathbb{P}}_\alpha(p)$ and $y \in x(p)$, then $y \in V^{\mathbb{P}}_\beta(p)$ for some $\beta \in \alpha$. Therefore $\phi(y)$ holds for all $y \in_p x$. By (66), $\phi(x)$ holds. Therefore $\phi(x)$ holds for all $x \in V^{\mathbb{P}}_\alpha(p)$. Hence (67) follows by the induction on ordinals. □

Lemma 4.16. (CZF^-) Let $\mathbb{P}$ be a frame, $p \in \mathbb{P}$ and $x, y \in V^{\mathbb{P}}(p)$.

1. $x \in y(p)$ implies $\mathbb{s}_p(x) \in \mathbb{s}_p(y)$.

2. $\mathbb{s}_p(x) = \mathbb{s}''_p[x(p)]$.

3. $\mathbb{s}_p(\operatorname{up}(x, y)) = \{\mathbb{s}_p(x), \mathbb{s}_p(y)\}$.

4. $\mathbb{s}_p(\operatorname{Union}(x)) = \bigcup \mathbb{s}_p(x)$.

5. $\mathbb{s}_p(\operatorname{Power}(x)) \subseteq \mathcal{P}(\mathbb{s}_p(x))$.

Proof. The initial three statements directly follow from the definition of $\mathbb{s}_p$ and up. Hence we only give proof for the last two statements:

4. For the one inclusion, we have

$$\begin{array}{lll} \mathbb{s}_p(\mathrm{Union}(x)) & = \mathbb{s}_p''[\bigcup\{z(p) \mid z \in x(p)\}] & = \bigcup\{\mathbb{s}_p''[z(p)] \mid z \in x(p)\} \\ & = \bigcup\{\mathbb{s}_p(z) \mid z \in x(p)\} & \subseteq \bigcup \mathbb{s}_p''[x(p)] \\ & = \bigcup \mathbb{s}_p(x). & \end{array} \tag{68}$$

   For the remaining inclusion, observe that $w \in \mathbb{s}_p(x)$ iff $w = \mathbb{s}_p(y)$ for some $y \in x(p)$. Hence $\bigcup \mathbb{s}_p(x) \subseteq \bigcup\{\mathbb{s}_p(y) \mid y \in x(p)\} = \mathbb{s}_p(\mathrm{Union}(x))$.

5. By a direct calculation, we have

$$\begin{array}{ll} \mathbb{s}_p(\mathrm{Power}(x)) & = \mathbb{s}_p''[\{z \in V^{\mathbb{P}}(p) \mid \forall q \geq p : z(q) \subseteq x(q)\}] \\ & = \{\mathbb{s}_p(z) \mid z \in V^{\mathbb{P}}(p) \text{ and } \forall q \geq p : z(q) \subseteq x(q)\} \\ & \subseteq \{\mathbb{s}_p(z) \mid \mathbb{s}_p(z) \subseteq \mathbb{s}_p(x)\} \\ & \subseteq \mathcal{P}(\mathbb{s}_p(x)) \end{array} \tag{69}$$

   as $z(p) \subseteq x(p)$ implies $\mathbb{s}_p(z) \subseteq \mathbb{s}_p(x)$. □

5 DCom over Kripke models

5.1 Kripke models and DCom

We might hope that we could employ Kripke models to construct a model of NDCom. However, its possibility is quite unclear: the author cannot prove either we can construct these models via Kripke models, or we cannot use Kripke models as it preserves the Axiom of Double Complement. Nevertheless, we can apply Kripke models to produce partial results.

We will show that the Axiom of Double Complement is persistent under Kripke models with *linear* frames:

Theorem 5.1. Let $\mathbb{P}$ be a linear frame. If V satisfies ZF then $V^{\mathbb{P}}$ satisfies DCom.

The following lemma has critical role in the proof of Theorem 5.1:

Lemma 5.2. Let $\mathbb{P}$ be a linear order. If $p \in \mathbb{P}$, $z, x \in V^{\mathbb{P}}(p)$ and $p \Vdash \neg\neg(z \in x)$ then $\operatorname{krk} z < \operatorname{krk} x$ holds.

Proof. Let $\mathbb{P}$ be a linear order. $p \Vdash \neg\neg(z \in x)$ implies there is $q \geq p$ such that $\tau_{pq}(z) \in x(q)$. Thus $\operatorname{krk} \tau_{pq}(z) < \operatorname{krk} x$. By Lemma 4.13, $\operatorname{krk} z = \operatorname{krk} \tau_{pq}(z)$, so we have the desired result. □

Proof of Theorem 5.1. Let $x \in V^{\mathbb{P}}(p)$. Define a function y of domain $\uparrow p$ as

$$y(q) = \{z \in V^{\mathbb{P}}_{\operatorname{krk} x}(q) \mid \exists s \geq q : \tau_{qs}(z) \in x(s)\}. \tag{70}$$

We can show $y \in V^{\mathbb{P}}(p)$ by Lemma 4.3. We claim that y is a double complement of x: that is,

$$p \Vdash \forall z : \neg\neg(z \in x) \leftrightarrow z \in y. \tag{71}$$

Let $q \geq p$ and $z \in V^{\mathbb{P}}(q)$. For the one direction, assume that $r \Vdash \neg\neg(z \in x)$ holds for some $r \geq q$. By Lemma 5.2, we have $\operatorname{krk} z < \operatorname{krk} x$. Thus we have $z \in V^{\mathbb{P}}_{\operatorname{krk} x}(q)$ by Proposition 4.12 and the Closure lemma (Lemma 4.3). Furthermore, $r \Vdash \neg\neg(z \in x)$ implies the existence of $s \geq q$ such that $\tau_{qs}(z) \in x(s)$. Hence $z \in y(q)$.

For the remaining direction, let $q \Vdash z \in y$, so that $z \in y(q)$. By definition, we have $s \geq q$ such that $\tau_{qs}(z) \in x(s)$. Since s is comparable with any element of $\mathbb{P}$ due to linearity, we have $\forall r \geq q \exists s \geq s : \tau_{qs}(z) \in x(s)$. Thus $q \Vdash \neg\neg(z \in x)$.

□

Example 5.3. The previous theorem uses the bound of $\operatorname{krk} z$ for names z satisfying $z \in x^{\mathsf{CC}}$. This bound follows from the linearity of $\mathbb{P}$. If the frame is not linear, however, guessing the bound of $\operatorname{krk} z$ would be difficult. This example illustrates this situation.

Working over a model V of ZFC. Take $\mathbb{P} = {}^{<\omega}\kappa$ for some infinite cardinal κ. Define $a^p_\alpha \in V^{\mathbb{P}}(p)$ recursively on $\alpha < \kappa^+$, simultaneously on p as follows: if $\alpha = 0$, take $a^p_0(q) = \varnothing$ for all $q \geq p$. If $\alpha = \beta + 1$, take

$$a^p_{\beta+1}(q) = \begin{cases} \{a^p_\alpha\} & \text{if } q = p \\ \{\tau_{pq}(a^p_\alpha)\} & \text{if } q \neq p. \end{cases} \tag{72}$$

If $\gamma < \kappa^+$ is a limit ordinal, choose a cofinal sequence $\langle \gamma_\xi \mid \xi < \operatorname{cf}\gamma\rangle$ (we need the Axiom of Choice to choose it) and define a^p_γ as follows:

$$a^p_\gamma(q) = \begin{cases} \varnothing & \text{if } q = p \text{ or there is } \xi > \operatorname{cf}\gamma \text{ such that } q \geq p^\frown\langle\xi\rangle, \\ \{a^q_{\gamma_\xi}\} & \text{if } q = p^\frown\langle\xi\rangle \text{ for some } \xi < \operatorname{cf}\gamma, \\ \{\tau_{p^\frown\langle\xi\rangle,q}(a^{p^\frown\langle\xi\rangle}_{\gamma_\xi})\} & \text{if } q \gneq p^\frown\langle\xi\rangle \text{ for some } \xi < \operatorname{cf}\gamma. \end{cases} \tag{73}$$

By the Closure lemma, $a^p_\alpha \in V^{\mathbb{P}}_\alpha(p)$ for all $\alpha < \kappa^+$. We will see that $\operatorname{krk} a^p_\alpha = \alpha$ holds for all p and $\alpha < \kappa^+$ by induction on α: this is trivial when $\alpha = 0$. If $\alpha = \beta + 1$, we have $\operatorname{krk} a^p_{\alpha+1} = \operatorname{krk} a^p_\alpha + 1$ as the right-hand-side bounds other Kripke ranks occured in Lemma 4.13. If $\gamma < \kappa^+$ is a limit ordinal with the chosen cofinal sequence $\langle \gamma_\xi \mid \xi < \operatorname{cf}\gamma\rangle$ then we have

$$\operatorname{krk} a^p_\gamma = \sup\{\operatorname{krk} a^{p^\frown\langle\xi\rangle}_{\gamma_\xi} + 1 \mid \xi < \operatorname{cf}\gamma\} = \gamma \tag{74}$$

by combining with Lemma 4.13 and some calculation.

We claim that any a^p_α eventually stabilizes to a^p_n for some $n < \omega$. In other words, for each $\alpha < \kappa^+$ and $q \geq p$, there is $r \geq q$ such that $\tau_{pr}(a^p_\alpha) = \tau_{pr}(\alpha^p_n)$ for some n. It obviously holds when $\alpha < \omega$. Now consider the case $\alpha = \beta + 1$. Then for any $q \geq p$, $a^p_{\beta+1}(q) = \{\tau_{pq}(a^p_\beta)\}$. Take $r \geq q$ and $n < \omega$ such that $\tau_{pr}(a^p_\beta) = \tau_{pr}(a^p_n)$. Then for $s \geq r$,

$$\begin{aligned} a^p_{\beta+1}(s) &= \{\tau_{ps}(a^p_\alpha)\} = \{\tau_{rs}(\tau_{pr}(a^p_\alpha))\} &(75)\\ &= \{\tau_{rs}(\tau_{pr}(a^p_n))\} = \{\tau_{ps}(a^p_n)\} = a^p_{n+1}(s). &(76)\end{aligned}$$

Therefore, $\tau_{pr}(a^p_{\beta+1}) = \tau_{pr}(a^p_{n+1})$. It remains to consider the case of limit ordinals. Let γ is a limit ordinal and $\langle \gamma_\xi \mid \xi < \operatorname{cf}\gamma\rangle$ be the cofinal sequence of γ. It is sufficient to just consider $q \geq p^\frown\langle\xi\rangle$ for some ξ. If $\xi > \operatorname{cf}\gamma$, then $\tau_{pq}(a^p_\gamma)$ would be the empty name. If $\xi < \operatorname{cf}\gamma$, then $\tau_{pq}(a^p_\gamma) = \{\tau_{p^\frown\langle\xi\rangle,q}(a^{p^\frown\langle\xi\rangle}_{\gamma_\xi})\}$. Take $r > q$ and $n < \omega$ such that $\tau_{p^\frown\langle\xi\rangle,r}(a^{p^\frown\langle\xi\rangle}_{\gamma_\xi}) = \tau_{p^\frown\langle\xi\rangle,r}(a^{p^\frown\langle\xi\rangle}_n)$. Therefore,

$$\begin{aligned} \tau_{pr}(a^p_\gamma) &= \tau_{p^\frown\langle\xi\rangle,r}(\tau_{p,p^\frown\langle\xi\rangle}(a^p_\gamma)) = \tau_{p^\frown\langle\xi\rangle,r}(a^{p^\frown\langle\xi\rangle}_\gamma) &(77)\\ &= \lambda\!\!\lambda s \in\uparrow r.\{\tau_{p^\frown\langle\xi\rangle,s}(a^{p^\frown\xi}_{\gamma_\xi})\} = \lambda\!\!\lambda s \in\uparrow r.\{\tau_{p^\frown\langle\xi\rangle,s}(a^{p^\frown\xi}_n)\} &(78)\\ &= \tau_{p^\frown\langle\xi\rangle,r}(a^{p^\frown\xi}_{n+1}). &(79)\end{aligned}$$

Moreover, we can prove $a^{p^\frown\langle\xi\rangle}_n = \tau_{p,p^\frown\langle\xi\rangle}(a^p_n)$ by induction n. Thus, we have $\tau_{pr}(a^p_\gamma) = \tau_{pr}(a^p_{n+1})$.

Now define $T \in V^{\mathbb{P}}(\bot)$ as $T(p) = \{\tau_{\bot,p}(a^\bot_n) \mid n < \omega\}$. By previous arguments, we have $\operatorname{krk} T = \omega$ and $\bot \Vdash \neg\neg(a^\bot_\alpha \in T)$ for all $\alpha < \kappa^+$. However we have $\operatorname{krk} a^\bot_\alpha = \alpha$, which can take any value between ω and κ^+.

5.2 Nonstability of V_ω

Kripke models provide a model of $\mathsf{IZF} + \mathsf{DCom}$ + 'V_ω is not stable' as we promised. Before to explain the full construction, we examine an sample name. Consider the sample poset $\mathbb{P} \subset {}^\omega 2$ described in Figure 1. Now consider the name a over $\mathbb{P}$ defined by

$$a(p) = \{\tau_{\bot,p}(\check{k}) \mid k < n(p)\}, \tag{80}$$

where $n(q)$ is the number of 0 in p in front of the first 1. If there is no 1, then $n(p)$ is just a number of 0 appearing in p. (For example, we have $n(001) = 2$ and $n(010010001) = 1$.) Then the following holds:

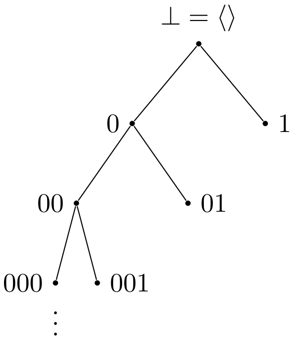

Figure 1: The sample poset $\mathbb{P}$

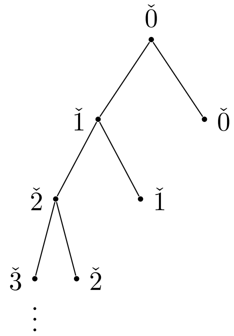

Figure 2: A description of the sample name a. A label of each node p roughly describes the stage set $a(p)$.

1. $p \Vdash \neg\neg(a \in \check{\omega})$ and
2. $\perp \nVdash a \in V_{\check{\omega}}$.

However, a could not act as a counterexample of stability of V_ω, since a is eventually a member of ω. More precisely, each leaf of $\mathbb{P}$ thinks a is a natural number. We need to 'iterate' our construction, so that each node of a frame candidate contains a name, which stabilizes to a natural number eventually as a did. We will see that the iterated construction can be 'performed' over the full binary tree ${}^{<\omega}2$.

Consider the model V of ZFC and take the full binary tree ${}^{<\omega}2$ as a frame. By Theorem 4.10 and 5.1, it satisfies $\mathsf{IZF} + \mathsf{DCom}$. Now consider the ${}^{<\omega}2$-name $a_p \in V^{{}^{<\omega}2}(p)$ defined by

$$a_p(p^\frown q) = \{\tau_{\perp, p^\frown q}(\check{k}) \mid k < n(q)\}, \tag{81}$$

where $n(q)$ is the number of 0 in front of the first 1 in q, as we defined before. Then

Lemma 5.4. $p \Vdash \neg\neg(a_p \in \check{\omega})$ and $p \nVdash a_p \in V_{\check{\omega}}$.

Proof. For each $q \in {}^{<\omega}2$, if 1 appears first in q at the kth position, then $p^\frown q \Vdash a_p = \check{k}$. If not, so q is the sequence of k consecutive zeros, then $p^\frown q^\frown\langle 1\rangle \Vdash a_p = \widecheck{k+1}$. Therefore, for each extension q of p, we can find $r \geq q$ such that $r \Vdash a_p \in \check{\omega}$.

Suppose that $p \Vdash a_p \in V_{\check{\omega}}$. We can prove from IZF that $V_\omega = \bigcup_{n\in\omega} \mathcal{P}^n(1)$, where $\mathcal{P}^n(1)$ is the n times iterated power set of 1. Hence there is $n \in \omega$ such that $p \Vdash a_p \in \mathcal{P}^n(\check{1})$. Especially, we have $q \Vdash a_p \in \mathcal{P}^n(\check{1})$ for any $q \geq p$. Thus, we can deduce the following equation from Lemma 4.16:

$$\mathbb{s}_q(a_p) \in \mathcal{P}^n(1). \tag{82}$$

Take $q = p^\frown\langle \underbrace{0, \cdots, 0}_{n+1 \text{ times}}, 1\rangle$. Since $q \Vdash a_p = \widecheck{n+1}$, we have $n+1 \in \mathcal{P}^n(1)$, a contradiction. $\square$

Theorem 5.5. In $V^{{}^{<\omega}2}$, $\perp \Vdash \neg(V_{\check{\omega}}^{\mathsf{CC}} \subseteq V_{\check{\omega}})$.

Proof. By Lemma 5.4, we have $p \nVdash \neg\neg(a_p \in V_{\check{\omega}}) \to a_p \in V_{\check{\omega}}$ for all $p \in {}^{<\omega}2$. Thus we have

$$\perp \Vdash \neg(\forall z : \neg\neg(z \in V_{\check{\omega}}) \to z \in V_{\check{\omega}}). \qquad \square \tag{83}$$

In Proposition 3.4, we showed that the combination of Δ_0-WELM and MP proves the stability of V_ω. We can see that just assuming MP without Δ_0-WELM is not sufficient to show Proposition 3.4, since $V^{{}^{<\omega}2}$ satisfies MP:

Proposition 5.6. $V^{{}^{<\omega}2}$ satisfies MP.

Proof. Suppose that $p \Vdash \forall n \in \check{\omega} : \phi(n) \vee \neg\phi(n)$ and $p \Vdash \neg\neg\exists n \in \check{\omega}\phi(n)$. Hence we have $m \in \omega$ such that $p \Vdash \neg\neg\phi(\check{m})$. Furthermore, m must satisfy either $p \Vdash \phi(\check{m})$ or $p \Vdash \neg\phi(\check{m})$. As the latter one and $p \Vdash \neg\neg\phi(\check{m})$ implies an absurdity, we have $p \Vdash \phi(\check{m})$. $\square$

5.3 Models of $\mathsf{IZ} + \mathsf{NDCom}$

It is unclear whether we can use Kripke models to construct a model of $\mathsf{IZF} + \mathsf{NDCom}$. In spite of that, we can construct a model of $\mathsf{IZ} + \mathsf{NDCom}$ by means of Kripke models. The idea of the construction is to construct a rapidly growing Kripke set, so that former stages of the Kripke set cannot cover the size of latter stages of the Kripke set. We construct a model by reducing the size of each step of the hierarchy.

The detail of the construction is as follows: Let V be a model of ZFC. Take $\mathbb{P} = \omega$ and consider the Kripke model $V^{\mathbb{P}}$. Fix a countable increasing sequence of limit ordinals $\langle \alpha_n \mid n < \omega \rangle$ greater than ω, and consider the following submodel $M = \langle M(n) : n < \omega \rangle$ defined by

$$x \in M(n) \iff x \in V^{\mathbb{P}}(n) \text{ and } \mathbb{s}_m(x) \in V_{\alpha_m} \text{ for all } m \geq n. \tag{84}$$

We can see (internal to $V^{\mathbb{P}}$) that M is a transitive subclass of $V^{\mathbb{P}}$. That is, externally, if $x, y \in V^{\mathbb{P}}(n)$, $n \Vdash y \in x$ and $x \in M(n)$ then $y \in M(n)$.

We need to define the forcing relation $\Vdash_M$ over M. The definition is simliar except for quantifiers: for example, $p \Vdash_M \forall x \phi(x)$ iff $q \Vdash_M \phi(a)$ for all $q \geq p$ and $a \in M(q)$.

Theorem 5.7. *M is a model of IZ.*

Proof. Extensionality follows from the transitivity of M. For $\in$-induction, observe that we can prove

$$\forall q \geq p \forall x \in M(q) \cap V^{\mathbb{P}}(q) : q \Vdash_M \phi(x) \tag{85}$$

by the similar argument in the proof of Theorem 4.10.

Pairing and Union follows from Lemma 4.16: In the case of Pairing, if $x, y \in M$ so $\mathbb{s}_n(x), \mathbb{s}_n(y) \in V_{\alpha_n}$ for all $n \in \omega$, then $\mathbb{s}_n(\operatorname{up}(x, y)) = \{\mathbb{s}_n(x), \mathbb{s}_n(y)\} \in V_{\alpha_n}$ as α_n is a limit ordinal.

Infinity holds since $\alpha_n > \omega$ for all $n \in \omega$ implies $\check{\omega} \in M$. (The fact that α_m is a limit ordinal for each $m \geq n$ is essential.)

For Separation, observe that if $x, y \in V^{\mathbb{P}}(n)$ and $n \Vdash y \subseteq x$ then $\mathbb{s}_m(y) \subseteq \mathbb{s}_m(x)$ for all $m \geq n$. Therefore M is in fact closed under subsets of a given set.

For Power set, let $x \in M(n)$, so $\mathbb{s}_m(x) \in V_{\alpha_m}$ for all $m \geq n$. By Lemma 4.16, $\mathbb{s}_m(\operatorname{Power}(x)) \subseteq \mathcal{P}(\mathbb{s}_m(x)) \in V_{\alpha_m}$ for each $m \geq n$ as α_m is a limit ordinal. Thus $\mathbb{s}_m(\operatorname{Power}(x)) \in V_{\alpha_m}$ for all $m \geq n$, as V_{α_m} is closed under subsets of its elements. □

Theorem 5.8. *M satisfies NDCom.*

Proof. For each ordinal ξ and $n \in \omega$, define $a_{\xi,n} \in V^{\mathbb{P}}(0)$ by

$$x_{\xi,n}(m) = \begin{cases} 0 & \text{if } m < n \\ \{\tau_{0,m}(\check{\eta}) \mid \eta < \xi\} & \text{if } m \geq n \end{cases}. \tag{86}$$

We can show that $m \Vdash x_{\xi,n} = \check{\xi}$ for all $m \geq n$ and

$$\mathbb{s}_m(x_{\xi,n}) = \begin{cases} 0 & \text{if } m < n \\ \xi & \text{if } m \geq n \end{cases}. \tag{87}$$

Therefore, $a_{\xi,n} \in M$ if $\xi < \alpha_n$. Now define $a \in V^{\mathbb{P}}(0)$ such that

$$a(n) = \{\tau_{0,m}(x_{\xi,m}) \mid m < n \wedge \alpha_{m-1} \leq \xi < \alpha_m\}. \tag{88}$$

(Take $\alpha_{-1} = 0$ for notational convenience.) We can see that $\mathbb{s}_0(a) = 0$ and $\mathbb{s}_{n+1}(a) = \alpha_n$ so $a \in M(0)$. We will show that a witnesses NDCom.

Suppose that $b \in M(n)$ is a 'superset' of the double complement of a: that is,

$$n \Vdash \forall z : \neg\neg(z \in a) \to z \in b. \tag{89}$$

We can see that $m \Vdash \neg\neg(z \in a)$ is equivalent to $\exists k \geq m : k \Vdash z \in a$ due to the linearity of the frame. Hence $m \Vdash \neg\neg(x_{\xi,m} \in a)$ iff $\xi < \sup_{n<\omega} \alpha_n$. Therefore, $b(m)$ must contain all of $\tau_{0,m}(x_{\xi,m})$. However, this is impossible as it implies $\mathbb{s}_m(b) \supseteq \sup_{n<\omega} \alpha_n$, contradicting with $\mathbb{s}_m(b) \in V_{\alpha_m}$. □

Some readers might wonder the validity of Collection scheme over M. Unfortunately, M does not even satisfy the Replacement scheme whatever $\langle \alpha_n \mid n < \omega\rangle$ is:

Theorem 5.9. *M satisfies the negation of an instance of Replacement.*

Proof. Consider the following formula:

$$\phi(x, y) \equiv \forall z : \neg\neg(z \in x) \leftrightarrow z = y. \tag{90}$$

In terms of double complement, $\phi(x, y)$ is just $x^{\mathsf{CC}} = y$. Let $b \in V^{\mathbb{P}}(0)$ be a $\mathbb{P}$-name given by $b(0) = \{x_{\xi,1} \mid \xi < \alpha_0\}$ and $b(n) = \tau''_{0,n}[b(0)]$, where $x_{\xi,n}$ is a $\mathbb{P}$-name defined in the proof of Theorem 5.8.

We claim first that $0 \Vdash \forall x \in a \exists y : x^{\mathsf{CC}} = y$. Note that no new element is added to b after level 1. Thus it suffices to show that

$$\forall x \in b(0) : 0 \Vdash \exists y : x^{\mathsf{CC}} = y. \tag{91}$$

We will make use of the concrete description of $x^{\mathsf{CC}}_{\xi,1}$ given by the proof of Theorem 5.1: we can show that $\operatorname{krk} x_{\xi,1} = \xi$ by induction on ξ. Hence the following Kripke set is a double complement of $x_{\xi,1}$:

$$y_{\xi,1}(n) = \{z \in V^{\mathbb{P}}_{\xi}(n) \mid \exists m \geq n : \tau_{n,m}(z) \in x_{\xi,1}(n)\}. \tag{92}$$

We will prove the following inclusion relations:

$$\xi \subseteq \mathbb{s}_0(y_{\xi,1}) \subseteq V_\xi. \tag{93}$$

For the former inclusion, observe that if $\eta < \xi$ then $\check{\eta} \in y_{\xi,1}(0)$, which follows from $\tau_{0,1}(\check{\eta}) \in x_{\xi,1}(1)$. For the latter inclusion, we will prove the following general statement: if $p \in \mathbb{P}$ and $x \in V^{\mathbb{P}}_{\xi}(p)$, then $\mathbb{s}_p(x) \subseteq V_\xi$. Assume that it holds for all $\eta < \xi$. For $x \in V^{\mathbb{P}}_{\xi}(p)$, then we have $x(p) \subseteq \bigcup_{\eta<\xi} V^{\mathbb{P}}_{\eta}(p)$. Therefore

$$\mathbb{s}''_p[x(p)] \subseteq \textstyle\bigcup_{\eta<\xi} \mathbb{s}''_p[V^{\mathbb{P}}_{\eta}(p)] \subseteq \displaystyle\bigcup_{\eta<\xi} V_\eta \subseteq V_\xi \tag{94}$$

The second inclusion comes from the inductive assumption. Thus $\mathbb{s}_p(x) = \mathbb{s}''_p[x(p)] \subseteq V_\xi$.

The upper bound in (93) ensures that if $\xi < \alpha_0$ then $y_{\xi,1} \in M(0)$. Therefore, on the one hand, $\forall x \in b(0) \exists y \in M : x^{\mathsf{CC}} = y$ holds. On the other hand, however, if there is $c \in M(0)$ such that $\forall x \in b(0) \exists y \in c(0) : 0 \Vdash x^{\mathsf{CC}} = y$, then $c(0)$ must contain $y_{\xi,1}$ for all $\xi < \alpha_0$. By the lower bound in (93), we have $\mathbb{s}_0(c) \notin V_\xi$ for all $\xi < \alpha_0$, contradicting with the definition of M. $\square$

6 Metamathematics on ADCom

Lubarsky [18] constructed a Kripke model over the frame $\mathbb{P} = \mathrm{Ord}$ to construct a model of $\mathsf{CZF}+\mathsf{Sep}+\neg\mathsf{Pow}$. We will call this model *Lubarsky's first model*. One can prove that Lubarsky's first model also satisfies ADCom, and this is the reason why we explain the whole construction of the model.

Lubarsky mentioned in [18] that he could construct a similar model by choosing $\mathbb{P} = \omega$ alternatively and requiring Kripke sets to be eventually constant. He also stated that this approach could be construed as 'taking a cofinal ω-sequence through Ord and cutting the full model down to those nodes.' [3] He does not construct his Kripke model in this way, as he expected neither two constructions are essentially harder than the other. However, we prefer the latter one (taking $\mathbb{P} = \omega$ and restricting Kripke sets to be hereditarily constant) than the former one (namely, assuming $\mathbb{P} = \mathrm{Ord}$.) There are two reasons for the preference: First, we want to work over CZF due to the issue of consistency strength. Second, we want to require $\mathbb{P}$ a linearly ordered class, but Ord is not provably linearly ordered over CZF. Assuming Ord a linearly ordered class yields $\mathsf{\Delta_0}-\mathsf{LEM}$, and we want to avoid it.

Consider the Kripke model V^ω, whose frame is the set of natural numbers ω with the usual ordering $\leq$. For each stage p and m, we want to define a subclass $K^m(p) \subseteq V^\omega(p)$ which satisfies

$$K^m(p) = \{x \in V^\omega(p) \mid \forall r \geq q \geq m : x(r) = \tau''_{qr}[x(q)] \wedge \forall q \geq p : x(q) \subseteq K^m(q)\}, \tag{95}$$

that is, x 'stops expanding hereditarily' after stage m. We need an inductive definition to give a precise definition for $K^m(p)$:

[3] We can find this description from the arXiv version of [18], not the original one.

Definition 6.1. Let Φ_m be an inductive definition defined by, $\langle a, x\rangle \in \Phi_m$ iff

1. $x \in V^\omega$,
2. $\forall p \in \operatorname{dom} x : x(p) \subseteq a$ and
3. $\forall q \geq p \geq m : p, q \in \operatorname{dom} x \to x(q) = \tau''_{pq}[x(p)]$.

Note that we can replace the last condition by the following sentence:

3′. $\forall k \in \omega : (\operatorname{dom} x =\uparrow k) \to \forall p \geq \max\{m,k\} : x(p) = \tau''_{\max\{m,k\},p}[x(\max\{m,k\})]$.

By Class Inductive Definition theorem, each Φ_m defines a least Γ_{Φ_m}-closed class K^m. Now take $K^m(p) = \{x \in K^m \mid \operatorname{dom} x =\uparrow p\}$. Finally, let $K(p) = \bigcup_{m\in\omega} K^m(p)$.

We may expect $K^m(p)$ satisfies (95) and behaves like the usual Kripke structures. The following sequence of lemmas state our expectation is valid:

Lemma 6.2 (Closure lemma for K)**.** Let x be a function whose domain is $\uparrow p$ for some $p \in \omega$. If x satisfies the defining formula in (95), then $x \in K^m(p)$.

Proof. Suppose that $x(q) \subseteq K^m(q)$ for all $q \geq p$. Take $a = \bigcup_{q\geq p} x(q)$. Then we can see that $\langle a.x\rangle \in \Phi_m$. Since $a \subseteq K^m$, we have $x \in K^m(p)$. □

We cannot generalize our lemma by just requiring $x(q) \subseteq K(q)$ for all $q \geq p$. We will see a counterexample in Theorem 6.11.

Lemma 6.3. 1. $K(p) \subseteq V^\omega(p)$ for all $p \in \omega$.

2. $K^m(p) \subseteq K^n(p)$ if $m \geq n$.
3. The transition map τ_{pq} restricted over $K^m(p)$ is a map from $K^m(p)$ to $K^m(q)$.
4. If a is a set then $\check{a} \in K^0(0)$.

Proof. The first and second statement follow from the very definition of $K(p)$. For the third statement, observe that τ_{pq} is just a restriction so it does not change anything except for the domain of an input. We can show the last statement by applying set induction on a: Suppose that $\check{b} \in K^0(0)$ for all $b \in a$. It is obvious that $\check{a}$ stops expanding hereditarily after stage 0. Hence, Lemma 6.2 ensures $\check{a} \in K^0(0)$. □

The forcing relation $\Vdash_K$ over K is defined similarly with some modifications. For example, $p \Vdash_K \forall x\phi(x)$ iff $q \Vdash_K \phi(x)$ for all $q \geq p$ and $x \in K(q)$.

We examine some structural properties of K before to describe the main result. We have decomposed the usual Kripke universe $V^{\mathbb{P}}$ into a hierarchy $\langle V^{\mathbb{P}}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ indexed by ordinals. Furthermore, the hierarchy satisfies a closure condition (i.e. Condition (d) of Definition 4.1.) We also want to decompose K similarly to apply induction on ordinals, to prove statements about K. The natural way to decompose K is to take $K^m_\alpha(p) := K^m(p) \cap V^\omega_\alpha(p)$. It is easy to see that their union is $K^m(p)$. The following lemma states $K^m_\alpha(p)$ satisfies an analogue of Condition (d):

Lemma 6.4. $x \in K^m_\alpha(p)$ iff x is a function of domain $\uparrow p$ such that x stops expanding hereditarily after stage m, satisfies monotonocity and $x(q) \subseteq \bigcup_{\beta\in\alpha} K^m_\beta(q)$ holds for all $q \geq p$.

Proof. Just decompose the sentence $x(q) \subseteq \bigcup_{\beta\in\alpha} K^m_\alpha(q)$ into $x(q) \subseteq \bigcup_{\beta\in\alpha} V_\alpha(q)$ and $x(q) \subseteq K^m(q)$. Then the equivalence follows from Lemma 6.2 and Condition (d) of Definition 4.1. □

Lubarsky observed that if a Kripke set $a \in K(m)$ does not change after stage m, then τ_{mp} behaves like an isomorphism for $p > m$ over a. He also mentioned that the isomophisom over a can be extended to the whole $K(m)$, by hereditarily translating the domain of Kripke names. Lubarsky's observation is necessary to prove K satisfies Separation. Thus, we provide it in a concrete form.

Definition 6.5. The translation function $\mathbb{t}_s : V^\omega \to V^\omega$ is defined recursively as follows: let $p \in \omega$ and $x \in V^\omega(p)$. $\mathbb{t}_s(x)$ is a function of domain $\uparrow (p+s)$ such that

$$\mathbb{t}_s(x)(q+s) = \mathbb{t}''_s[x(q)] = \{\mathbb{t}_s(z) \mid z \lhd x \wedge \operatorname{dom} z =\uparrow q\} \tag{96}$$

for all $q \geq p$.

Note that $\mathbb{t}_s$ is defined on recursion over $\lhd$, so $\mathbb{t}_s$ is well-defined.

The translation function $\mathbb{t}_s$ 'translates' the domain of a Kripke set $x \in V^\omega(p)$ to $\uparrow (p+s)$ hereditarily. If $p \geq s$, we can imagine to translate domain of $x \in V^\omega(p)$ to $\uparrow (p-s)$. We can see that $\lhd$ is progressive over $V^\omega(\geq p) := \bigcup_{q\geq p} V^\omega(q)$, and it allows to define a downward translation function:

Definition 6.6. The downward translation function $\mathbb{t}_{-s} : V^\omega(\geq s) \to V^\omega$ is defined recursively as follows: Let $p \geq s$ and $x \in V^\omega(p)$. Then $\mathbb{t}_{-s}$ is a function of domain $\uparrow (p-s)$ such that

$$\mathbb{t}_{-s}(x)(q) = \mathbb{t}''_{-s}[x(q+s)] = \{\mathbb{t}_{-s}(z) \mid z \lhd x \wedge \operatorname{dom} z = \uparrow (q+s)\} \tag{97}$$

for all $q \geq p-s$.

The following lemma describes basic facts on the translation function.

Lemma 6.7. Let $p, q \in \omega$.

1. $\mathbb{t}_s$ is a bijection between $V^\omega(p)$ and $V^\omega(p+s)$.
2. $\mathbb{t}_s(\tau_{pq}(x)) = \tau_{p+s,q+s}(\mathbb{t}_s(x))$ for all $x \in V^\omega(p)$ and $p \leq q$.
3. If $x \in K^m(p)$ for $m \geq p$ then $\mathbb{t}_s(x) \in K^{m+s}(p+s)$.
4. $\mathbb{t}_s$ is a bijection between $K(p)$ and $K(p+s)$.

Proof. 1. In fact, $\mathbb{t}_s : V^\omega \to V^\omega(\geq s)$ and $\mathbb{t}_{-s} : V^\omega(\geq s) \to V^\omega$ are inverses of each other. It can be shown by induction on $\lhd$.

We will show that $\mathbb{t}_{-s} \circ \mathbb{t}_s$ is the identity map. The other equality can be shown analogously. Assume inductively that $\mathbb{t}_{-s} \circ \mathbb{t}_s(y) = y$ holds for all $y \lhd x$. For $q \geq p$ and $x \in V^\omega(p)$, we have

$$\mathbb{t}_{-s} \circ \mathbb{t}_s(x)(q) = \mathbb{t}''_{-s}[\mathbb{t}_s(x)(q+s)] = \mathbb{t}''_{-s}[\mathbb{t}''_s[(x(q)]] = x(q) \tag{98}$$

since $y \lhd x$ for all $y \in x(q)$. Hence $\mathbb{t}_{-s} \circ \mathbb{t}_s(x) = x$ for all $x \in V^\omega$. Moreover, by examine the domains of Kripke sets, we can see that the restriction $\mathbb{t}_s \restriction V^\omega(p)$ and $\mathbb{t}_{-s} \restriction V^\omega(p+s)$ are inverses of each other.

2. Let $z \in \mathbb{t}_s(\tau_{pq}(x))(r+s)$ for $p \leq q \leq r$, so there is $w \in \tau_{pq}(x)(r) = x(r)$ such that $z = \mathbb{t}_s(w)$. Hence $z \in \mathbb{t}''_s[x(r)] = \mathbb{t}_s(x)(r+s) = \tau_{p+s,q+s}(\mathbb{t}_s(x))(r+s)$. This shows $\mathbb{t}_s(\tau_{pq}(x))(r+s) \subseteq \tau_{p+s,q+s}(\mathbb{t}_s(x))(r+s)$. Showing the reverse inclusion is analogous, so we omit it.

3. If $x \in K^m(p)$, then $x(q) = \tau''_{mq}[x(p)]$ for all $q \geq m$. We can show $\mathbb{t}_s(x)(q+s) = \tau''_{m+s,q+s}[\mathbb{t}_s(x)(m+s)]$ for $q \geq m$, so $\mathbb{t}_s(x) \in K^{m+s}(p+s)$.

4. We can show the following facts. The proof is similar to what we have already done, so we omit it.

   (a) $\mathbb{t}_{-s}(\tau_{p+s,q+s}(x)) = \tau_{p,q}(\mathbb{t}_{-s}(x))$ for all $x \in V^\omega(p+s)$ and $p \leq q$.

   (b) If $x \in K^{m+s}(p+s)$ for $m \geq p$, then $\mathbb{t}_{-s}(x) \in K^m(p)$.

   Therefore, $\mathbb{t}_s : K(p) \to K(p+s)$ and $\mathbb{t}_{-s} : K(p+s) \to K(p)$. Since the composition of two functions is the identity, they are inverses of each other. □

The following lemma shows that the transition function and the translation function coincide for Kripke sets which do not change at any stage:

Lemma 6.8. Let $x \in K^p(p)$. Then $\mathbb{t}_s(x) = \tau_{p,p+s}(x)$.

Proof. By the proof of 3 of Lemma 6.7, we have $\mathbb{t}_s(x)(q+s) = \tau''_{p+s,q+s}[\mathbb{t}_s(x)(p+s)]$ for $q \geq p$. Therefore, we can see that $\mathbb{t}_s(x) = \tau_{p,p+s}(x)$ is equivalent to $\mathbb{t}_s(x)(p+s) = x(p+s)$ for $x \in K^p(p)$.

We will show that $\mathbb{t}_s(x)(p+s) = x(p+s)$ by induction on rank: assume that $x \in K^p_\alpha(p)$ and the equality holds for all $y \in K^p_\beta(p)$ and $\beta \in \alpha$. Especially, we have $\mathbb{t}_s(y) = \tau_{p,p+s}(y)$ for all $y \in x(p)$. Therefore, $\mathbb{t}_s(x)(p+s) = \mathbb{t}''_s[x(p)] = \tau''_{p,p+s}[x(p)] = x(p+s)$. □

The following lemma states $\mathbb{t}_s$ behaves like an isomorphism between $K(p)$ and $K(p+s)$:

Lemma 6.9. Let $p \in \omega$ and $\vec{a} \in K(p)$.

1. If $p \Vdash_K \phi(\vec{a})$, then $p + s \Vdash_K \phi(\mathbb{t}_s(\vec{a}))$.
2. If $p \geq s$ and $p \Vdash_K \phi(\vec{a})$, then $p - s \Vdash_K \phi(\mathbb{t}_{-s}(\vec{a}))$.

Proof. We will prove them simultaneously by induction on ϕ, uniform to p, s and $\vec{a}$. We will only consider $\mathbb{t}_{+s}$, as the case for $\mathbb{t}_{-s}$ is similar.

1. Atomic formulas: assume that $p \Vdash_K a_0 = a_1$, which is equivalent to $a_0 = a_1$. Since $\mathbb{t}_{\pm s}$ are one-to-one, this is equivalent to $\mathbb{t}_s(a_0) = \mathbb{t}_{\pm s}(a_1)$, so we have $p \pm s \Vdash_K \mathbb{t}_s(a_0) = \mathbb{t}_s(a_1)$.

   If $p \Vdash_K a_0 \in a_1$, so $a_0 \in a_1(p)$, then $\mathbb{t}_s(a_0) \in \mathbb{t}''_s[a_1(p)] = \mathbb{t}_s(a_1)(p+s)$. Thus $p + s \Vdash_K \mathbb{t}_s(a_0) \in \mathbb{t}_s(a_1)$.

2. Binary connections: cases for $\wedge$ and $\vee$ are easy to prove, so we omit it. We will concentrate on the case $\to$.

   Assume that $p \Vdash_K (\phi \to \psi)(\vec{a})$ holds: That is, for any $q \geq p$, $q \Vdash_K \phi(\tau_{pq}(\vec{a}))$ implies $q \Vdash_K \psi(\tau_{pq}(\vec{a}))$. Furthermore, let $q + s \Vdash_K \phi(\tau_{p+s,q+s}(\mathbb{t}_s(\vec{a})))$ holds for $q \geq p$, which is equivalent to $q + s \Vdash_K \phi(\mathbb{t}_s(\tau_{p,q}(\vec{a})))$. By inductive assumption on ϕ, we have $q \Vdash_K \phi(\mathbb{t}_{-s}(\mathbb{t}_s(\tau_{p,q}(\vec{a}))))$, so $q \Vdash_K \phi(\tau_{p,q}(\vec{a}))$. Therefore, $q \Vdash_K \psi(\tau_{pq}(\vec{a})$ and we have $q + s \Vdash_K \psi(\mathbb{t}_s(\tau_{pq}(\vec{a}))$ by the inductive hypothesis on ψ.

   In sum, $q + s \Vdash_K \phi(\tau_{p+s,q+s}(\mathbb{t}_s(\vec{a})))$ implies $q + s \Vdash_K \psi(\tau_{p+s,q+s}(\mathbb{t}_s(\vec{a})))$ for all $q \geq p$, which means $p + s \Vdash (\phi \to \psi)(\mathbb{t}_s(\vec{a}))$.

3. Quantifiers: the case $\exists$ is easy to check. For $\forall$, assume that $p \Vdash_K \forall x \phi(x, \vec{a})$, which is equivalent to $q \Vdash_K \phi(x, \tau_{pq}(\vec{a}))$ for all $q \geq p$ and $x \in K(q)$. Therefore, we have $q + s \Vdash_K \phi(\mathbb{t}_s(x), \mathbb{t}_s(\tau_{pq}(\vec{a}))$ for all $x \in K(q)$. Since $\mathbb{t}_s$ is an onto function from $K(q)$ to $K(q+s)$, we have $q + s \Vdash_K \phi(y, \tau_{p+s,q+s}(\mathbb{t}_s(\vec{a})))$ for all $y \in K(q+s)$. Since $q \geq p$ is arbitrary, we have $p+s \Vdash_K \forall x \phi(x, \mathbb{t}_s(\vec{a}))$.

□

We are ready prove that K satisfies CZF:

Theorem 6.10. K satisfies CZF.

Proof. We can show Extensionality directly. For $\in$-induction, we can use the argument which is described in the proof of $\in$-induction in Theorem 4.10, with Lemma 6.4.

The main idea of a proof for Pairing, Union and Separation are the same: we shall prove that operations Union, up and Sep are closed under $K(p)$. For up, suppose that we have $x, y \in K(p)$. Without loss of generality, we can assume that there is $m \geq p$ such that $x, y \in K^m(p)$. We claim that $\mathrm{up}(x, y) \in K^m(p)$ by applying Lemma 6.2: it is obvious that $\mathrm{up}(x, y)(q) \subseteq K^m(q)$ for all $q \geq p$. Moreover, if $q \geq m$ then

$$\mathrm{up}(x, y)(q) = \tau''_{pq}\{x, y\} = \tau''_{mq}\{\tau_{pm}(x), \tau_{pm}(y)\} = \tau''_{mq}[\mathrm{up}(x, y)(m)]. \tag{99}$$

Therefore, $\mathrm{up}(x, y)$ stops expanding hereditarily after stage m. The case for Union is similar, so we omit it.

For Sep, it is obvious that $\mathrm{Sep}(x; \vec{y}) \subseteq K^m(p)$ if $x, \vec{y} \in K^m(p)$. Without loss of generality, assume that $m \geq p$. To show $\mathrm{Sep}(x; \vec{y})$ stops expanding after stage m, we must check that $\mathrm{Sep}(x; \vec{y})(q) \subseteq \tau''_{mq}[\mathrm{Sep}(x; \vec{y})(m)]$ for all $q \geq m$.

Let $z \in \mathrm{Sep}(x; \vec{y})(q)$. Then $z \in x(q)$ and $q \Vdash_K \phi(z, \tau_{pq}(\vec{y}))$. For notational convenience, let $s = q - m$. Since $q \Vdash_K \phi(z, \tau_{pq}(\vec{y}))$, we have $m \Vdash_K \phi(\mathbb{t}_{-s}(z), \mathbb{t}_{-s}(\tau_{pq}(\vec{y}))$. Observe that $\tau_{pm}(\vec{y} \in K^m(m)$, so we can apply Lemma 6.8 and we have

$$\mathbb{t}_{-s}(\tau_{pq}(\vec{y}) = \mathbb{t}_{-s}(\tau_{mq}(\tau_{pm}(\vec{y})) = \mathbb{t}_{-s}(\mathbb{t}_s(\tau_{pm}(\vec{y})) = \tau_{pm}(\vec{y}). \tag{100}$$

Therefore, $m \Vdash_K \phi(\mathbb{t}_{-s}(z), \tau_{pm}(\vec{y})$. Moreover, $\mathbb{t}_s(\mathbb{t}_{-s}(z)) = z \in x(q) = \tau''_{mq}[x(m)] = \mathbb{t}''_s[x(m)]$. By injectivity of $\mathbb{t}_s$, we can conclude $\mathbb{t}_{-s}(z) \in x(m)$. Hence, we can conclude $\mathbb{t}_{-s}(z) \in \mathrm{Sep}(x; \vec{y})(m)$. Since $z \in x(q) \subseteq K^m(q) = K^q(q)$, $z \in \tau''_{mq}[\mathrm{Sep}(x; \vec{y})(m)]$. (Note that our proof works not only for Bounded separation, but also for Full separation if our background theory has Full separation.)

For Strong Collection and Subset Collection, we analyze the proof of Theorem 4.10 and modify it. For Strong Collection, let $\phi(x, y)$ be a formula whose parameters belong to $K(p)$. Let $a \in K(p)$ and $q \geq p$ satisfies $q \Vdash_K \forall x \in a \exists y \phi(x, y)$. Take $m > q$ large enough that a and parameters of ϕ belongs to $K^m(p)$. By the assumption on ϕ, we have

$$\forall r \geq q \forall x \in a(r) \exists y : y \in K(r) \wedge r \Vdash_K \phi(x, y). \tag{101}$$

By letting $A = \bigcup_{q \leq r \leq m} \{r\} \times a(r)$, we have the following analogue of (43):

$$\forall \pi \in A \exists y : y \in K(\jmath_0 \pi) \wedge \jmath_0 \pi \Vdash_K \phi(\jmath_1 \pi, y). \tag{102}$$

By Strong Collection, we can find C satisfying analogues of (44) and (45).

We will define $b \in K(q)$ as we define b in (46) for $r \leq m$. For $r > m$, take $b(r) := \tau''_{mr}[b(m)]$. That is, our b is defined as follows:

$$b(r) = \begin{cases} \{\tau_{sr}(y) \mid y \in C \wedge q \leq s \leq r \wedge \operatorname{dom} y = \uparrow s\} & \text{if } r \leq m, \\ \tau''_{mr}[b(m)] & \text{if } r > m. \end{cases} \tag{103}$$

We claim that $q \Vdash \forall x \in a \exists y \in b \phi(x, y)$. Let $x \in a(r)$ for $r \geq q$. If $r \leq m$, then the proof in Theorem 4.10 works. Now assume that $r > m$. Then $x = \tau_{mr}(x_0)$ for some $x_0 \in a(m)$. Hence there is $y_0 \in b(m)$ such that $m \Vdash_K \phi(x_0, y_0)$. Therefore, $r \Vdash_K \phi(x, \tau_{mr}(y_0))$. This proves our desired result as $\tau_{mr}(y_0) \in b(r)$.

It remains to show that the Axiom of Subset Collection is valid in K. Since Subset Collection is equivalent to Fullness under Strong Collection, it is sufficient to check that Fullness is valid in K.

Let $a, b \in K(p)$. Take a large $m \geq p$ which satisfies $a, b \in K^m(p)$. Follow the proof of Theorem 4.10 with some necessary modification. We define A_q instead as follows: $A_q = \bigcup_{q \leq r \leq m} \{r\} \times a(r)$ and $B = \bigcup_{p \leq r \leq m} b(r)$. By Collection, we can find $\mathbb{C}$ such that, for each q such that $p \leq q \leq m$, there is $\mathcal{F} \in \mathbb{C}$ which satisfies (53). For each $p \leq q \leq m$, define $\mathcal{Z}_q$ as before. For given $p \leq q \leq m$ and $C \in \mathcal{Z}_q$, define $z_{q,C}(r)$ same as (55) if $q \leq r \leq m$. For $r > m$, define $z_{q,C}(r) = \tau''_{mr}[z_{q,C}(m)]$. Then we can see that $z_{q,C} \in K^m(q)$. We finally let

$$c(q) = \begin{cases} \{\tau_{sq}(z_{s,C}) \mid p \leq s \leq q \wedge C \in \mathcal{Z}_q\} & \text{if } q \leq m, \\ \tau''_{mq}[c(m)] & \text{if } q > m. \end{cases} \tag{104}$$

We claim that c witnesses Fullness. We will show that

$$p \Vdash_K \forall x : a \rightrightarrows b \exists y \in c : (y : a \rightrightarrows b) \wedge y \subseteq x. \tag{105}$$

Let $q \geq p$, $x \in K(q)$ and $q \Vdash_K x : a \rightrightarrows b$. Observe that a and b are not expanding after the stage m. Consider x' of domain $\uparrow q$ given by

$$x'(r) = \begin{cases} x(r) & \text{if } r \leq m, \\ \tau''_{mr}[x(m)] & \text{if } r > m. \end{cases} \tag{106}$$

We will see that x' is a 'support' of x that does not expand hereditarily after stage m.

By definition, x' stops expanding hereditarily after stage m. Furthermore, every elements of x is not expanding hereditarily after stage m as $q \Vdash x \subseteq a \times b$, and so does elements of x'. Therefore, $x' \in K^m(q)$. Moreover, it is easy to see that $q \Vdash_K x' \subseteq x$. Before to consider x' instead of x, we need to see that $q \Vdash_K x' : a \rightrightarrows b$ holds. We shall prove the following sentence:

$$q \Vdash_K \forall u \in a \exists v \in b : \operatorname{op}(u, v) \in x'. \tag{107}$$

Let $r \geq q$ and $u \in a(r)$. If $r \leq m$, then this follows from $q \Vdash_K x : a \rightrightarrows b$. If $r > m$, then $u = \tau_{mr}(u_0)$ for some $u_0 \in a(m)$, and we can find $v_0 \in b(m)$ such that $\operatorname{op}(u_0, v_0) \in x(m)$. Therefore, $\operatorname{op}(u, \tau_{mr}(v_0)) = \tau_{mr}(\operatorname{op}(u_0, v_0)) \in \tau''_{mp}[x(m)] = x'(r)$.

Hence, there is $\mathcal{F} \in \mathbb{C}$ and $C \in \mathcal{F}$ satisfying (53) for x'. It remains to show that $q \Vdash z_{q,C} \subseteq x'$, but this follows from $C \subseteq R_x$ and the fact that both $z_{q,C}$ and x' stops expanding after stage m. $\square$

Theorem 6.11. K validates ADCom.

Proof. Consider the following name:

$$\check{1}_p(q) = \begin{cases} \varnothing & \text{if } p < q \\ \{\check{0}\} & \text{if } p \geq q. \end{cases} \tag{108}$$

We can see that $\check{1}_p \in K^p(0)$ and $0 \Vdash_K \neg\neg(\check{1}_p = \check{1})$ holds. We prove the latter one: $0 \Vdash_K \check{1}_p \subseteq \check{1}$ is obvious. For the double negation of the remaining inclusion, observe that $0 \Vdash_K \neg\neg\phi$ iff there is p such that $p \Vdash_K \phi$ due to linearity of the Kripke frame. Thus K thinks the double complement of $\{1\}$ contains all of $\check{1}_p$ for all $p \in \omega$, if it exists.

However, if a Kripke set contains all of $\check{1}_p$, then it is a subset of none of K^m for all $m \in \omega$. Hence no Kripke sets contain all of $\check{1}_p$. Therefore, no Kripke name is a double complement of $\{1\}$. This proves WDec(1) does not exist, and so ADCom holds by Theorem 3.17. ☐

Notice that the whole construction is conveyed over CZF, so we can derive the following equiconsistency result:

Corollary 6.12. CZF and CZF + ADCom are equiconsistent. ☐

7 Double Complement over realizability

It is natural to ask what axioms and propositions are compatible with DCom, NDCom, and ADCom. The aim of this subsection is to establish the persistency for these principles under modest assumptions.

We will not devote to explain basic facts and notations on realizability models. We will use notations and theorems that come from [19] and [25]. Therefore, readers who are not familiar with McCarty-styled realizability need to consult with the mentioned articles.

The following lemma shows our cumulative hierarchy on $\mathcal{A}$-names are closed under the internal equality on $V(\mathcal{A})$. Its proof is available in Chapter 2, Lemma 6.2. of [19]:

Lemma 7.1 (The closure lemma)**.** (CZF) Let $\mathcal{A}$ be a pca and $x, y \in V(\mathcal{A})$. Then the following holds for any ordinal α:

1. If $V(\mathcal{A}) \models z \in x$ and $x \in V(\mathcal{A})_\alpha$, then there is $\beta < \alpha$ such that $z \in V(\mathcal{A})_\beta$.
2. If $x \in V(\mathcal{A})_\alpha$ and $V(\mathcal{A}) \models x = y$ then $y \in V(\mathcal{A})_\alpha$. ☐

The following lemma ensures we can apply separation for $\Vdash \phi(x)$, when $\phi(x)$ is a bounded formula. See Lemma 4.5. of [25] for its proof:

Lemma 7.2. (CZF) Let $\phi(x)$ be a bounded formula with parameters from $V(\mathcal{A})$. If $x \subseteq V(\mathcal{A})$ is a set, then

$$\{\langle e, c\rangle : e \in \mathcal{A} \land c \in x \land e \Vdash \phi(c)\} \tag{109}$$

is a set. ☐

We can prove that DCom is preserved under realizability by making use of the closure lemma. We need Σ_1-separation or Regular Extension Axiom (REA) in our proof.

Theorem 7.3. (CZF + DCom + Σ_1-Sep or CZF + DCom + REA) Let $\mathcal{A}$ be a pca. Then $V(\mathcal{A}) \models$ DCom.

Proof. Let x be an $\mathcal{A}$-name. By Lemma 7.1, we can find an ordinal α such that

$$\forall z \in V(\mathcal{A}) : V(\mathcal{A}) \models z \in x \implies z \in V(\mathcal{A})_\alpha. \tag{110}$$

Define $y = \{\mathbf{0}\} \times V(\mathcal{A})^{\mathsf{CC}}_\alpha \cap V(\mathcal{A})$. Before to proceed the proof, we have to show that

$$V(\mathcal{A})^{\mathsf{CC}}_\alpha \cap V(\mathcal{A}) = \{x \in V(\mathcal{A})^{\mathsf{CC}}_\alpha \mid x \in V(\mathcal{A})\} \tag{111}$$

is a set. If Σ_1-Sep holds, then we may use the fact that $x \in V(\mathcal{A})$ is a Σ_1-formula. We need the following fact when the case REA holds: if B is a regular set then $B \cap V(\mathcal{A})$ is a set. (See Lemma 6.1

of [25] for its proof.) Take a regular set B that contains $V(\mathcal{A})^{\mathsf{CC}}_\alpha$. Then we have $V(\mathcal{A})^{\mathsf{CC}}_\alpha \cap V(\mathcal{A}) = V(\mathcal{A})^{\mathsf{CC}}_\alpha \cap B \cap V(\mathcal{A})$, so $V(\mathcal{A})^{\mathsf{CC}}_\alpha \cap V(\mathcal{A})$ is a set.

It remains to show that

$$V(\mathcal{A}) \models \forall z : \neg\neg(z \in x) \to z \in y. \tag{112}$$

Suppose that $e \Vdash \neg\neg(z \in x)$, which is equivalent to $\neg\neg(\exists f \in \mathcal{A} : f \Vdash z \in x)$. Since $\neg\neg(V(\mathcal{A}) \models z \in x)$, we have $\neg\neg(z \in V(\mathcal{A})_\alpha)$. Hence $z \in V(\mathcal{A})^{\mathsf{CC}}_\alpha$. Therefore, we can see

$$\mathbf{p00} \Vdash \neg\neg z \in y. \tag{113}$$

Now we can see that $\lambda e.\mathbf{p00}$ is a realizer of DCom. ☐

Note that we use Pow in this proof to ensure $V(\mathcal{A})_\alpha$ is a set. Since DCom implies Pow, we can use the power sets freely. The next theorem, which shows NDCom is absolute under McCarty-styled realizability, also uses Pow. Unlike the previous result, we have to assume Pow separately:

Theorem 7.4. ($\mathsf{CZF} + \mathsf{Pow} + \mathsf{NDCom}$) $V(\mathcal{A}) \models \mathsf{NDCom}$.

Proof. Let a be an instance of NDCom, so no set is a double complement of a. Now assume that $V(\mathcal{A})$ believes $\check{a}$ has a double complement: formally, there is $b \in V(\mathcal{A})$ such that

$$V(\mathcal{A}) \models \forall x : \neg\neg(x \in \check{a}) \to x \in b. \tag{114}$$

Take $c = \{x \mid \exists e \in \mathcal{A} : e \Vdash \check{x} \in b\}$. We will prove that c is a set by showing the following general statement: for each ordinal α and a set x, if $\check{x} \in V(\mathcal{A})_\alpha$ then $x \subseteq V_\alpha$.

Its proof uses the induction on x: suppose that we have $\forall \beta \in \mathrm{Ord} : \check{y} \in V(\mathcal{A})_\beta \to y \subseteq V_\beta$ for all $y \in x$. Now assume that $\check{x} = \{\langle \mathbf{0}, \check{y}\rangle \mid y \in x\} \in V(\mathcal{A})_\alpha$. Then for each $y \in x$, there is $\beta \in \alpha$ such that $\check{y} \in V(\mathcal{A})_\beta$. By the inductive assumption, $y \subseteq V_\beta$ for some $\beta \in \alpha$. Hence $x \subseteq \bigcup_{\beta\in\alpha} \mathcal{P}(V_\beta) = V_\alpha$.

Therefore, c is equal to $\{x \in V_\alpha \mid \exists e \in \mathcal{A} : e \Vdash \check{x} \in b\}$ for some $\alpha \in \mathrm{Ord}$. (Just take α such that $b \in V(\mathcal{A})_\alpha$.) By Lemma 7.2, c is a set. Moreover, c includes a double complement of a by the following calculation:

$$\begin{array}{rcl} \neg\neg(x \in a) & \Longrightarrow & \neg\neg(V(\mathcal{A}) \models \check{x} \in \check{a}) \\ & \Longleftrightarrow & V(\mathcal{A}) \models \neg\neg(\check{x} \in \check{a}) \\ & \Longrightarrow & V(\mathcal{A}) \models \check{x} \in b \\ & \Longrightarrow & x \in c, \end{array} \tag{115}$$

which contradicts with the assumption that a does not have a double complement. ☐

It remains to show that ADCom is persistent under realizability. Since ADCom is incompatible with Pow, the persistency should not rely on power sets. This proof mimics that of Lemma 6.22. of [31].

Theorem 7.5. (CZF) If WDec(1) does not exist in V, then it also does not exist in $V(\mathcal{A})$. In other words, ADCom is absolute under realizability.

Proof. Assume the contrary that WDec(1) does not exist in V, but $V(\mathcal{A})$ thinks WDec(1) exists. From $\mathrm{WDec}(1) = \{0\} \cup \{x \subseteq 1 \mid \neg\neg(x = 1)\}$, we can deduce that the existence of WDec(1) is equivalent to the existence of $\{x \subseteq 1 \mid \neg\neg(x = 1)\}$: one direction is trivial. For the remaining direction, use $\{x \subseteq 1 \mid \neg\neg(x = 1)\} = \mathrm{WDec}(1) \setminus \{0\}$.

Therefore, there is $a \in V(\mathcal{A})$ and $e \in \mathcal{A}$ such that

$$e \Vdash \forall x : x \subseteq 1 \land \neg\neg(x = 1) \to x \in a. \tag{116}$$

Now consider $W = \{\{0 \mid x \text{ is inhabited}\} \mid \exists f \in \mathcal{A} : \langle f, x\rangle \in a\}$. We claim that $\mathrm{WDec}(1) \subseteq W$, so we have a contradiction. Let $c \in \mathrm{WDec}(1)$. Define

$$x_c = \{\langle \mathbf{0}, \varnothing\rangle \mid 0 \in c\}. \tag{117}$$

We will show that $V(\mathcal{A}) \Vdash x_c \in a$. If $c = 1$, then we can see that $\mathbf{p}(\lambda f.\mathbf{p00})(\lambda f.\mathbf{p00}) \Vdash x_c = 1$. Therefore $\neg\neg(c = 1)$ implies $\neg\neg(\exists g : g \Vdash x_c = 1)$, which is equivalent to $\mathbf{0} \Vdash \neg\neg(x_c = 1)$. (In fact, we can replace $\mathbf{0}$ to any $e \in \mathcal{A}$.) It is easy to see that $\lambda f.\mathbf{p00} \Vdash x_c \subseteq 1$. Combining with (116), we have

$$t \Vdash x_c \in a. \tag{118}$$

for $t = e \cdot \mathbf{p}(\lambda f.\mathbf{p00})\mathbf{0}$. Hence, there is $z \in V(\mathcal{A})$ such that $\langle (t)_0, z\rangle \in a$ and $(t)_1 \Vdash z = x_c$. We can see that

$$0 \in c \iff \langle \mathbf{0}, \varnothing\rangle \in x_c \tag{119}$$
$$\iff z \text{ is inhabited,} \tag{120}$$

so $c = \{0 \mid z \text{ is inhabited}\} \in W$. □

The following preservation results allow proving various compatibility results. For example, our preservation results prove the theorems in [14][4] and more:

Theorem 7.6. If ZF is consistent, then the following set of axioms are consistent with $T_0 = \mathsf{IZF} + \mathsf{RDC} + \mathsf{PAx}$ or $T_1 = \mathsf{CZF} + \mathsf{REA} + \mathsf{RDC} + \mathsf{PAx}$ respectively:

(a) $\mathsf{DCom} + \mathsf{CT}_0 + \mathsf{MP} + \mathsf{IP} + \mathsf{UP}$

(b) DCom+ There is a D-infinite set D such that D and ${}^D D$ are equipotent.

Proof. We start from a model V of ZFC for T_0, or the model constructed in [3] for T_1. Let consider the realizability models $V(\mathcal{A})$. We know that $V(\mathcal{A})$ satisfies IZF or CZF if V does. (See Chapter 3 of [19] and [25] for its proof.) Since V satisfies RDC and PAx, $V(\mathcal{A})$ also satisfy RDC and PAx by Theorem 3.1.12 of [7]. In the case of REA over CZF, apply Theorem 6.2 of [25].

The consistency of T_i $(i = 0, 1)$ with (a) comes from the combination of Theorem 7.3 and Chapter 3 and 4 of [19], or Theorem 7.1 and 9.2. of [25] under the Kleene realizability. (b) follows from Theorem 7.3 and Theorem 4.1.1 of [7] under $\mathcal{A} = D_\infty$. □

We may show the consistency of $\mathsf{IZF} + \mathsf{DCom}$ with Brouwnian principles in the same vein. However, it requires the development of some details on realizability models that satisfy continuity principles. We may establish these details by applying facts on [7] and [5], but we will not make any progress in this article.

8 Questions

Despite the results in this paper, lots of questions remain open. The major problem is the consistency of the *negation* of DCom over IZF:

Question 8.1. Is ¬DCom and NDCom consistent with IZF?

If the answer is yes, then we can show that ¬DCom and NDCom is compatible with principles like Church's thesis, by applying appropriate realizability models. Unfortunately, there is no obvious way to realize semi-classical principles like LPO, and it brings another question:

Question 8.2. If ¬DCom and NDCom consistent with IZF, then do they consistent with semi-classical principles? Especially, do they consistent with LPO or WLEM?

We examine Kripke models to produce models that are related to DCom. We have not examine models that produced by forcing (see [11] or [6]) or set realizability (see [23]). What is known is that the forcing over $\mathsf{P}^{\neg\neg}(1)$ produces a classical model of set theory (See Lemma 4.1. of [11] or Chapter IV of [6].) Therefore, there is a forcing poset that forces DCom. We may ask how about the case ¬DCom:

Question 8.3. Can we produce a model of $\mathsf{IZF} + \neg\mathsf{DCom}$ via forcing or set realizability?

Proposition 3.15 states the existence of double complement of sets that has at least two elements, implies Pow. Moreover, we can prove the double negation of $\{1\}$ is $\{x \subseteq 1 \mid x \neq 0\}$, so we can ask whether the negation of Pow implies subsets of 1 are the only sets which have its double complement. In other words, ¬Pow may imply ADCom. However, we stated that even Question 3.14 is open:

Question 8.4. Does the negation of the Axiom of Power Set imply ADCom?

[4] Note that Hahanyan [14] uses a different axiomatic system we have considered. However, we can translate the results of Hahanyan via methods in Chapter VII, Section 1 of [5].

If V is a model of ZF, then we can relate properties of $\mathbb{P}$ and semi-classical properties valid in $V^{\mathbb{P}}$. For example, we can see that $\mathbb{P}$ is directed iff $V^{\mathbb{P}} \models \mathsf{WLEM}$. Linearity of $\mathbb{P}$ is related to the validity of *Dirk Gently's Principle* (DGP) over $V^{\mathbb{P}}$:

$$(\phi \to \psi) \vee (\psi \to \phi). \tag{121}$$

We proved that DCom is still valid in $V^{\mathbb{P}}$ if $\mathbb{P}$ is linear, and it leads to the following question:

Question 8.5. Does $\mathsf{IZF} + \mathsf{DGP}$ prove DCom?

We can see that the model in Section 5.3 satisfies DGP, so $\mathsf{IZ} + \mathsf{DGP}$ does not prove DCom. However, the case for IZF is open.

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